Multi-focusing and sustained dissipation in the dissipative nonlinear Schrödinger equation

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Abstract

The possibility of physically relevant singular solutions of the nonlinear Schrödinger equation with sustained dissipation into the singularity is considered through numerical study of a nonlinear dissipative regularisation and its small dissipation limit. A new form of such dissipative solutions is conjectured for certain parameter ranges where this behaviour was previously not expected, including the two-dimensional case of laser self-focusing, involving a multi-focusing mechanism. The space and time structure of such solutions for very small values of the nonlinear dissipation parameter is studied numerically and compared to a conjectured mechanism related to a new family of stationary singular solutions of the NLSE. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Nonlinear; Schrödinger equation; Dissipative

1. Background

The nonlinear Schrödinger equation (NLSE)

$$\frac{\partial \psi}{\partial t}(t, x) = \mathrm{i}\Delta \psi(t, x) + \mathrm{i}|\psi(t, x)|^{2\sigma}\psi(t, x), \quad t \geq 0, x \in \mathbb{R}^D \tag{1}$$

is a generic model for the slowly varying envelope of a wave-train in conservative, dispersive, mildly nonlinear wave phenomena. Physical applications include the collapse of various wave-modes in plasmas where $D = 3$ as well as laser self-focusing where $D = 2$.

This wave collapse (or self-focusing) is manifested in solutions of the NLSE by the development of large gradients in small regions and in some cases, singularities in finite time.

All the main physical examples have the cubic nonlinearity so most references will be to the cubic Schrödinger equation (CSE)

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$$\frac{\partial \psi}{\partial t}(t, x) = \mathrm{i}\Delta\psi(t, x) + \mathrm{i}|\psi(t, x)|^2\psi(t, x) \tag{2}$$

with generalisations to other σ given in square brackets (e.g. [·]).

2. Singular solutions and dissipative regularisation

For $D \geq 2$ [$\sigma D \geq 2$], it is possible for solutions to develop singularities at some finite time t_0 as shown by Vlasov, Petritshev and Talanov [1] and Glassey [2].

The few known explicit singular solutions and numerical solutions strongly suggest that generically, this takes the form of a single point focusing singularity.

Physically, such singularities will be prevented by the regularising effect of mechanisms ignored in the above equation, and one important case is nonlinear dissipation through multi-photon absorption modelled with dissipation coefficient, βI^μ, dependent on the intensity $I = |\psi|^2$, with μ depending on the number of photons involved in the interaction: this gives the dissipative nonlinear Schrödinger equation (DNLSE)

$$\frac{\partial \psi}{\partial t} = \mathrm{i}\Delta\psi + \mathrm{i}|\psi|^{2\sigma}\psi - \beta|\psi|^{2\mu}\psi. \tag{3}$$

The typical physical case is $\mu = 3$ but the details of the dissipation are not clear, and so one hopes for results that are somewhat independent of the details of the dissipative term. Thus, for the most part, the remaining discussion further specialises to $\sigma = 1$, $\mu = 3$, with generalisations following in brackets.

3. Collapse types

If dissipation arrests focusing and leads to defocusing, scaling arguments suggest that this will happen when the focusing and dissipation terms are in balance which occurs for $I \approx 1/\sqrt{\beta}$ at which point the time scale of the evolution is $1/I, \approx \beta^{1/2}$ and the approximate length scale is $1/\sqrt{I}, \approx \beta^{1/4}$.

Thus, one expects dissipation to occur principally in a dissipative core where $r \lesssim \beta^{1/4}$ at a rate per unit volume of $\beta I^4 \approx 1/\beta$ suggesting that the total loss of power, $N = \|\psi\|_2^2$, will scale like

$$\delta N \approx \beta^{(D-2)/4}$$

This allows for a nonzero limiting dissipation only in the case $D = 2$, consistent with arguments based on solutions in the nondissipative case which suggest that a fixed quantum of power would be dissipated in such an event: this scenario has been called strong collapse, while the expected absence of limiting dissipation is called weak collapse.

However, if stationary singular solutions of the NLSE exist with dissipation into a point singularity, it is possible that focusing solutions would persist near these rather than rapidly defocusing, giving increased total dissipation at a rate $O(\beta^{(D-4)/4})$: this collapse scenario has been called super-strong, or distributed.

4. Stationary singular solutions of the NLSE

Stationary solutions of (NLSE) singularity were found by Malkin [3] and Zakharov et al. [4] for $3 < D \leq 4$ $[1/\sigma + 2 < D \leq 2/\sigma + 2]$.

In the external case $D = 4$ $[D = 2/\sigma + 2$, e.g. $D = 3$, $\sigma = 2]$ there are exact explicit solutions

$$\psi(r) = B \frac{\exp\left[\mathrm{i} \log r \sqrt{B^2 - 1}\right]}{r} \tag{4}$$

where $r = \|x\|$, $B > 1$. Both the amplitude and phase are as for the large r behaviour of the self-similar blow-up solutions of the CSE as described by LeMesurier et al. [5] for $D > 2$, making it feasible for focusing solutions to get close to these singular ones. The parameter B is related to a power flux into the singularity of

$$P \overset{\mathrm{def}}{=} \lim_{r \to 0} |\psi|^2 r^{D-1} \frac{\mathrm{d}}{\mathrm{d}r} \arg \psi = B^2 \sqrt{B^2 - 1} \tag{5}$$

Note that this is the one case where the scaling argument above suggests a total dissipation rate that does not scale as a negative power of β, allowing the possibility that the limit as $\beta \to 0$ of the dissipation rate of the DNLSE matches the rate of a singular solution of this kind.

For $3 < D < 4$ solutions are known only as asymptotic expansions, still with amplitude $O(1/r)$ and the possibility of a flux into the singularity, but for $D = 3$ the solution form changes to one with amplitude $O(1/(r \log r))$ (slightly slower than the decay rate of self-similar focusing solutions of the CSE) and has zero dissipation rate.

For $2 < D < 3$ $[2/\sigma < D < 1/\sigma + 2]$, existence of another class of dissipative singular solutions was first observed in [6], based on existence results of Lions [7]. The construction starts with asymptotic expansions for real-valued (and so nondissipative) solutions, of the form

$$\psi(r) = \frac{B}{r^\alpha}(1 + b_1 r^\delta + b_2 r^{2\delta} + \cdots) \tag{6}$$

with $\alpha = D - 2$, $\delta = 2(3 - D)$ and B the sole free parameter: Lions proved, for integer dimensions at least, the existence of singular solutions matching the first two terms of this form.

