Regularisation and control of self-focussing in the 2D cubic Schrödinger equation by linear potentials

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Abstract

The self-focussing singularity of the Cubic Schrödinger Equation arises in nonlinear optics (in two dimensions), models of Bose-Einstein condensates (in three and two dimensions) and many other situations.

The 2D case is very sensitive to perturbations of the equation and so solutions can be regularized in a number of ways. Here the effect of linear potentials is considered, such as could arise in models of optical fibres with narrow cores of different refractive index.

The most intriguing conclusion is the inhibition of collapse by attractive linear potentials, in a way that can lead to a coherent oscillating beam, as opposed to simply causing dispersion or dissipation as has been seen with other regularizing mechanisms.

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1 Background

The Cubic Schrödinger Equation

\[
\frac{\partial \psi}{\partial t}(t, x) = i[\Delta \psi(t, x) + |\psi(t, x)|^2 \psi(t, x)], \quad x \in \mathbb{R}^D
\]

(1)

is a generic model for the slowly varying envelope of a wave-train in conservative, dispersive, mildly nonlinear wave phenomena.

In the analogy to the true Schrödinger equation, the negation of the factor multiplying \( \psi \) in the nonlinear term has the role of a self-induced potential, so in the plus sign case, this potential is attractive, leading to positive feedback, self-focussing or wave collapse and the possibility of singular collapse: collapse continuing all the way to a [point] singularity in finite time. Only this self-focussing case will be considered from now on.

The CSE itself arises in model of laser propagation, and Bose-Einstein condensates are modelled by the Gross-Pitaevskii Equation, which simply adds a linear attractive potential, or “trap”, usually modelled as a quadratic, plus linear dissipation:

\[
\frac{\partial \psi}{\partial t}(t, x) = i[\Delta \psi + |\psi|^2 \psi + |x|^2 \psi] - \beta |\psi|^{2\mu} \psi - \gamma \psi.
\]

Bose-Einstein condensates are most commonly a 3D situation, but special situations can restrict the condensate to a surface or quasi two dimensional form, so \( D = 2 \) is also of interest.

A possibly more realistic modelling would have the potential flatten out at large distances, and one model for this is a NLS equation with Gaussian potential (NLSGP):

\[
\frac{\partial \psi}{\partial t}(t, x) = i[\Delta \psi + |\psi|^2 \psi - \beta |\psi|^2 |\psi|^2] - \beta |\psi|^{2\mu} \psi - \gamma \psi
\]

(2)

where \( h > 0 \) corresponds to an attractive potential. This equation could also apply to laser propagation, modelling a core of different refractive index due to doping in an optical fibre, or the nonlinear effect of another beam.

**Self-focussing in nonlinear Schrödinger equations** has been extensively studied since it was first reported by Chiao, Garmire and Townes in
1964 [1] and the possibility of singular collapse was proven by Vlasov, Petritshev and Talanov in 1971 [13]; much background relevant to this paper can be found in [10], [7] and their bibliographies.

The self-focusing is manifested in solutions of the CSE by the development of large intensities and gradients in regions that must be proportionately small due to conservation of the $L^2$-norm, or power, leading to significant challenges in numerical simulations (as discussed below), and even greater challenges for experimentalists.

Physically singularity formation will be prevented by the regularizing effect of various phenomena neglected in the basic CSE model. For example conservative dispersion due to saturation of the nonlinear effect is particularly important in laser propagation models ([10]) and dissipation is important in plasma physics and Bose-Einstein condensates ([8,5]).

Here, in contrast, a possibility of regularization by attractive potentials, without dissipating or dispersing the solution, will be studied.

2 Self-similar singular solutions

For $D \geq 2$, it is possible for solutions to develop singularities at some finite time $t_0$ as shown by Vlasov et al [13], while for $D < 2$ Ginibre and Velo guaranteed global existence for reasonably initial data [3].

