Regularization and control of self-focusing in the 2D cubic Schrödinger equation by attractive linear potentials

Brenton leMesurier

Department of Mathematics
College of Charleston
Charleston, SC 29424, USA

Peter Christiansen

Department of Informatics and Mathematical Modelling
Danish Technical University, Bygning 321
Lyngby 2800, Denmark

Abstract

The self-focusing singularity of the attractive 2D Cubic Schrödinger Equation arises in nonlinear optics and many other situations, including certain models of Bose-Einstein condensates.

This 2D case is very sensitive to perturbations of the equation and so solutions can be regularized in a number of ways. Here the effect of linear potentials is considered, such as could arise in models of optical fibres with narrow cores of different refractive index, wave-guides induced in a nonlinear medium by another beam, and as part of the Gross-Pitaevskii model of Bose-Einstein condensates.

It is observed that in critical dimension only, one can have inhibition of collapse by *attractive* linear potentials, without dissipation, and that this can lead to a stable oscillating beam, as opposed to the dispersion or dissipation seen with previously studied regularizing mechanisms.

---

[1] Thanks to Yuri Gaididei and Jens Juul Rasmussen for suggestions and discussions, and to Alan Newell and Alejandro Aceves for useful comments on an earlier version of this paper. Parts of this work were supported by a South Carolina Research Initiative Grant and by the Department of Informatics and Mathematical Modelling at the Danish Technical University.

Preprint submitted to Elsevier Preprint *30 January 2003*

1 Introduction

Self-focusing and wave collapse has been extensively studied in the [attractive] Cubic Schrödinger Equation (CSE)

$$\frac{\partial \psi}{\partial z}(z,x) = i[\Delta\psi(z,x) + |\psi(z,x)|^2\psi(z,x)], \quad x \in \mathbb{R}^D \tag{1}$$

and variants, and good introductions can be found in the articles of Rasmussen and Rypdal [12,13] and Berge[1], and the book of Sulem and Sulem[16]. (The propagation variable is herein called z, as in the principal case of laser propagation it signifies position along a beam. However note that in other applications mentioned, it is time-like, so t is used below where the physics demands it.) Some background on work that is more recent and more specific to the aims of the current paper can be found in [9,10]. The wide interest arises because the CSE is a generic model for the slowly varying envelope of a wave-train in conservative, dispersive, mildly nonlinear wave phenomena.

In the analogy to the true Schrödinger equation, the negation of the factor multiplying ψ in the nonlinear term has the role of a self-induced potential, so in the plus sign case used here, this potential is attractive, leading to positive feedback, *self-focusing* or *wave collapse* and the possibility of *singular collapse*: collapse continuing all the way to a [point] singularity in finite time.

The CSE itself arises in model of laser propagation, while Bose-Einstein condensates are modelled by the **Gross-Pitaevskii Equation,** which simply adds a linear attractive potential or "trap", usually modelled as a quadratic, and dissipation terms:

$$\frac{\partial \psi}{\partial t}(t,x) = i[\Delta\psi + |\psi|^2\psi + \|x\|^2\psi] - \beta|\psi|^{2\mu}\psi - \gamma\psi.$$

Bose-Einstein condensates are most commonly a 3D situation, but special situations such as "pancake geometry" can restrict the condensate to a surface or quasi two dimensional form, so $D = 2$ is also of interest.

When in laser propagation one adds a central core of higher refractive index, the model gains an attractive linear potential: a square well for an abrupt transition, but in the case of graded index, a possible model is a NLS equation with Gaussian potential (NLSGP):

$$\frac{\partial \psi}{\partial z}(z,x) = i[\Delta\psi + |\psi|^2\psi - h_p e^{-\|x\|^2/(2w_p^2)}\psi] \tag{2}$$

with sign $h_p > 0$ for the attractive case. This equation could also apply to the nonlinear guiding effect of another Gaussian beam; in the BEC case, replacing the quadratic trap potential by a Gaussian is possibly a more realistic model allowing that the force causing condensation flattens out at large distances.

In this paper the dissipative terms of the GP equation are not present, since the interesting observations involve collapse inhibition in a conservative equation; however adding them to simulations does nothing but further help the collapse inhibition.

Self-focusing singularity formation in nonlinear Schrödinger equations has been extensively studied since the physical phenomenon of self-focusing was first reported by Chiao, Garmire and Townes in 1964 [2] and the possibility of singular collapse was proven by Vlasov, Petritshev and Talanov in 1971 [17]. Self-focusing is manifested in solutions of the CSE by the development of large intensities and gradients in regions that must be proportionately small due to conservation of *power* (L^2-norm squared); the collapsing spatial scale leads to significant challenges in numerical simulations as discussed below, and even greater challenges for experimentalists, making careful simulation important.

Physically, singularity formation will be prevented by the regularizing effect of various phenomena neglected in the basic CSE model. For example conservative dispersion due to saturation of the nonlinear effect is particularly important in laser propagation models [6] and dissipation is important in plasma physics and Bose-Einstein condensates [9]. Here, in contrast, we study a possibility of regularization by attractive potentials, without dissipating or dispersing the solution.

2 Self-similar singular solutions

For $D \geq 2$, Vlasov et al [17] showed that solutions will develop singularities at some finite value Z of the propagation variable when the energy $H = \int(|\nabla\psi|^2)/2 - (|\psi|^4/4)dx$, a conserved quantity, is negative. On the other hand, in the critical case $D = 2$, global existence is guaranteed with the power $N = \int|\psi|^2 dx$, another conserved quantity, is less than a threshold N_c, explained further below.