One can get then get expansions for solutions $\psi = A\,\mathrm{e}^{\mathrm{i}\theta}$ with arbitrary inward flux P by combining a real solution of

$$A'' + \frac{D-1}{r}A' + |A|^2 A - \frac{P^2}{r^{2D-2}A^3} = 0$$

with the phase given by $\theta' = -Pr^{1-D}/A^2$: the extra term in the equation affects only later terms in the expansion, and any obstructions can be resolved with standard logarithmic corrections that do not affect the leading order behaviour. The dissipation term adds power loss at the rate $\beta \int I^4 r^{D-1}\,\mathrm{d}r$ $[\beta \int I^{\mu+1} r^{D-1}\,\mathrm{d}r]$.

The role of such solutions in describing solutions of the DNLSE can be expected to be confined to a focusing shoulder region where the focusing nonlinearity is significant but the dissipative term is not lying between the dissipative core and the outer region where intensity is too low to cause significant focusing: $1 \ll I \ll 1/\sqrt{\beta}$ and hence $\beta^{1/4} \ll r \ll 1$.

5. Numerical methods

To resolve solutions well on the extremely fine spatial scales that develop near the focus while respecting boundary conditions, a modification of earlier "dilation rescaling" methods [5,8–10] is used here. For the radially symmetric case, the spatial variable $r \in [0, r_{max}]$ is related to a computational variable ρ on a fixed grid by

$$r = f(\rho, l(t)), \quad \rho \in [0, 1] \tag{7}$$

so the transformed equation is

$$\psi_t = \mathrm{i}\Delta\psi + \mathrm{i}|\psi|^{2\sigma}\psi - |\psi|^{2\mu}\psi + \psi_r r_l l_t. \tag{8}$$

Note that all derivatives of ψ including those in the Laplacian are still with respect to the physical co-ordinate r, not ρ.

The transformation function should be odd, increasing, achieve a desired scale length l near the focus by having $f_\rho(\rho, l)|_{\rho=0} = l$, and fix the outer boundary by having $f(1, l) = r_{max}$. The form used here is

$$f(\rho, l) = l \sinh(k(l)\rho) \tag{9}$$

where $k(l)$ is determined by the condition $f(1, l) = r_{max}$ to fix the outer boundary. The length scale $l(t)$ is based on the functional

$$l^*(\psi(t, \cdot)) = C \frac{\int |\psi||\nabla\psi|^D r^{D-1} \, \mathrm{d}r}{\int |\nabla\psi|^{D+1} r^{D-1} \, \mathrm{d}r}. \tag{10}$$

This functional is designed to be convergent for all relevant values of D and σ in the presence of the behaviour $|\psi| \approx r^{-1/\sigma}$ that develops as the singularity is approached, and to be numerically stable (which simpler measurements at the origin only are not).

To get stable, manageable implicit time stepping schemes, the evolution of $l(t)$ is decoupled from the main evolution equation, determining its values through a time step before that step is started, using

$$\frac{\mathrm{d}l}{\mathrm{d}t} = \frac{l_n^* - l_{n-1}^*}{t_n - t_{n-1}} + \frac{l_n^* - l_n}{t_n - t_{n-1}}, \quad \text{on } [t_n, t_{n+1}] \tag{11}$$

where $l_n^* = l^*(t_n)$ etc.

The time discretisation is done by a partially implicit second order accurate PC method so as to avoid solving nonlinear equations with implicit time differencing: the Laplacian term is handled by the implicit trapezoid rule scheme and the other terms by the two stages of the modified Euler method.

5.1. Accuracy checks

The NLSE has several conserved quantities that can be used to check the accuracy of solutions, in particular the power $N = \|\psi\|_2^2$. In the dissipative case power is not conserved but has a simple evolution equation: this can be integrated and the values checked against the actual power. Other functionals can be treated similarly but it is best to keep to ones that do not involve spatial derivatives: thus the $L^{2(\sigma+1)}$ norm is also checked, and data are discarded when either error exceeds an appropriate tolerance.

This has worked well in practice: the stage of a run where this test fails corresponds well to the start of significant divergence of the solution from results of computations with more refined discretisations.
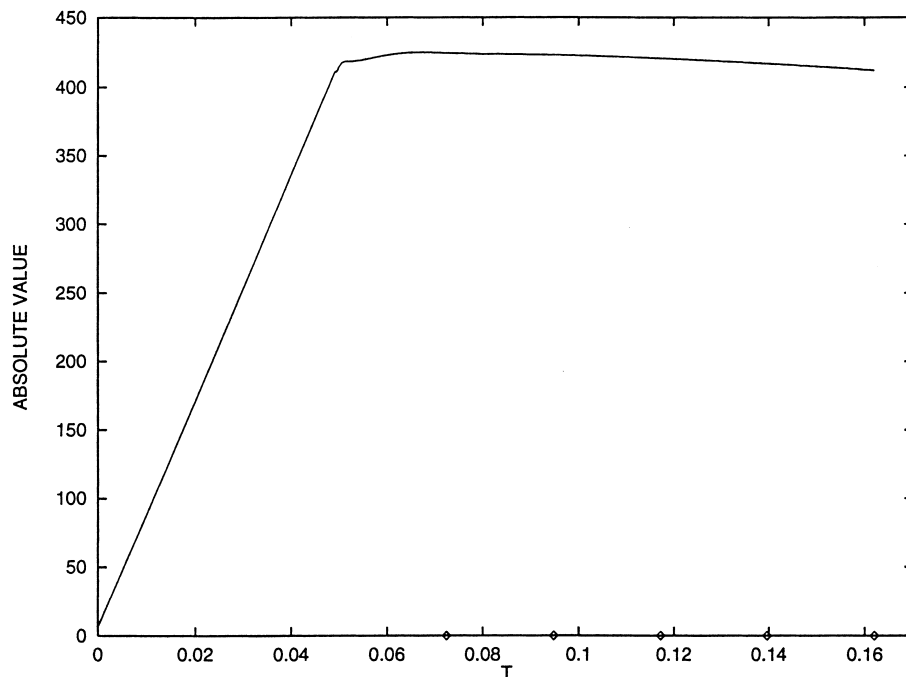
Fig. 1. $D = 4$, $h = 6$, $\beta = 10^{-10}$: maximum intensity, with marks on the axis at the times of the spatial profiles in the following figures.

6. Numerical results and observations

In this section $\sigma = 1$ throughout, the numerical results presented are for initial data $\psi_0 = h \, e^{-r^2}$, and have $\mu = 3$. The last is justified by the observation in [6] and other unpublished computations that larger μ values give qualitatively very similar results at "corresponding" β values, meaning ones that give the same maximum amplitude (and hence similar space and time scales). Note also that power and other spatial integrals are computed using the unnormalised volume measure $r^{D-1} \, dr$.

