Beam stabilization in the 2D nonlinear Schrödinger equation with attractive potential by beam splitting and radiation

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The effect of attractive linear potentials on self-focusing in waves modelled by the Nonlinear Schrödinger equation is considered. It is shown that the attractive potential can prevent both singular collapse and dispersion that are generic in the the Nonlinear Schrödinger equation in the critical dimension, and can lead to a stable oscillating beam. This is observed to involve a splitting of the beam into an inner part that is oscillatory and of sub-critical power, and an outer dispersing part.

Analysis is given in terms of rate competition between the linear and nonlinear focusing effects, radiation losses and known stable periodic behaviour of certain solutions in the presence of attractive potentials.

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I. INTRODUCTION

Variants of the nonlinear Schrödinger equation (NLS) with attractive potentials arise in many models of weakly nonlinear dispersive waves and the nonlinear phenomenon of self-focusing or wave collapse. The simplest NLS equation with only a cubic nonlinearity has unusual behaviour such as formation of self-focusing singularities in finite time when there are two or more dimensions transverse to the propagation variable. The problem is most acute for critical transverse dimension $D = 2$, which is fraught with instabilities: small perturbations in initial data or various small additional terms modelling physical features ignored in the basic cubic Schrödinger equation (CSE) can change solutions between singular collapse and dispersion.

Most such perturbations of the equation are higher order nonlinear effects, but here we consider the effects of an attractive potential, giving what will be called the Nonlinear Schrödinger Equation with Potential (NLSP),

$$i\frac{\partial \psi}{\partial t} + \nabla^2 \psi + |\psi|^2 \psi - U(\mathbf{x}) \psi = 0$$  \hspace{1cm} (1)

We shall concentrate on the two-dimensional case with $\psi = \psi(t, x, y)$, $\nabla^2 = \partial_x^2 + \partial_y^2$. This arises in models of CW laser propagation in waveguides, molecular excitations in a lattice near an inhomogeneity [1]. The Gross-Pitaevskii model [2] for attractive quasi-2D Bose-Einstein condensates, and magnetic nano-structures [3]. The standard Gross-Pitaevskii Equation for BEC’s in a confining trap has a quadratic potential, and usually also linear and nonlinear dissipation terms, but the latter terms are unimportant to the current study. Note that, in some of the applications, $t$ represents propagation distance rather than time. The cubic nonlinearity represents intensity dependent refractive index in the optical case; exciton-phonon interaction in molecular excitations; and interaction between the Bose particles in BEC: the last is repulsive in most cases, but attractive for $^7$Li atoms for example. The potential $U(\mathbf{x})$ represents a spatially
 dependent refractive index for laser propagation; spatial inhomogeneity for molecular excitations; and the confining
field or “trap” used in forming a BEC.

In situations where the initial beam is wide relative to the potential, one convenient model for the potential is a
gaussian

\[ U(r) = -h_p e^{-r^2/(2w_p^2)} \]

of depth \( h_p \), and width \( w_p \). This is plausible for a graded index fibre laser wave guide, and for molecular excitations,
where the potential represents the effect of a small impurity in the molecular lattice.

Recent numerical observations [4–7] show that such an attractive gaussian potential can have the counter-intuitive
effect of inhibiting or limiting collapse, while also preventing dispersion, leading instead to oscillatory beams that avoid
both the extremes of singular collapse and dispersion that are the only generic outcomes in the focusing 2D cubic
Schrödinger equation (CSE; the above equation with no potential). This happens for various potentials narrower than
the initial beam but sufficiently deep, with an initial focusing followed by a train of focusing/defocusing oscillations.
For example, Fig. 1 shows the amplitude evolution for a case in which the CSE develops a singularity at \( t \approx 5.73 \).

All simulations here are performed with gaussian initial data centered at the origin, normalized to height \( h \) and
width 1: \( \psi(0, r) = h e^{-r^2/2}, \) \( r = |x| \) and one can achieve this collapse inhibition with \( h \) up to 2.5, whereas without
the potential, singularity formation is observed for all \( h > h_c \approx 1.93 \). The numerical methods used are described in [5, 8].