In the **supercritical** case $D > 2$ there are explicit radially symmetric self-similar solutions

$$\psi(z,r) = \frac{1}{[K(Z-z)]^{1/2}} \exp\left[\frac{i}{K}\ln\left(\frac{Z}{(Z-z)}\right)\right] Q\left(\frac{r}{[K(Z-z)]^{1/2}}\right) \quad (3)$$

where $r = \|x\|$, K is a positive constant and $Q(r)$ is a solution of

$$-\left(Q'' + \frac{D-1}{r}Q'\right) + Q - |Q|^2 Q - iK(Q + rQ') = 0, \quad Q'(0) = 0. \tag{4}$$

The existence of suitable solutions Q has been shown by Kopell and Landman [4] though only for D sufficiently near 2, and the few known explicit singular solutions and numerical solutions strongly suggest that generically, singularities takes the form of a single point *focusing singularity*, asymptotic to such a solution.

Critical Collapse, "Townes Solitons" and non-linear bound states. In the critical case $D = 2$ no such non-trivial singular solutions exist because solutions for Q satisfying the boundary conditions can be shown to have $K = 0$ and then Eqn. (3) fails. However, there is a strong non-rigorous argument for occurrence of solutions asymptotic to these, with the spatial profile Q replaced by the so called ground-state, the unique positive solution R_0 of

$$-\left(R'' + \frac{1}{r}R'\right) + R - R^3 = 0, \quad R'(0) = 0 \tag{5}$$

and with growth rate is faster by a log-log correction $\sqrt{\ln\ln\frac{1}{Z-z}}$, as shown by leMesurier, Landman et al [7,5] using a limit of the supercritical self-similar singular solutions above, and independently in a different way by Fraiman [3].

The ground state R_0 is at the threshold of collapse in several ways, having $H = 0$ and $N = N_c$. It gives an important one-parameter family of steady state solutions

$$\psi(z, r) = e^{i\lambda z} R_0(\lambda^2 r) \tag{6}$$

under the dilation rescaling symmetry of the CSE. The name *Townes soliton* is sometimes used for the ground state R_0, but will here be used for these related steady state solutions. These solutions are unstable: since R_0 has energy $H = 0$, small perturbations can give $H < 0$ and hence singular collapse.

On the other hand, it is at the borderline of stability: Shatah and Strauss [15] established that a member of such a one-parameter family of steady states such as in Eqn. (6) is orbitally stable if and only if the power is strictly increasing in the frequency parameter λ, whereas for the Townes solitons it is of course constant.

Rose and Weinstein [14] have shown that when certain attractive potentials are added to the 2D CSE, such families of steady state solutions exist, and this orbital stability condition is satisfied on the entire family. These steady states are constructed using bifurcation from "linear" bound states, of the (linear) Schrödinger equation with the same potential, and so such bound states must exist for the nonlinear bound state to be guaranteed, irrespective of orbital stability. Once existence is established, in the "narrow deep" limit $\lambda \to \infty$ these steady states are asymptotic to the above family of Townes solitons steady states of the CSE. Thus $N_c = \|R_0\|_2^2$ is an upper limit on the power of such steady states.

In the present situation, for $D = 2$, their result applies to sufficiently large gaussian potentials. Noting that by dilation rescaling, all Gaussian potentials with the same value of $w_p h_p$ are equivalent, existence of linear bound states is known to require $w_p h_p \geq \textit{const.} \approx 0.5$, whereas collapse inhibition and oscillatory solutions will be seen below also for far smaller potentials. (In the super-critical case $D = 3$, such steady states also exist, but orbital stability is at most guaranteed only for the "wide, shallow" limit $\lambda \approx 0$ of the solution family.)

3 Numerical Methods

To resolve solutions well on the extremely fine spatial scales that develop near the focus while respecting boundary conditions, a modification of earlier "dilation rescaling" methods [8,11,6,18] is used here. For the radially symmetric case, the spatial variable $r \in [0, r_{max}]$ is related to a computational variable ρ on a fixed grid by

$$r = f(\rho, l(z)),\ \rho \in [0, 1].$$

The transformed equation with linear potential $V(x)$ is

$$\psi_z = i\Delta\psi + i|\psi|^2\psi + iV(r)\psi + (r_l l_z)\psi_r$$

Note that all derivatives of ψ including those in the Laplacian are still with respect to the physical coordinate r, not ρ.

Choice of rescaling function. The transformation function should be odd, increasing, achieve a desired scale length l near the focus by having $f_\rho(\rho, l)|_{\rho=0} = l$, and fix the outer boundary by having $f(1, l) = r_{max}$. The form used here is $f(\rho, l) = l \sinh(k(l)\rho)$, where $k(l)$ is determined by the condition $f(1, l) = r_{max}$ to fix the outer boundary.

Determining the length scale. The length scale $l(t)$ is based on the functional

$$l^*(\psi(z,.)) = C\frac{\int |\psi||\nabla\psi|^D r^{D-1} dr}{\int |\nabla\psi|^{D+1} r^{D-1} dr}. \tag{7}$$

This functional is designed to be convergent for all relevant values of D in the presence of the behaviour $|\psi| \approx r^{-1}$ that develops as the singularity is approached, and to be numerically stable (which simpler measurements at the origin only are not).