The case $D = 4$ is a useful theoretical starting point as it has the most clear-cut behaviour, and is already well studied in [4,6,11,12], with the numerical evidence supporting the scenario of "super-strong collapse" through a quasi-stationary state described above. The basic pattern is shown by the amplitude which is nearly constant at the focus (Fig. 1) and as a function of radius r (Fig. 2, which uses intensity $I = |\psi|^2$), and by the dissipation rate which is almost steady, slowing only when the total power available for dissipation is substantially diminished (Fig. 3).

Further, graphs of radial cross-sections at a succession of times all show a very slow and monoatomic decay of the conjectured quasi-stationary state as power is lost, and as discussed shortly, the spatial structure confirms that the form fits the conjecture based on the singular stationary solutions in the focusing shoulder, modified in the dissipative core.

The main example used here has $h = 6$ and very small $\beta = 10^{-10}$, for which the dissipative core is expected to occur for intensity $I \gtrsim 10^5$; the figures show that this inequality holds for $r \lesssim 0.01$. The r^{D-1} weight in the distribution of volume means that there is also a significant proportion of the dissipation at radii somewhat beyond the nominal boundary of the dissipative core, but still with most of it occurring
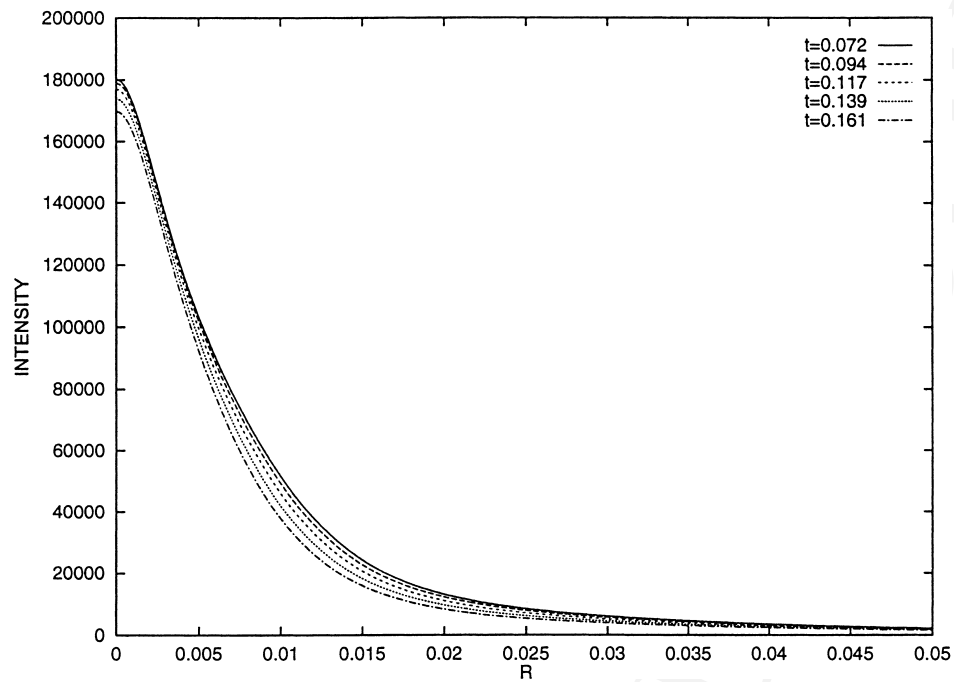
Fig. 2. $D = 4$, $h = 6$, $\beta = 10^{-10}$: spatial profiles of intensity $|\psi|^2$.
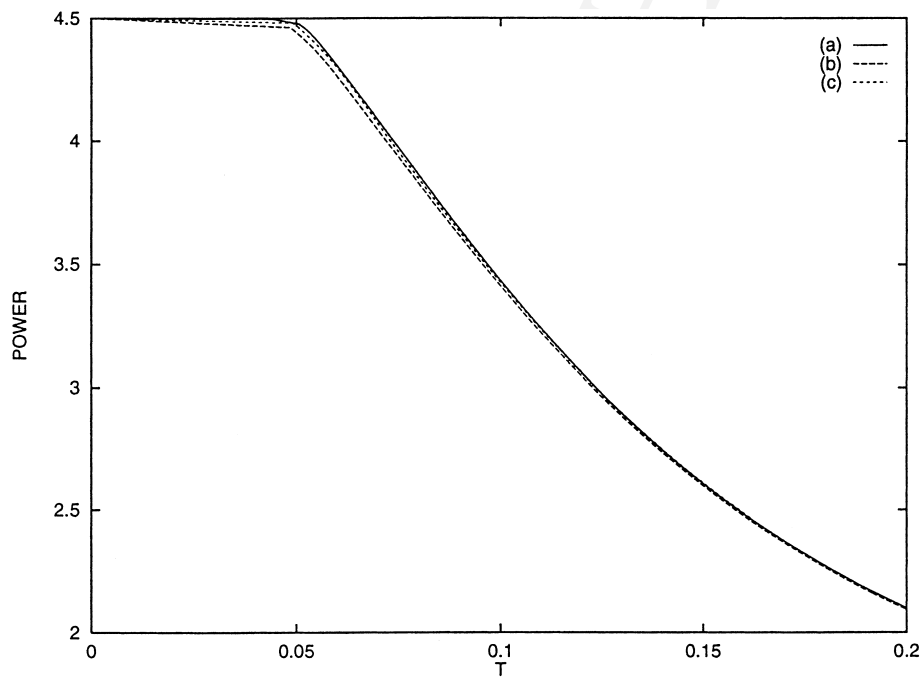
Fig. 3. $D = 4$, $h = 6$,: power as function of time for (a) $\beta = 10^{-6}$; (b) $\beta = 10^{-8}$; (c) $\beta = 10^{-10}$, showing dissipation rate nearly independence of time and β.
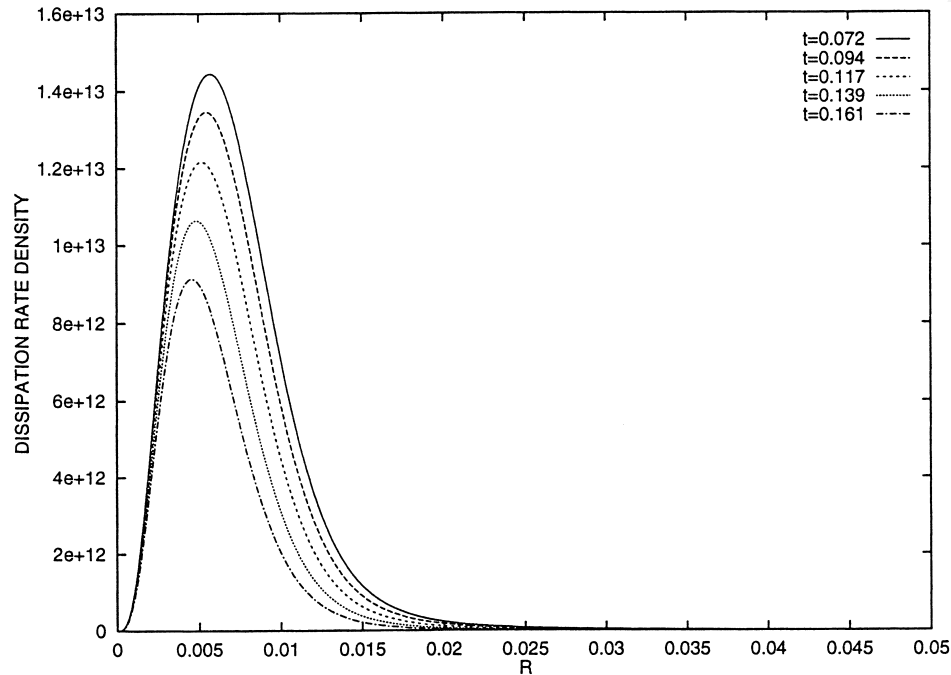
Fig. 4. $D = 4$, $h = 6$, $\beta = 10^{-10}$: dissipation rate per unit r, concentrate in $r < 0.02$.