Here we present an analysis for explaining these observations along with additional numerical studies of the structure
of the solutions. The analysis is based on a scaling argument and a collective coordinate calculation. For this, some
background is needed; sources for results stated without citation are the articles of Rasmussen and Rypdal [9, 10], the
book of Sulem and Sulem [11], as well as [1, 12–14] for more recent results involving also the influence of quadratic
potentials.
Note that since our initial data is of the form $he^{-r^2/2}$, the conserved quantities of the CSE are the "power", $N = \int |\psi|^2 dx = \pi \hbar^2$ and the Hamiltonian, "energy", $H = H_0 := \int \frac{1}{2} |\nabla \psi|^2 + \frac{1}{4} |\psi|^4 \, dx = N \left( 1 - \frac{N}{4\pi} \right)$. Thus the sufficient condition for global existence holds for $h < h_{cr} := \sqrt{N_{cr} / \pi} \approx 1.927$, and singular collapse is guaranteed for $H < 0$ holds for $N > 4\pi$, or $h > 2$. $N_{cr}$ is the critical power, which is given as the norm of the stationary ground state solution of the CSE. Also, the variance $V = \int r^2 |\psi|^2 \, dx$ can be related to the beam width defined by $\bar{r}$ by $V = N\bar{r}^2$.

When a potential is added, exact results on singular collapse in the NLSP are only known for the case of an attractive quadratic potential, which, however, is also a relevant approximation for a beam concentrated near the bottom of an attractively radial symmetric potential. The critical potential (2) can be shifted by a harmless constant and then approximated near the origin by $U(r) = \Omega_0^2 r^2 / 4$, $\Omega_0^2 = 2\hbar_p / w_{np}^2$, so we will discuss quadratic potentials in this form. The power $N$ is conserved with the same form as before, the conserved energy takes the form $H = H_0 + (\Omega_0 / 2)^2 V$ and the variance satisfies

$$V(t) = \left( V(0) - \frac{2H}{\Omega_0^2} \right) \cos 2\Omega_0 t + \frac{2H}{\Omega_0^2} \tag{3}$$

so collapse is guaranteed if $H_0(0) = H - \Omega_0^2 V(0)/4 \leq 0$. The only change from the CSE is the extending the singularity formation sufficient condition $H_0(0) < 0$ on the initial data to also include the case of equality.

Note that if no singularity forms, the variance is bounded and in fact periodic. Another kind of periodic solutions have been established by Rose and Weinstein [15]; orbitally stable steady state solutions $e^{i\lambda t} \psi(r)$ exist with $N < N_{cr}$ for a class of potentials including the quadratic form here, and gaussians large enough for the corresponding linear Schrödinger equation to have a bound state: for a detailed discussion see Chadan et al [16].

II. COLLAPSE INHIBITION: COMPETING COLLAPSE LENGTH SCALES AND RADIATION FROM SOLITON-LIKE SOLUTIONS

A. An argument for inhibition with approximate threshold, based on competing length scales

We first discuss the inhibition of the collapse by the narrow potential using a simple scaling argument. The basic idea consists in comparing the time scale $T_{pot}$ of the the linear focusing in the potential with the time scale $T_{nl}$ for the self-focusing of the beam. When the potential is narrower than the initial beam, $w_p < 1$, and $T_{pot} \ll T_{nl}$ the beam may be split into two parts before the self-focusing takes place. If the inner part separates substantially from the outer, and the power in each of the two parts is less than the critical power $N_{cr}$ needed for formation of a focusing singularity, the initially fast focusing of the inner part driven by the potential would not be capable of continuing to singularity formation: instead its width could be expected to follow the sinusoidal form in Eq. 3. Also the outer part would be subcritical and radiate outwards.