Time discretization. To get stable, manageable implicit time stepping schemes, the evolution of $l(t)$ is decoupled from the main evolution equation, determining its values through a time step before that step is started, using

$$\frac{dl}{dz} = \frac{l_n^* - l_{n-1}^*}{t_n - z_{n-1}} + \frac{l_n^* - l_n}{z_n - z_{n-1}}, \text{ on } [z_n, z_{n+1}] \tag{8}$$

where $l_n^* = l^*(z_n)$ etc.

The time discretization is then done by the implicit trapezoid rule due to its good properties for conservative and Hamiltonian-form equations such as nonlinear Schrödinger equations. Note that a one-step method is essential due to the non-smooth time dependence of the rescaling. The resulting nonlinear system is solved by a PC iteration on only the nonlinear, non-stiff terms. That is, all linear terms of the nonlinear discrete system are expressed exactly at each iteration.

Accuracy Checks. The NLS Equation has several conserved quantities that can be used to check the accuracy of solutions, in particular the power $N = \|\psi\|_2^2$. Other functionals that are not conserved can be used by deriving evolution equations that can be integrated, and the values checked against the actual evolution. It is best to keep to functionals that do not involve spatial derivatives, and here the evolution of the L^4 norm is also checked, and data are discarded when either functionals' error exceeds an appropriate tolerance.

Other checks are done by comparing various refinements of discretizations in the transverse and propagation variables, and corroborate the above functional evolution tests: the stage of a run where the functional evolutions becomes inaccurate occurs no later than when the results with more refined discretizations show comparable deviations.

4 Numerical Results

All numerical results presented here are for gaussian initial data of height h and RMS width 1: $\psi_0 = he^{-r^2/2}$.

Collapse thresholds for Gaussian beams. To assess the inhibition of singular collapse by narrow attractive potentials, one must first observe when collapse occurs in the unmodified CSE, as theoretical results such as the sufficient condition $H < 0$ for $D \geq 2$ and the necessary condition $N \geq N_c$ for $D = 2$ only do not determine this completely. In the case of gaussian initial data $\psi_0 = he^{-r^2/2}$ in critical dimension 2, it is known only that blowup cannot occur for $h \lessapprox 1.93$ (since then $N < N_c$), and must occur for $h > 2$ (for then the energy H is negative).

Experiments show that collapse to a singularity occurs for h above some threshhold h_c, dependent on dimension, with dispersion for smaller h values.

For $D = 2$, $h_c \approx 1.95$, for which power $N \approx 1.02N_c$ and energy $H \approx 0.009$ (Fig. 1). [Here and in all figures, amplitude curves that go past the top of the figure represent strong evidence of real blowup, specifically reaching well past amplitude of $1,000$.]

For $D = 3$ one gets $h_c \approx 2.077$, with energy still comfortably positive: $H \approx 1.4$ (Fig. 2).

2D case: inhibition of singular collapse by a small attractive potential. It is not surprising that for $h \gtrapprox h_c$, a small repulsive potential in the NLSGP modification can inhibit collapse.

For a narrow *attractive* potential in the NLSGP one might expect only an acceleration of the collapse, and this does happen for sufficiently small potentials. However, with potentials narrower than the initial self-induced potential of the initial data, increasing h_p often causes the eventual failure of this initial focusing, with the peak dispersing (though typically refocusing at later times, as discussed below.)