for $r \lesssim 0.02$ as shown by the "dissipation rate density" $I^4 r^{D-1}$ $[I^{\mu+1} r^{D-1}]$ in Fig. 4 (for higher powers of μ this "dissipation core boundary" would be even sharper). The focusing shoulder should then extend from there until $r \approx 1$. The log–log graph of intensity (Fig. 5) shows a good fit to the conjectured B^2/r^2 form over the range of $01 \le r \le 1$ identified above as the focusing shoulder, and the phase also fits the expected $\log r$ behaviour over the same interval (Fig. 6: note that all phase values are relative to the value at the origin, and in fact it is the value at larger radii that is really more constant).

The second main point is that most of the dissipation occurs in and near the dissipative core while most of the total power is outside this region (Fig. 7), so that the mechanism of sustained dissipation splits into a combination of a focusing flux (see Eq. (5)) from the shoulder to the core and then dissipation within the latter: this mimics the conjectured limiting case where the dissipation occurs only at the singularity point itself.

The physically relevant cases $D = 3$ and $D = 2$: unfortunately the picture for $2 \le D \le 3$ is more complicated. Firstly as, β diminishes, the initial focus is sustained for a decreasing interval of time before a defocusing occurs (Fig. 8); as one might expect since the dissipation rate is expected to exceed the focusing flux for small enough β. Further the total dissipation before defocusing decreases as β does, apparently converging to zero (Fig. 9).

However, as these figures also show, the beam can refocus after collapse of the initial focus and as shown in [6] and elsewhere, the time scale of refocusing also diminishes with β, raising the question of what the total dissipation is through multiple focii over a fixed interval of time $[T, \tau]$, where T is the singularity time for $\beta = 0$. In other words, does the limit $\beta \to 0$ for fixed $\tau > T$ still give a total remaining power that decreases as a function of τ, as the evidence strongly suggests for $D = 4$?
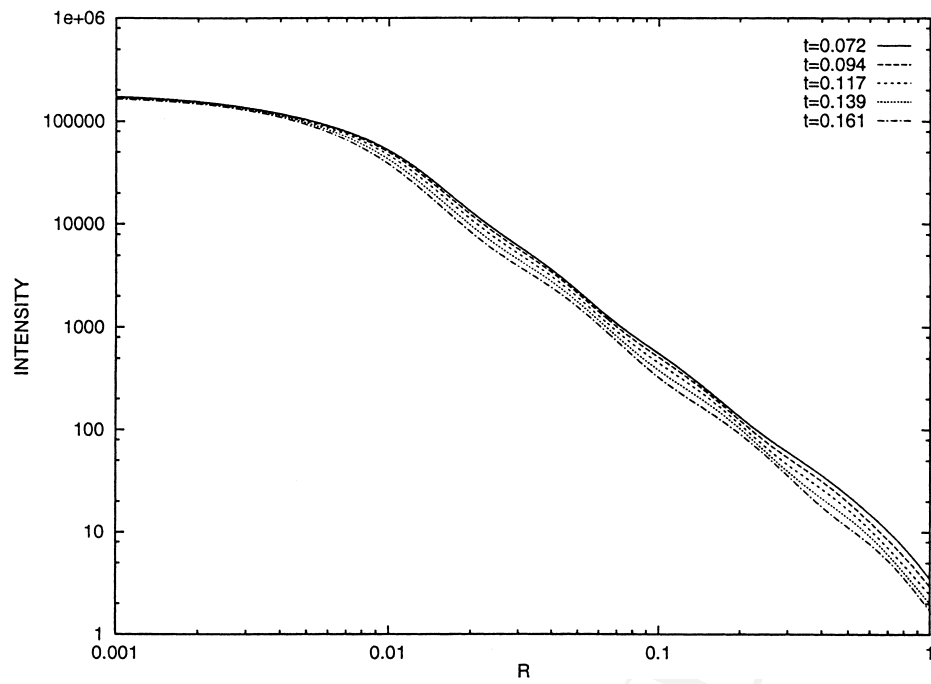
Fig. 5. $D = 4$, $h = 6$, $\beta = 10^{-10}$: log–log intensity curves, showing $O(1/r^2)$ form in the focusing shoulder.
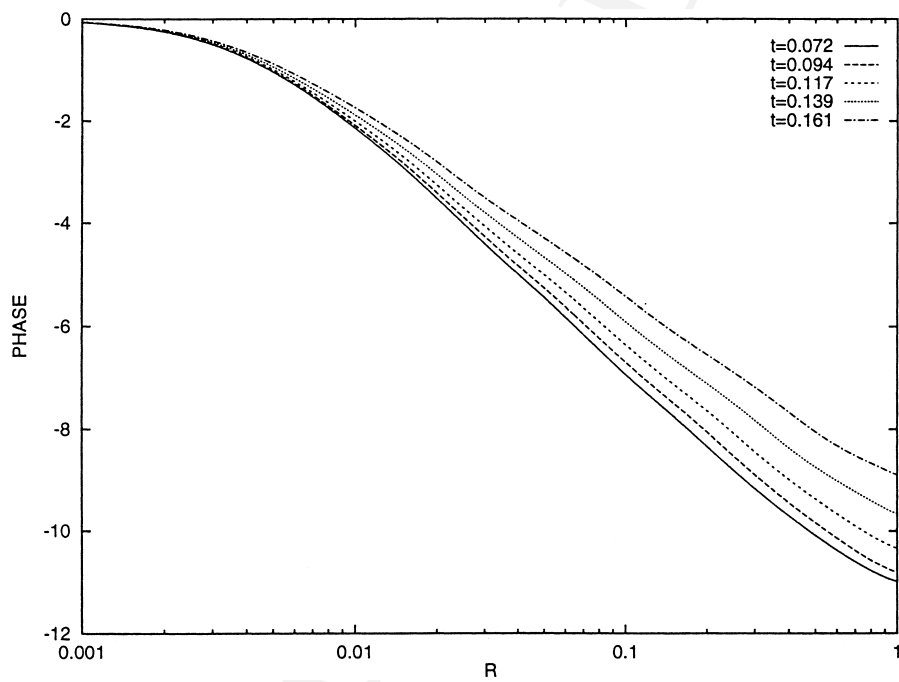