The time scales $T_{pot}$ and $T_{nl}$ are estimated as follows: For the CSE with real-valued initial data (plane waves), the parabolic evolution of the variance implies that the singularity time must satisfy $T \leq T_{nl} := \sqrt{N / 4\pi}$. For the same initial data with the quadratic potential, the sinusoidal variance evolution (3) gives the bound $T \leq T_{pot} := \frac{\pi w_p}{2\sqrt{2}h_p}$.

It is interesting to note that this crude upper bound depends only on the linear potential, once the nonlinearity has initiated focusing of the initial plane wave. However, the quadratic potential approximation only applies when a significant part of the beam’s power lies within the concave part of the gaussian potential, so this estimate should only apply when $h_p$ is not much smaller than 1.

Thus, the scaling argument suggests qualitatively that if the linear focusing effect has a shorter length scale, $T_{pot} < T_{nl}$, equivalent to

$$h_p > \frac{w_p^2}{2} \pi^2 \left( \frac{\hbar^2}{\hbar_{cr}^2} - 1 \right) \tag{4}$$

rapid linear focusing of the central part of the beam initially within the potential could cause it to split from the outer part of the beam that is focusing only under the slower nonlinear effect. If the splitting is sufficient we may expect the inhibition of the collapse and a behavior as sketched above: a quasi-steady beam with oscillating width (and amplitude) following Eq. (3) and trapped in the potential. This beam will be surrounded by and expanding wave form in the outer region.

In fact numerical results do show this spatial splitting with inner part having power just below $N_{cr}$, as seen in the next section. So we observe that this condition for splitting and thereby collapse arrest is in qualitative agreement with the numerical results when $w_p \lesssim 1$. 


TABLE I: Shallowest potentials for collapse inhibition at various small beam widths

<table>
<thead>
<tr>
<th>$h_p$</th>
<th>1.3</th>
<th>1.8</th>
<th>4</th>
<th>10</th>
<th>40</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_p$</td>
<td>.35</td>
<td>.2</td>
<td>.1</td>
<td>.05</td>
<td>.02</td>
<td>.01</td>
</tr>
</tbody>
</table>

However, for $w_p \ll 1$, $h_p$ for arresting the collapse does not scale like 4 at all. For smaller $w_p$, an initial self-focusing can be expected to follow roughly the dilation transform form seen without a potential, i.e., $|\psi(t, x)| = R(x/L(t))/L(t)$, where $R$ designates the ground state solution of the stationary CSE (see also the following section). Thus, roughly $|\psi(t, 0)| w' = |\psi(0, 0)| = h$ (the initial width is 1) and when the beam reaches the width $w' \approx w_p$, the amplitude is $h' \approx h/w_p$, after which the potential-focusing takes over. Reversing this dilation would correspond to the case with the original initial data, but with the potential given as $w_p' \approx 1$, $h_p' \approx h_p w_p$ resulting in a rescaling of the splitting condition to

$$h_p w_p > \frac{\pi^2}{2} \left( \frac{h^2}{h^2_{cr}} - 1 \right).$$

This can be compared to the values in Table I.

This mechanism for an initial inhibition of singularity formation does not rule out the possibility that self-focusing could bring the power of the inner part above the critical threshold on its slower time scale, after the initial inner focusing has produced one or more inner oscillations; this would lead to the pattern of one or more oscillations followed by singularity formation that is also observed above for potential just large enough to inhibit singularity formation in the initial beam collapse. Thus, to determine whether singularity formation can be permanently prevented, radiation needs to be considered, and a more detailed model of the radiation mechanism removing power from the region of the potential is needed.