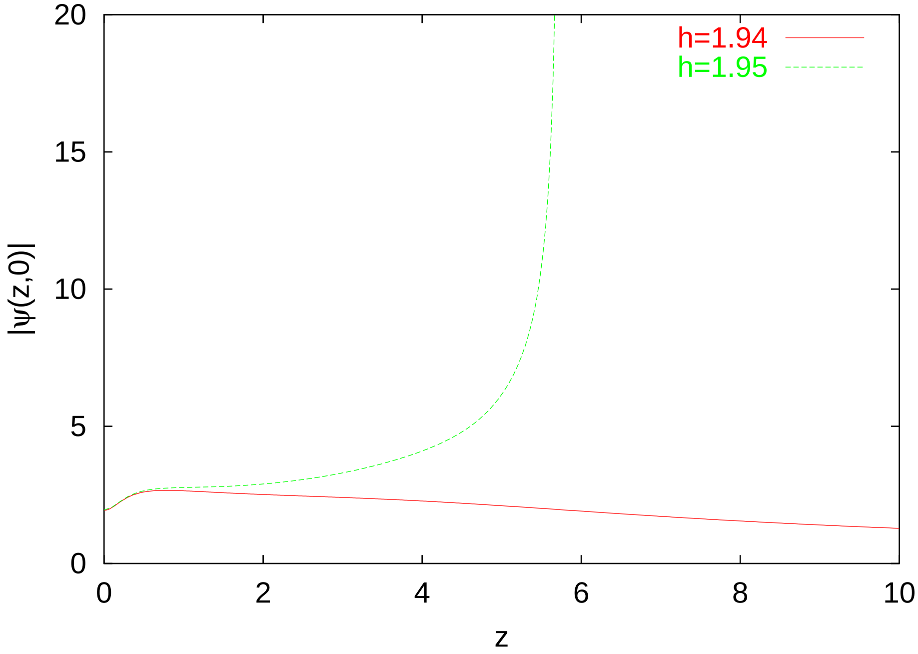

Fig. 1. $D = 2$, threshold beam power for CSE collapse

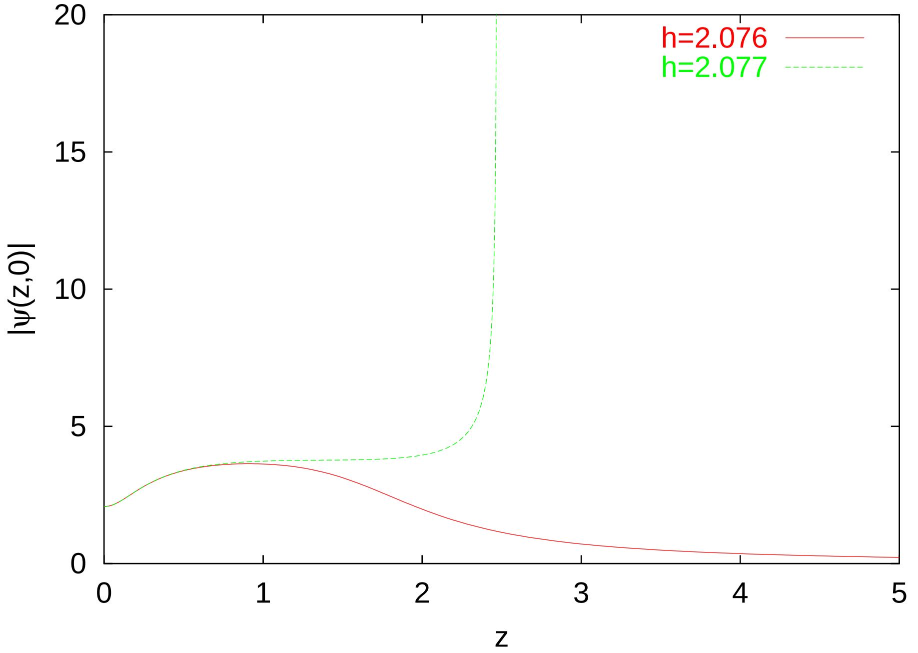

Fig. 2. $D = 3$, threshold beam power for CSE collapse

For example, in the near-threshhold case $h = 1.95$, this defocusing occurs with potential depth as small as $h_p = 0.11$ for the optimal width of about $w_p = 0.5$ (Fig. 3). This should be compared to the initial self-induced potential (minus the intensity) which is a somewhat wider and far deeper gaussian of width $1/\sqrt{2}$ and height 3.8025.