Fig. 6. $D = 4$, $h = 6$, $\beta = 10^{-10}$: log-linear relative phase curves, showing $O(\log r)$ form in the focusing shoulder.
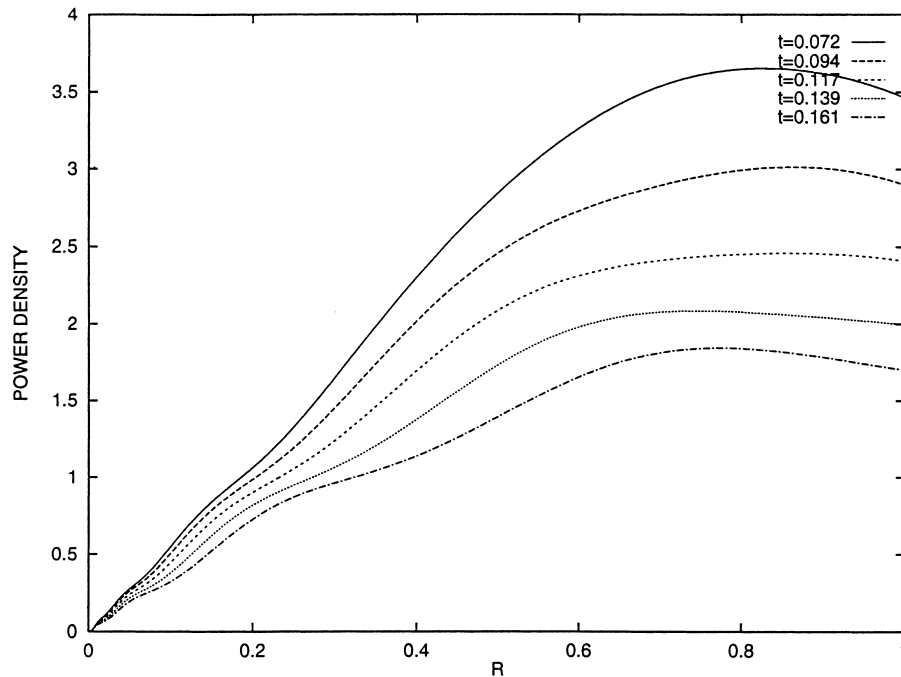
Fig. 7. $D = 4, h = 6, \beta = 10^{-10}$: power distribution per unit radius, almost all outside dissipative core.

For $D = 3$ the densities of dissipation rate power and are again largely separated (Figs. 10 and 11), so one can still describe the process as an interaction between different phenomena in the core and shoulder regions. The boundary of the dissipation core is at about $r = 0.05$. During the focus collapse, waves develop in the intensity travelling out to the inner part of the shoulder ($r \approx 0.1$), and these hold considerably more power than the core (Figs. 11 and 12).

Also, the phase flattens out in the dissipation core (Fig. 13), as it must: the phase gradient is a factor in the intensity flux density of Eq. (5). However, further out, in the inner part of the shoulder, the phase difference from the origin hardly changes: the faster phase growth at the origin during focusing has slowed or stopped but has not been significantly reversed, so this flattening pushes out the region of steep phase gradient into the inner shoulder. This phase gradient is in the region where the intensity waves reach, and is seen to drive them back to the core after a short time: this persistent phase gradient structure could explain why refocusing occurs far faster than the initial focus formation, in which the phase gradient had to develop from nothing.