### B. Radiation model for relaxation to sub-critical inner bound states

The collective coordinates method used for example to derive the form of the singular solutions of the CSE [17] and collapse inhibition by an attractive potential with moving off-centre beam [6] can be used to study the evolution of beam width, describing the inner, focusing, part of the beam and its interaction with radiation to the outer, dispersing part. Omitting details, which are similar to these cited precedents, one describes the behaviour in terms of a beam width scale variable $L$ and an excess of power over critical in the inner region, $\Delta$, related to the original equation through

$$\psi(t, x) = \frac{1}{L(t)} \Phi(\tau, \xi) \exp \left( i \tau + i \frac{\dot{L}}{L(t)} \frac{r^2}{4} \right), \quad \xi = \frac{x}{L(t)}, \quad \frac{\partial}{\partial t} = \frac{1}{L^2(t)}, \quad r = |x|$$

and

$$\Delta = \frac{N_s - N_{cr}}{M}$$

Here, the width scale is determined by the condition that the transformed profile $\Phi$ is almost stationary, and in the inner regions is approximated by the Townes soliton $R$, the stationary state profile for the CSE given by the positive radially symmetric solution of

$$-\nabla^2 R + R - R^3 = 0,$$

$N_s$ represents the power within some inner region relative to the width scale,

$$N_s = \int_{|x| \leq \xi \cdot L(t)} |\psi(t, x)|^2 \, dx = \int_{|\xi| \leq \xi_s} |\Phi(\tau, \xi)|^2 \, d\xi$$

$N_{cr} \approx 11.67$ is the power of the Townes soliton, the critical power needed for singularity formation, and

$$M = \frac{1}{4} \int |\xi|^2 \psi^2 \, d\xi \approx 3.4$$
By considering the radiation rate for the core power we eventually get the equations

\[ \dot{L} = -\frac{\Delta}{L^3} - \frac{1}{2M} \frac{\partial}{\partial L} \mathcal{V}(L, x) \]  
(10)

\[ \Delta = -\frac{\gamma}{L^2} \left( \frac{N_c}{M} + \Delta \right) \]  
(11)

with the effective potential

\[ \mathcal{V}(L) = \int U(Lx)R^2(|x|) \, dx, \]  
(12)

\[ \gamma = H(\beta) \exp\{-\pi/\sqrt{\beta}\}, \quad \beta = \Delta + \frac{L^3}{2M} \frac{\partial \mathcal{V}}{\partial L} \]  
(13)

and \( H(\beta) \) is the Heaviside function. These equations show that the width dynamics controls the tunneling rate and in this way modifies the core power kinetics.

One can eliminate the integral get a closed ODE approximation by replacing \( R(r) \) with its Gaussian approximation in the form

\[ R(r) \approx R_g(r) = \sqrt{\frac{N_c}{\pi B^2}} \exp\left( -\frac{r^2}{2B^2} \right) B^2 \approx 0.8. \]  
(14)

resulting in

\[ \frac{1}{M} \dot{\mathcal{V}}(L) = -\frac{N_c}{M} \frac{h_p^2 w_p^2}{2w_p^2 + B^2 L^2} = \frac{4.3 h_p^2 w_p^2}{2.5 w_p^2 + L^2} \]  
(15)

However, as the resulting system can still only be studied numerically, an alternative is to evaluate the effective potential using either numerical approximation of \( R(r) \), or the more accurate approximation

\[ R(r) \approx R_s(r) = K_s \text{sech}(r/B). \]  
(16)

In comparisons, the sech version consistently gave somewhat better fits to the NLSP, so only this is used here. We compared to the case \( h = 1.95, w_p = 0.5 \) where inhibition occurred for \( h_p \) above 0.12, and compared to that for \( h_p = 0.14 \) as shown in Fig. 1. The equivalent initial conditions for ODE’s are

\[ L(0) = 1.4, \dot{L}(0) = 0, \Delta(0) = 0.07, \]

but collapse inhibition does not occur until about \( h_p = 2.3 \), and to match more closely the PDE’s behaviour for \( h_p = 0.14, h_p = 5 \) is used in the ODE. For comparison the ODE amplitude data, \( 1/L \) is plotted in Fig. 2 as well as the power surplus \( \Delta \). This shows fairly good qualitative agreement with the full solution of the PDE, with the major discrepancy being an underestimation of the radiation rate, requiring the deeper potential. In particular the reduction of power in the inner region below the critical threshold is observed to be sharply the criterion for collapse inhibition in both the NLSGP and the collective coordinates model.