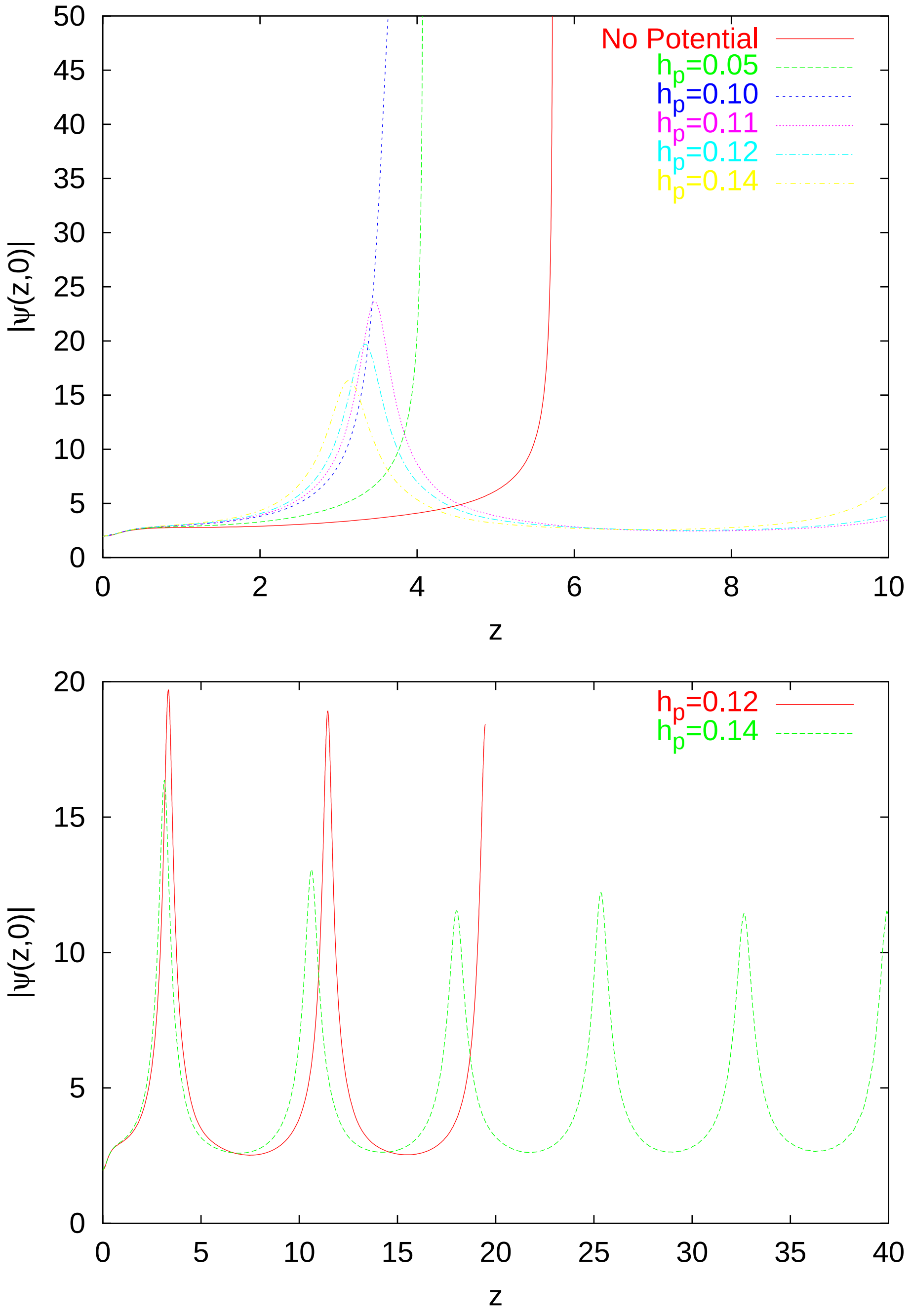

Fig. 3. $D = 2$, $h = 1.95$, $w_p = 0.5$: collapse inhibition at small z, then oscillations at larger z

Though this effect is most pronounced when the beam power is just a little above the threshold for collapse, it persists at least until the power is 30% above threshhold, with the smallest inhibiting potential becoming rapidly stronger and slowly narrower as h increases. Table 1 gives the shallowest inhibiting gaussian potential found for a range of initial data. As the product $w_p h_p$ must be about 0.5 or greater for the potential to support a Rose-

h	1.95	2	2.1	2.2	2.5
h_p	0.15	1.25	3.46	7	20
w_p	0.4	0.35	0.3	0.2	0.15
$h_p w_p$	0.06	0.44	1.04	1.4	3

Table 1
Shallowest potentials inhibiting collapse for various initial data: some are far too small for the existence of steady states to be guaranteed.

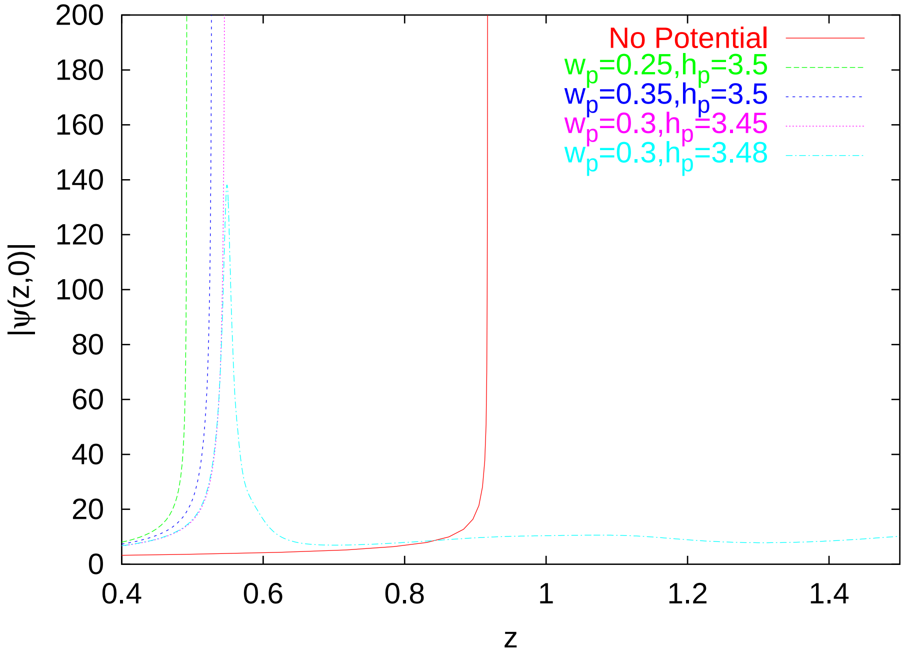

Fig. 4. Collapse inhibition for potentials of various widths and depths, $h = 2.1$

Weinstein nonlinear bound state, it is clear that collapse inhibition can occur in the absence of such a bound state. On the other hand, it is possibly noteworthy that the point at which the initial data attains the negative energy condition for guaranteed collapse ($h = 2$) is somewhat close to the level at which collapse inhibition occurs only for potentials with a nonlinear bound state.

The above effect seems to be truly a critical dimension phenomenon; no such collapse inhibition is seen in 3D NLS even for gaussian initial data only very slightly above the threshold for collapse, even when very large attractive potentials of various widths are tried (Fig. 6).

Spatial structure of 2D collapse inhibition. Spatial cross sections of the solution amplitude at various times in cases with an inhibiting potential can be compared to those for a collapsing solution of the CSE with the same