For the laser propagation case $D = 2$, with smaller $\beta = 10^{-10}$, the intensity looks similar but behaviour of the phase from the collapse of the initial focus to the rapid formation of the next (Fig. 14) has a somewhat different form, but with an even clearer indication of the preservation of the overall phase gradient across the shoulder. Bear in mind that the values are computed relative to the origin: in reality the values in the outer part are changing little except for a small oscillation propagating out, while the phase in the core is retarded during defocusing but then rapidly rebuilt as the phase gradient in the shoulder produces an inward flux and hence refocusing.
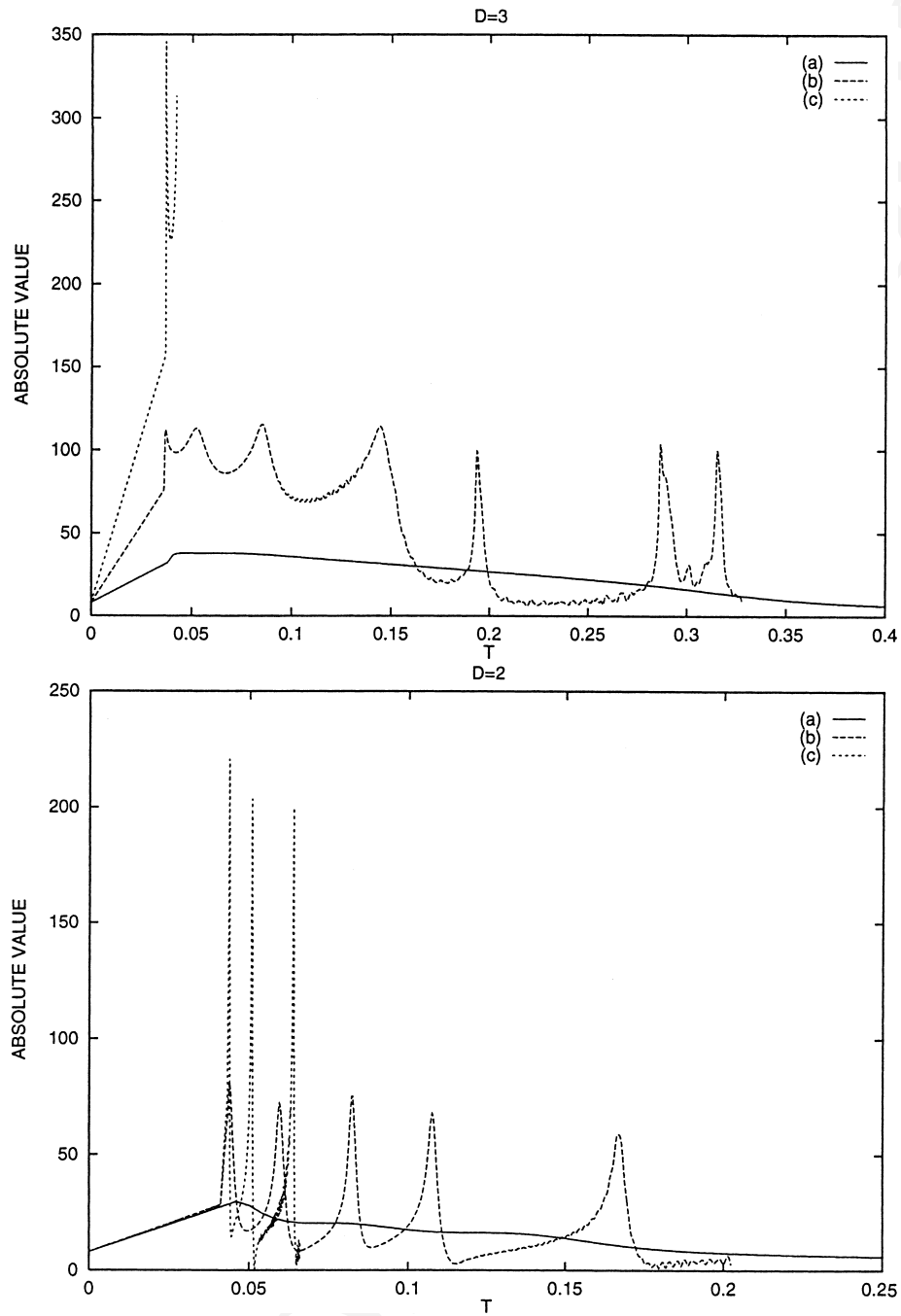
Fig. 8. Multi-focusing for $D = 3$ and 2: intensity evolution for (a) $\beta = 10^{-6}$; (b) $\beta = 10^{-8}$; (c) $\beta = 10^{-10}$.
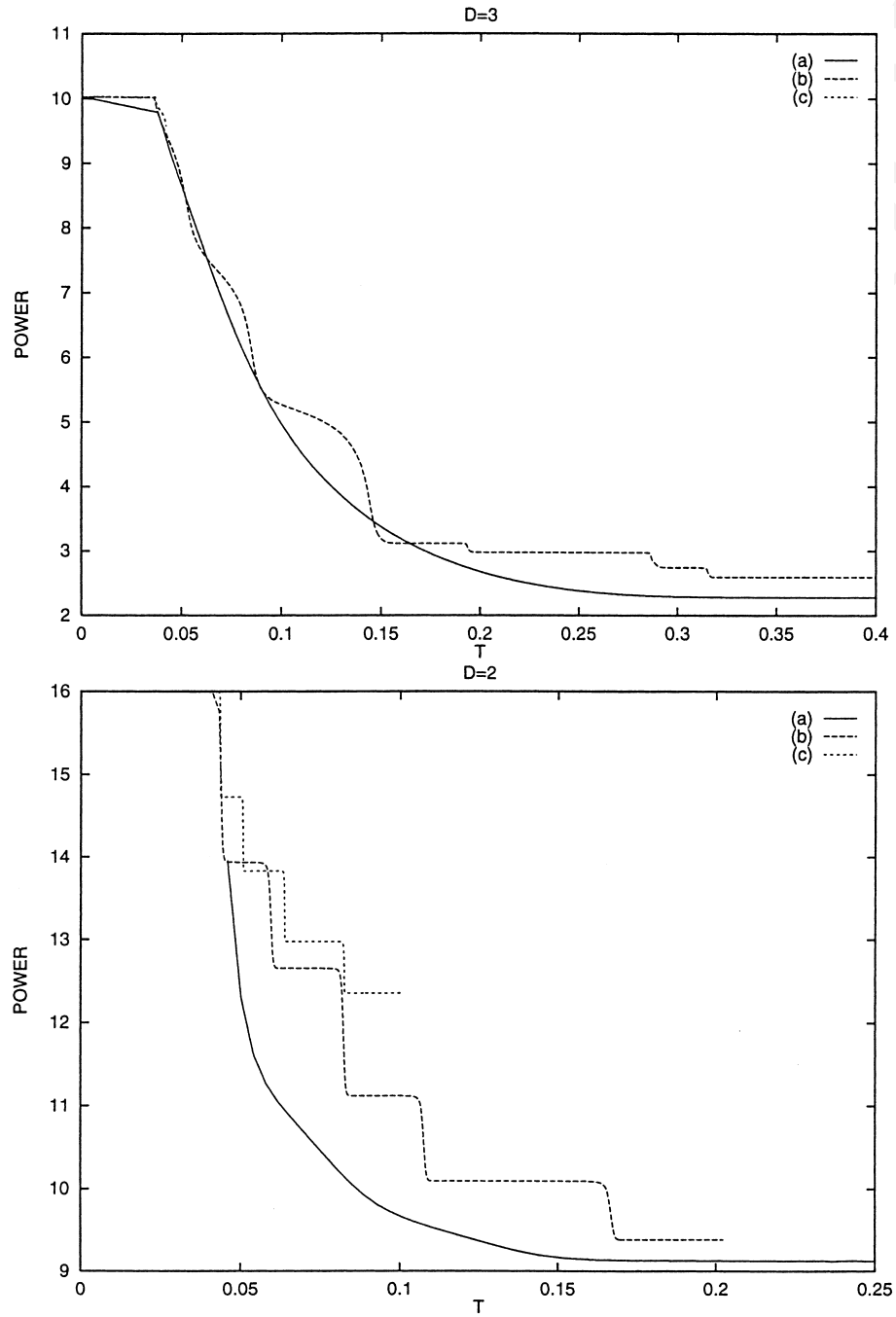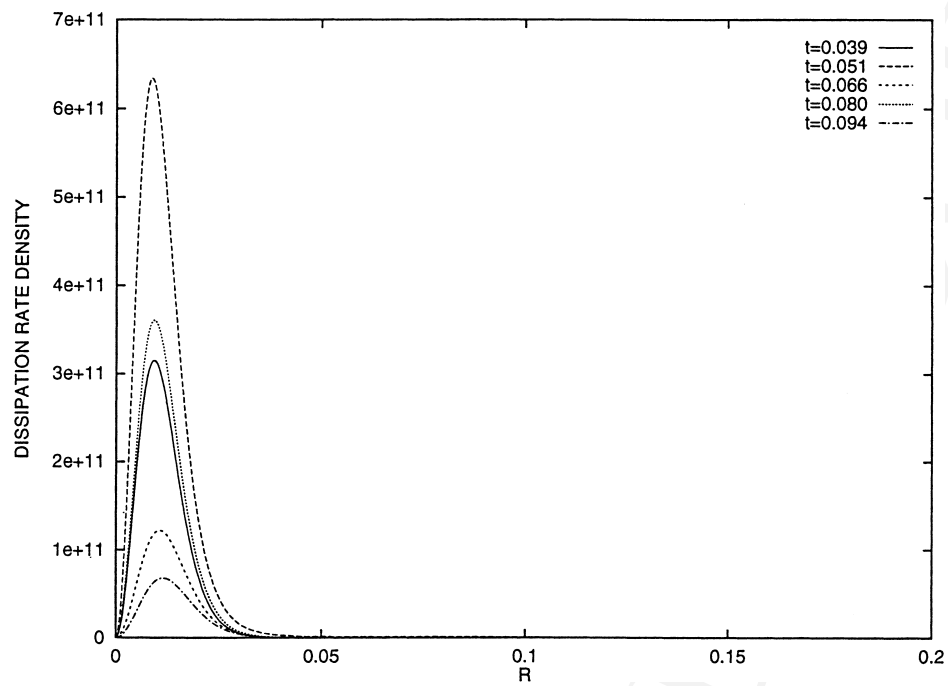
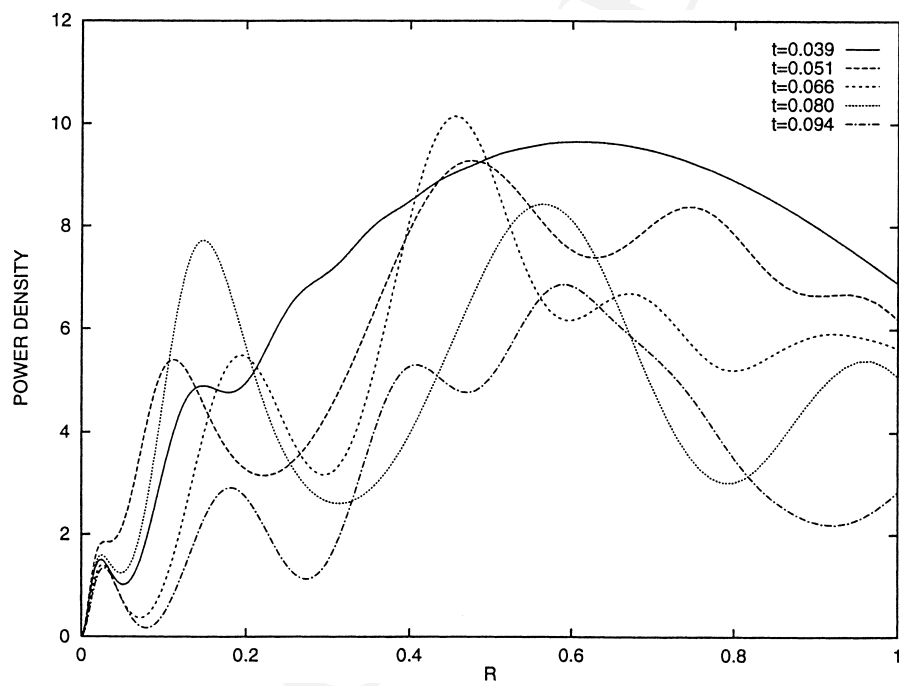Fig. 9. Multi-focusing for $D = 3$ and 2: power evolution for the same β values as above.