Explicit radially symmetric self-similar solutions arise in the supercritical case $D > 2$

$$\psi(t, r) = \frac{1}{[K(T - t)]^{1/2}} \exp \left[ \frac{i}{K} \ln \left( \frac{T}{(T - t)} \right) \right] \times Q \left( \frac{r}{[K(T - t)]^{1/2}} \right)$$

where $r = \|x\|$, $K$ is a positive constant and $Q(r)$ is a solution of

$$- \left( Q'' + \frac{D - 1}{r} Q' \right) + Q - |Q|^2 Q - iK(Q + rQ') = 0, \quad Q(0) = 0.$$

The few known explicit singular solutions and numerical solutions strongly suggest that generically, singularities takes the form of a single point focusing singularity, asymptotic to such a solution.
2.1 Critical Collapse and the “Townes Soliton”

In the critical case $D = 2$ such solutions do not exist as solutions for $Q$ must have $K = 0$, but there is a strong non-rigorous argument for occurrence of solutions asymptotic to these, with the spatial profile $Q$ replaced by the so called ground-state or Townes soliton, the unique positive solution $R_0$ of

$$- \left( R'' + \frac{1}{r} R' \right) + R - R^3 = 0, \quad R(0) = 0$$

(3)

and with growth rate is faster by a log-log correction $\sqrt{\ln \ln \frac{1}{T - t}}$ [2,11,4].

3 Numerical Methods

To resolve solutions well on the extremely fine spatial scales that develop near the focus while respecting boundary conditions, a modification of earlier “dilation rescaling” methods [6,12,10,14] is used. For the radially symmetric case, the spatial variable $r \in [0, r_{max}]$ is related to a computational variable $\rho$ on a fixed grid by

$$r = f(\rho, l(t)), \quad \rho \in [0, 1],$$

The transformed equation with linear potential $V(x)$ is

$$\psi_t = i \Delta \psi + i|\psi|^2 \psi + iV(r)\psi - \beta |\psi|^2 \psi + (r_i l_i) \psi_r$$

Note that all derivatives of $\psi$ including those in the Laplacian are still with respect to the physical coordinate $r$, not $\rho$.

3.1 Choice of rescaling function

The transformation function should be odd, increasing, achieve a desired scale length $l$ near the focus by having $f_\rho(\rho, l)|_{\rho=0} = l$, and fix the outer boundary by having $f(1, l) = r_{max}$. The form used here is

$$f(\rho, l) = l \sinh(k(l)\rho)$$

(4)
where $k(l)$ is determined by the condition
\[ f(1, l) = r_{\text{max}} \]
to fix the outer boundary.

### 3.2 Determining the length scale

The length scale $l(t)$ is based on the functional
\[
l^*(\psi(t, \cdot)) = C \frac{\int |\psi| |\nabla \psi|^{D+1}r^{D-1}dr}{\int |\nabla \psi|^{D+1}r^{D-1}dr}.
\] (5)

This functional is designed to be convergent for all relevant values of $D$ in the presence of the behaviour $|\psi| \approx r^{-1}$ that develops as the singularity is approached, and to be numerically stable (which simpler measurements at the origin only are not).

### 3.3 Time discretization

To get stable, manageable implicit time stepping schemes, the evolution of $l(t)$ is decoupled from the main evolution equation, determining its values through a time step before that step is started, using
\[
\frac{dl}{dt} = \frac{l^*_n - l^*_n}{t_n - t_{n-1}} + \frac{l^*_n - l_n}{t_n - t_{n-1}}, \text{ on } [t_n, t_{n+1}]
\] (6)

where $l^*_n = l^*(t_n)$ etc.

The time discretisation is then done by a partially implicit second order accurate PC method; a one-step method is essential due to the non-smooth time dependence of the rescaling. The Laplacian term is handled by the implicit trapezoid rule scheme and the other terms by the two stages of the modified Euler method.

### 3.4 Accuracy Checks

The NLS Equation has several conserved quantities that can be used to check the accuracy of solutions, in particular the power $N = ||\psi||_2^2$. In the dissipative
case power is not conserved but has a simple evolution equation: this can be integrated and the values checked against the actual power. Other functionals can be treated similarly but it is best to keep to ones that do not involve spatial derivatives: thus the $L^4$ norm is also checked, and data are discarded when either error exceeds an appropriate tolerance.

This has worked well in practice; the stage of a run where this test fails corresponds well to the start of significant divergence of the solution from results of computations with more refined discretisations.