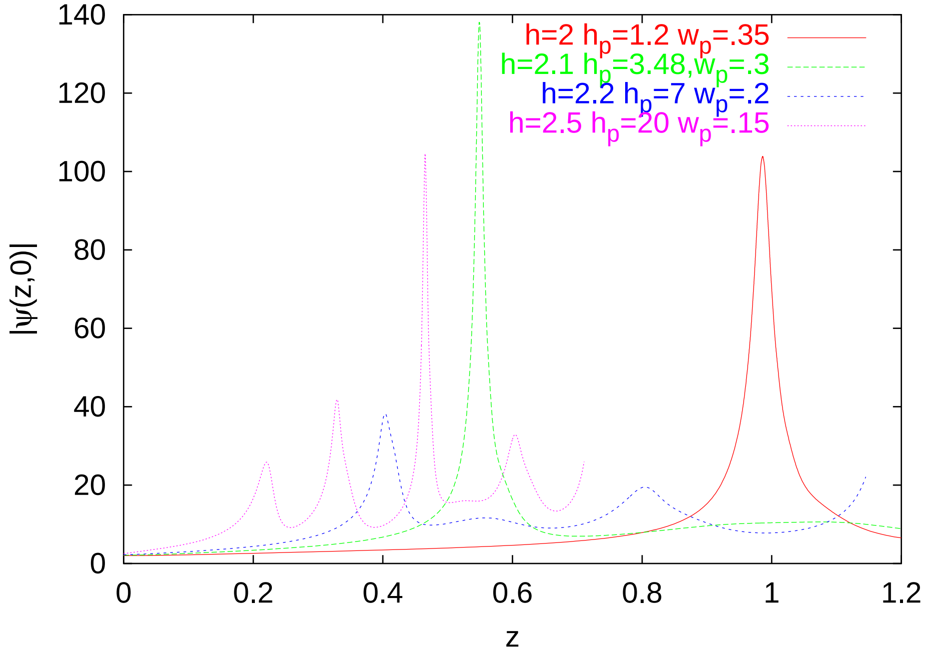

Fig. 5. Collapse inhibition for various initial data and potentials

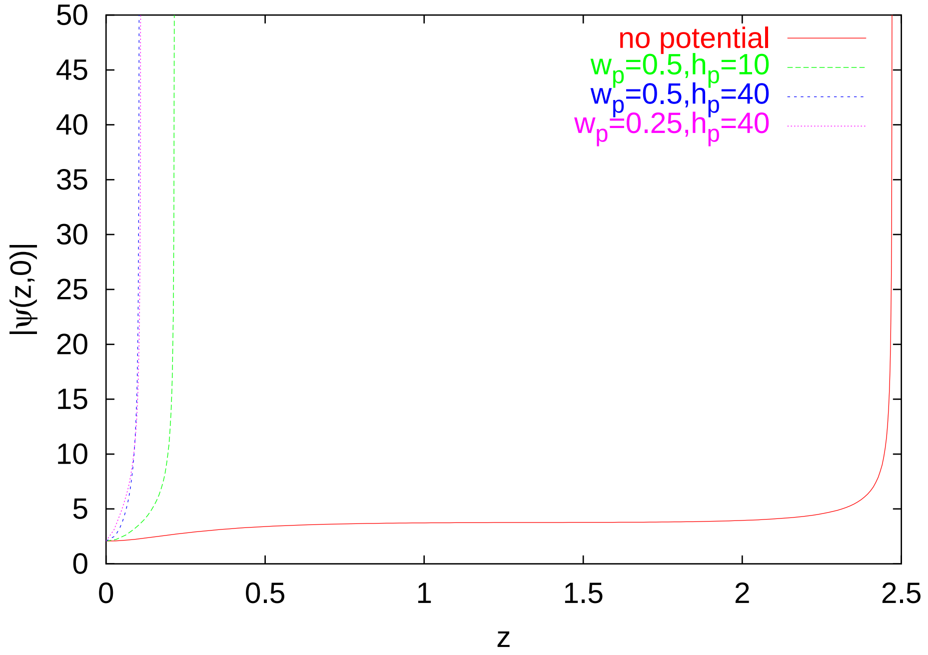

Fig. 6. Super-critical in dimension 3, $h = 2.077$: no inhibition of collapse

initial data (Fig's 7,8,9 for $h = 2.1, w_p = 0.3, h_p = 3.48$), and suggest a likely mechanism for the inhibition.

The potential accelerates the initial focusing of the part of the power distribution inside its well, but the resulting spike has less than critical power N_c and

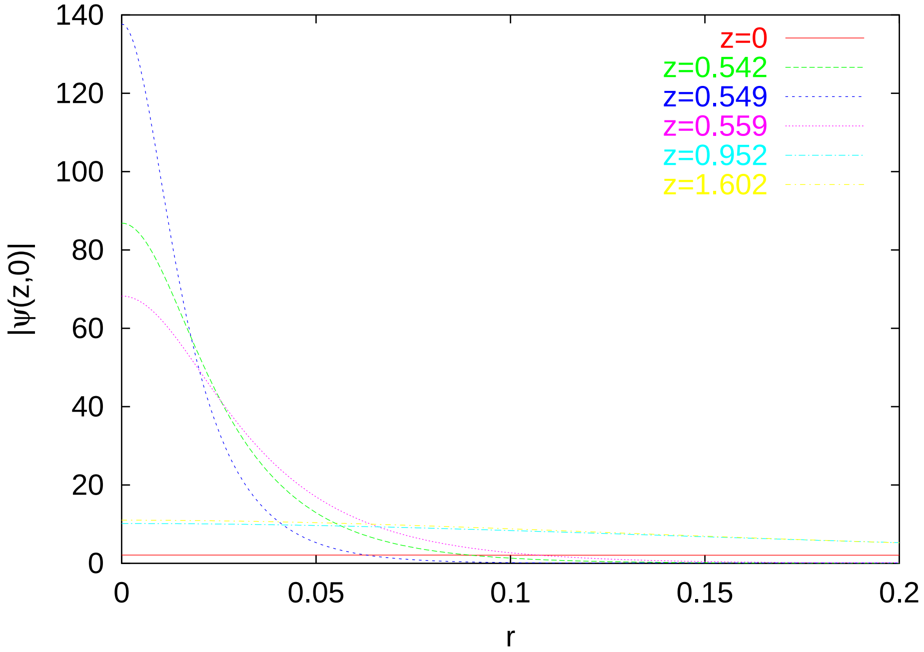

Fig. 7. Evolution of transverse structure, at z values close to successive peaks and troughs of focusing intensity: within the potential, oscillatory dilation close to Townes soliton form. $h = 2.1$, $w_p = 0.3$, $h_p = 3.48$