Fig. 10. $D = 3$, $h = 8$, $\beta = 10^{-8}$: dissipation density spatial profiles.

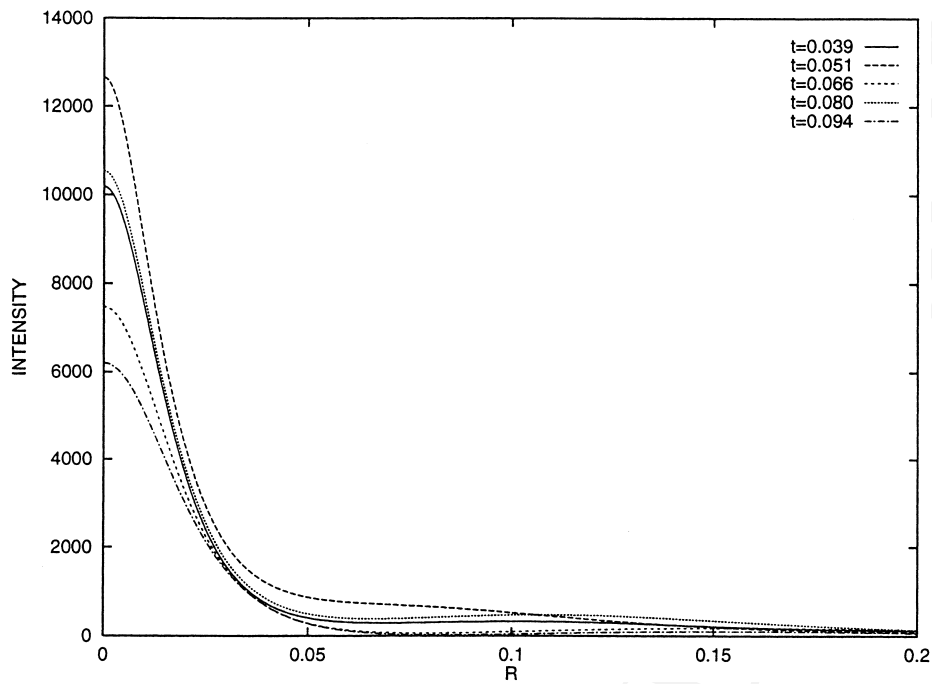Fig. 11. $D = 3$, $h = 8$, $\beta = 10^{-8}$: power density spatial profiles.