### III. NUMERICAL RESULTS ON RADIATION FROM OSCILLATING BEAMS

The main results here are a study of the observed transverse structure of solutions, which are seen to conform qualitatively to the above theoretical model.

Several observations can be made:

**Width oscillation in trapped beams with near ground state form** As shown in Fig. 3 for our “standard case” \( h = 1.95, h_p = 0.14, w_p = 0.5 \) (and for other case in [4, 5], when a potential prevents collapse or dispersal, the beam instead has oscillations in width and height, and width amplitude profiles following roughly a dilation pattern at small radii of the form

\[ |\psi(t, r)| \approx \frac{1}{L(t)} \tilde{R} \left( \frac{r}{L} \right) \]
FIG. 2: Collective coordinates approximations with the sech approximation of the Townes soliton, $L(0) = 1.4$, $L'(0) = 0$, $\Delta(0) = 0.07$, $w_p = 0.5$, $h_p = 5$. 
FIG. 3: $h = 1.95, h_p = 0.14, w_p = 0.5$ \(|\psi(t, r)\)| profiles showing oscillations in Townes soliton form: first height max solid, first min dotted, second max dashed, second min. dash-dot.

for an asymptically oscillatory width scale $L(t)$ and spatial profile $\hat{R}$ somewhat close to the ground state $R$ but with power slightly below the critical value; and therefore potentially close to stable bound states $u(r, L)$ of NLSP, at least when $L$ is small. This supports the approximation used for the inner region part of the collective coordinate model.

**Near null of intensity separating inner and outer parts** At each minimum of beam width in an oscillatory solution, the intensity drops very close to zero at a “separation point” at approximately the potential’s width, as shown for our standard case in Fig. 4, where the three curves at lower left are at the first three minima of beam width while the other two are at intervening width maxima. Fig. 5 shows the same for a beam with higher initial power, leading to considerably more power outside this separation point. In each case the intensity at the separation point is less than one thousandth of its maximum.

**Radiation** Beyond the central oscillating spike and this separation point, one observes waves radiating outwards, see Fig. 6, carrying power well beyond the potential and spreading to such low intensities that nonlinearities are unimportant, thus the self-focusing effect will not refocus this power. Accordingly we find that the radiation is well described by the linear dispersion relation of the CSE, which at large radii reduces to $\omega = k^2$. 
IV. CONCLUSIONS

- Modifying the 2D focusing Cubic Schrödinger Equation by the addition of a small attractive gaussian potential can in some case prevent singular wave collapse that would occur for the same initial data without it, leading to an oscillatory inner part with less than the minimum power required for singularity formation, plus radiation.

- The initial failure of singularity formation is qualitatively explained by the linear focusing due to the potential occurring faster than the self-focusing collapse, leading to less than the critical power for collapse entering the inner region while the rest is left outside the potential well.

- Further from the potential, there is radiation at each inward cycle of such oscillations, particularly the first, and this is also modelled via the lens transformation and a collective coordinates reduction, successfully predicting the main quantitative feature of beam width oscillations that decay somewhat, the power within the potential well becoming sub-critical and hence preventing singularity formation, and power loss form the central focus through radiation at each oscillation.

- Singularity inhibition can occur with extremely narrow potentials, where the initial power outside the potential region is itself above critical. It appears that initial self-focusing occurs with little affect from the potential, bringing this power into the potential region so that the above mechanism can then act.
FIG. 5: $h = 2.1, h_p = 3.48, w_p = 0.3, |\psi(t, .)|$ with extreme splitting; $|\psi(t, 0)| = 115$

FIG. 6: $h = 1.95, h_p = 0.14, w_p = 0.5 \Re(\psi) \text{ vs. } r$ at $t = 5, 10, 14.3$ showing radiation