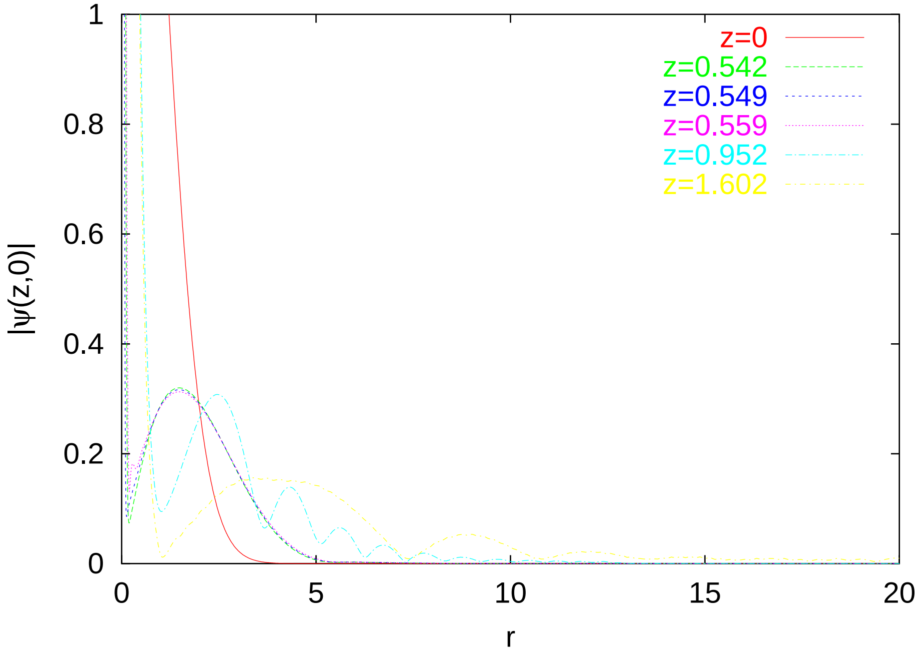

Fig. 8. As above but for radii going well beyond the potential width w_p: small dispersive waves from each oscillation

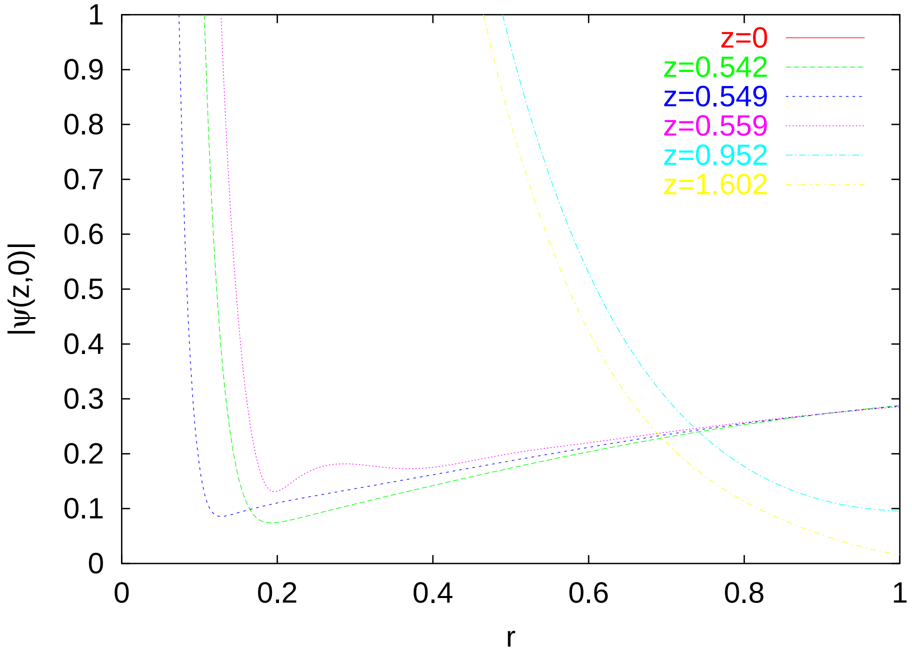

Fig. 9. As above, at small radii and low amplitudes: the near zero amplitudes for $r \approx 0.2 \approx (2/3)w_p$ at each focusing peak indicate almost complete separation of an inner peak trapped in the potential from the radiating outer part free of potential.

is more or less separated from the remaining "mass", so essentially evolves on its own: as for a solution with total power less than N_c, it can and does collapse to a finite degree but then disperses.

In this example, the intensity at the centre decreases almost to its initial value (Fig. 7), but there is still a peak trapped in the potential well (Fig's 8,9), which then refocuses, producing a second focusing peak of somewhat lower intensity.

In other cases seen in Fig's 3 and 5, the second focus is far closer to the first in intensity and the solutions exhibit a sustained oscillation between focusing and defocusing. The absence of any dissipation, or of any dispersive term (indeed there is a small attractive nonlinear effect outside the core that focuses) allows the mass to retain coherence in some cases.