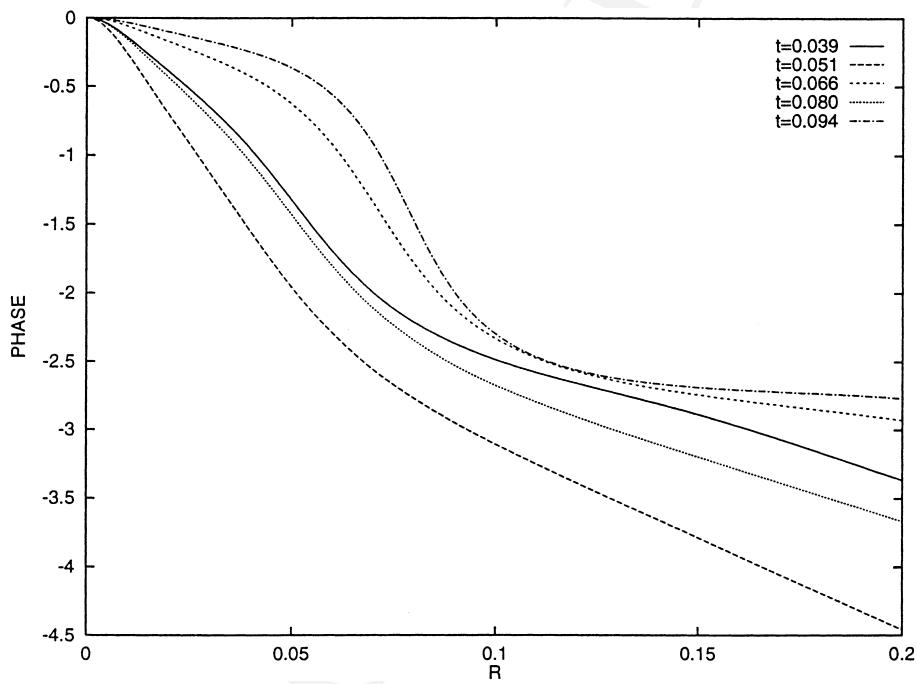Fig. 12. $D = 3$, $h = 8$, $\beta = 10^{-8}$: intensity spatial profiles.

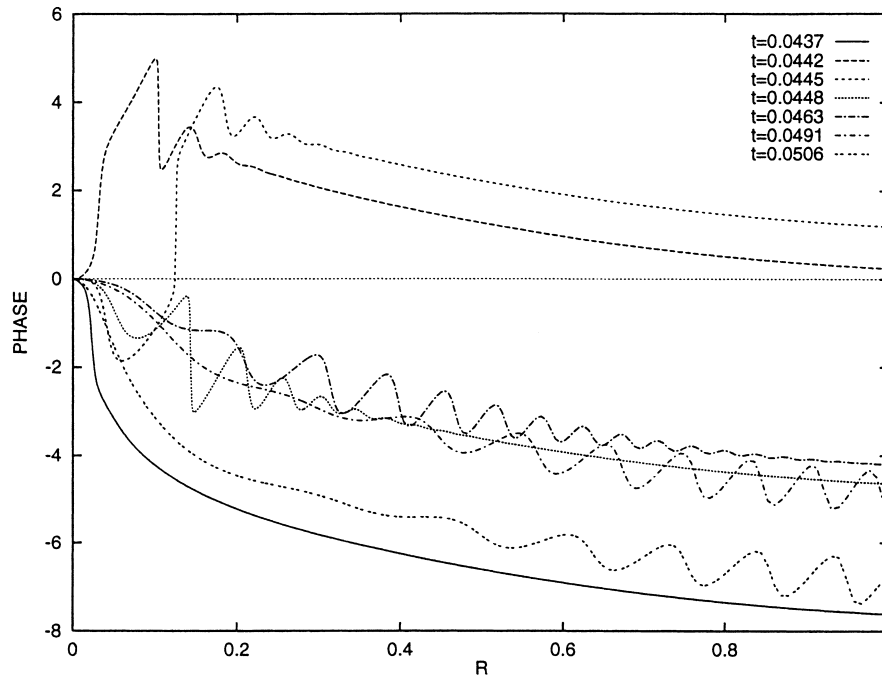Fig. 13. $D = 3$, $h = 8$, $\beta = 10^{-8}$: relative phase spatial profiles.

Fig. 14. $D = 2, h = 8, \beta = 10^{-10}$: relative phase spatial profiles.

Another difference is that during focusing peaks, a substantial proportion if the power is in the dissipative core pattern, but in the essentially nondissipative periods in between, the power is again almost entirely in the region where dissipation never occurs. Thus, the oscillations between dissipative bursts and nondissipative interludes can still be understood in terms of phase structures there produced mostly by focusing not dissipative effects, with some significant but small contributions from waves sent out from the dissipative core as focusing peaks collapse.

7. Conclusions

1. For $D = 4$ (or $D = 3$ with quintic nonlinearity, $\sigma = 2$) a quasi-steady-state approximating stationary singular solutions of the NLSE matches dissipation in a small central core at a rate essentially independent of the dissipation coefficient β, suggesting a natural dissipative singular solution in the limit $\beta \to 0$. As β diminishes, the shrinking core in which dissipation is significant contains an ever decreasing fraction of the total power, so that the mechanism divides cleanly into an inward flux in a shoulder where focusing but not dissipation occurs, and dissipation only in the core; mimicking the behaviour of the time independent singular dissipative solutions of the NLSE.
2. For the plasma physics case $D = 3$ the exact stationary singular form is on the threshold between different forms, the expected dissipation rate in the dissipative core goes to infinity as $\beta \to 0$, and the quasi-stationary state collapses after a short time interval. However, as before, the power is mostly in the nondissipative focusing shoulder and there, the $\log r$ phase structure developed during the initial

nondissipative focusing period and present in the stationary singular solutions persists, at least as a rapid drop-off of phase which supports the quick rebuilding of the phase gradient that in turn produces the inward focusing flux. This has only been studied in detail for a couple of focusing events, but evidence from longer time computations suggests that this multi-focusing can combine to produce total dissipation roughly independent of β and with overall time scale also essentially independent, so that a well-defined dissipative limit as $\beta \to 0$ is still plausible.

3. For the laser self-focusing case $D = 2$ there is no exact stationary singular form and the expected asymptotic form of singular solutions of the NLSE has no phase gradient. However, in the approach to singularity, those solutions are predicted [13] to have phase proportional to $\log r$ with a very slowly decreasing coefficient. In the DNLSE, this phase structure is seen clearly at the onset of dissipation, and is then preserved in the focusing shoulder. Thus, despite many theoretical and qualitative differences from the supercritical case $D > 2$, this phase structure can drive repeated rapid refocusing, and long time calculations suggest that again, the result in the limit of small β approximates a roughly steady dissipation rate, little dependent on β.

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