Thus beams can be constrained between singular collapse and broad dispersion. This could be potentially useful in controlling and guiding beams of high intensity or in highly nonlinear optical media.

As a final note, the dispersion of the focusing spike seems more or less complete at each oscillation, and so the time scale of the subsequent refocusings is comparable to that of the original: this should be contrasted to the situation with dissipative regularizations of the CSE studied in [9] and [10], where the amplitude and phase structure supporting power flux towards the origin survived the collapse of the focus, leading to successive refocusings far more rapid than the original one.

5 Conclusions

- Modifying the focusing Cubic Schrödinger Equation by the addition of a small attractive gaussian potential can in some case prevent singular wave collapse that would occur for the same initial data without it. This has potential applications to models of nonlinear laser propagation in the presence of wave guides, due to either doping of the medium, or interaction with another "guiding beam".
- This phenomenon is unique to the critical dimension of 2, where there is a minimum beam power needed for a solution to collapse; and collapse inhibition is most effective (occurs for smallest potential) when the power is only slightly above this threshold. However, inhibition still occurs when the power is significantly above the two thresholds for both the possibility and the guarantee of collapse in the CSE.
- Collapse inhibition is in general followed by oscillations, rather than dispersion or dissipation, as with most previously regularizing mechanisms. Thus in applications such as wave guides (fibres with cores of higher index), the phenomenon could be used to stabilize transmissions of intense beams, in situations where the highly nonlinear behaviour of such beams is interesting.
- Further from the potential, there is radiation at each inward cycle of such oscillations, particularly the first; this could be a mechanism by which the power remaining in the central, potential, region drops below the minimum needed for collapse.
- A mechanism for the oscillations is less clear however. In particular, collapse inhibition and oscillation can occur in cases where the potential is too small to sustain a bound state of the underlying linear Schrödinger equation, so that the orbitally stable steady states found by Rose and Weinstein[14] are unlikely to exist.

References

[1] Luc Bergé. Wave collapse in physics: Principles and applications to light and plasma waves. *Phys. Rep.*, 303(5–6):259–372, 1998.

[2] R.Y. Chiao, E. Garmire, and C.H. Townes. Self-trapping of optical beams. *Phys. Rev. Lett.*, 13:479–482, 1964.

[3] G.M. Fraĭman. Asymptotic stability of manifold of self-similar solutions in self-focusing. *Sov. Phys. JETP*, 61(2):228–233, 1986. (ZETF 88, (2) 390).

[4] N. Kopell and M Landman. Spatial structure of the focusing singularity of the nonlinear Schrödinger equation: a geometrical analysis. *SIAM J. Appl. Math.*, 55:1297–1323, 1995.

[5] M.J. Landman, G.C. Papanicolaou, C. Sulem, and P.L. Sulem. Rate of blowup for solutions of the nonlinear Schrödinger equation at critical dimension. *Phys. Rev. A*, 38(8):3837–3843, 1988.

[6] B.J. LeMesurier, G. Papanicolaou, C. Sulem, and P.L. Sulem. Focusing and multi-focusing solutions of the nonlinear Schrödinger equation. *Physica D*, 31:78–102, 1988.

[7] B.J. LeMesurier, G.C. Papanicolaou, C. Sulem, and P.L. Sulem. Local structure of the self-focusing singularity of the nonlinear Schrödinger equation. *Physica D*, 32:210–226, 1988.

[8] Brenton LeMesurier. *The focusing singularity of the nonlinear Schrödinger equation.* PhD thesis, New York University, 1986.

[9] Brenton LeMesurier. Dissipation at singularities of the nonlinear Schrödinger equation through limits of regularisations. *Physica D*, 138:334–343, 2000.

[10] Brenton LeMesurier. Multi-focusing and sustained dissipation in the dissipative nonlinear Schrödinger equation. *Mathematics and Computers in Simulation*, 55:503–517, 2001.

[11] D.W. McLaughlin, G.C. Papanicolaou, C. Sulem, and P.L. Sulem. The focusing singularity of the nonlinear Schrödinger equation. *Phys. Rev. A*, 34:1200–1210, 1986.

[12] J. J. Rasmussen and K. Rypdal. Blowup in nonlinear Schrödinger equations I: a general review. *Phys. Scr.*, 33:481, 1986.

[13] J. J. Rasmussen and K. Rypdal. Blowup in nonlinear Schrödinger equations II: similarity structure of the blow-up singularity. *Phys. Scr.*, 33:498, 1986.

[14] H. Rose and M.I. Weinstein. On the bound states of the nonlinear Schrödinger equation with a linear potential. *Physica D*, 30:207–218, 1988.

[15] J. Shatah and W.A. Strauss. Instability of nonlinear bound states. *Comm. Math. Phys.*, 100:173–190, 1985.

[16] Catherine Sulem and Pierre-Louis Sulem. *The nonlinear Schrödinger equation: self-focusing and wave collapse.* Springer, 1999.

[17] S.N. Vlasov, V.A. Petritshev, and V.A. Talanov. Averaged description of wave beams in linear and nonlinear media (the method of moments). *Radiophys. and Quant. Electr.*, 14:1062–1070, 1971.

[18] V.E. Zakharov and V.F. Shvets. Nature of wave collapse in the critical case. *JETP Lett.*, 47:275–278, 1988.