4 Numerical Results

To assess the inhibition of self-focussing collapse by narrow attractive potentials, one must first observe when collapse occurs in the unmodified CSE, as theoretical results such as the sufficient condition $H < 0$ for $D \geq 2$ and a necessary condition $N \geq N_c$ for $D = 2$ only do not determine this completely. In the case of Gaussian initial data $\psi_0 = he^{-(r^2/2)}$ in critical dimension 2, it is known only that blowup cannot occur for $h \lesssim 1.93$ (since then then $N < N_c$), and must occur for $h > 2$ (for then the energy $H < 0$).

Experiments show that self-focussing collapse to a singularity occurs for $h$ above some threshold $h_c$, dependent on dimension.

For $D = 2$, $h_c \approx 1.95$, for which power $N \approx 1.02N_c$ and energy $H \approx 0.009$ (Fig.1). [Here and in all figures, amplitude curves that go past the top of the figure represent real blowup, specifically reaching well past amplitude of 1, 000.]

For $D = 3$ one gets $h_c \approx 2.077$, with energy still comfortably positive ($H \approx 1.4$) (Fig.2).

4.1 2D: inhibition of self-focussing collapse by a small attractive potential

It is not surprising that for $h \gg h_c$, a small repulsive potential in the NLSGP modification can inhibit collapse. For a narrow attractive potential in (NLSGP) one might expect only an acceleration of the collapse, and this does happen for sufficiently small potentials, but with potentials narrower than the initial self-induced potential of the initial data, as $h_p$ increases one then often sees this initial, accelerated focussing fail with the peak dispersing (though typically refocussing as discussed below.)

For example, in the near-threshold case $h = 1.95$, this defocussing occurs
Fig. 1.

Fig. 2.
with potential depth as small as $h_p = 0.11$ for the optimal width of about $w_p = 0.5$ (Fig’s 3, 4). This should be compared to the initial self-induced potential (minus the intensity) which is a somewhat wider and far deeper gaussian: width $1/\sqrt{2}$, height $3.8025$.

Though this effect is most pronounced when the beam power is just a little above the threshold for collapse, it persists at least until the power is 30% above threshold, with the smallest inhibiting potential becoming rapidly stronger and slowly narrower as $h$ increases.

For example, with $h = 2$, the smallest inhibiting gaussian potential is about $h_p = 1.25, w_p = 0.35$; for $h = 2.1$, it is about $h_p = 3.46, w_p = 0.3$ (Fig.5); for $h = 2.2$, it is about $h_p = 7, w_p = 0.2$; and for $h = 2.5$, it is about $h_p = 20, w_p = 0.15$ (Fig. 6).

[NOTE for revision: FIGURES can be provided on $(w_p, h_p)$ space for these claims.]

4.2 3D NLSGP: No inhibition of collapse

The above effect seems to be truly a critical dimension phenomenon; no such collapse inhibition is seen in 3D NLS even for Gaussian initial data only very
Fig. 4.

Fig. 5.
slightly above the threshold for collapse, even when very large attractive potentials of various widths are tried (Fig. 7).

4.3 Spatial structure of 2D collapse inhibition

Spatial cross sections of the solution amplitude at various times in cases with an inhibiting potential can be compared to those for a collapsing solution of the CSE with the same initial data (Fig’s 8, 9, 10 for \( h = 2.1, w_p = 0.3, h_p = 3.48 \)), and suggest a likely mechanism for the inhibition.

The potential accelerates the initial focussing of the part of the power distribution inside its well, but the resulting spike has less than critical power \( N_c \) and is more or less separated from the remaining “mass”, so essentially evolves on its own: as for a solution with total power less than \( N_c \), it can and does collapse to a finite degree but then disperses.

In this example, the focus almost completely collapses during which time a significant part the mass outside the spike disperses, but there is still enough concentrated mass that the dispersing spike can merge with it from this a second focus, lower but still significant, does form.

In other cases seen in Fig’s 4 and 6, there is far less dispersion before the
D=3, h=2.077

Fig. 7.

D=2, h=2.1, wp=0.3, hp=3.48

Fig. 8.
D=2, h=2.1, wp=0.3, hp=3.48

Fig. 9.

D=2, h=2.1, wp=0.3, hp=3.48

Fig. 10.
central spike merges with the rest of the mass and thus a second (and subsequent) focus of comparable intensity to the first occurs: a sustained oscillation of focussing and focus collapse. The absence of any dissipation, or of any dispersive term (indeed there is a small attractive term even outside the core that focusses) allows the mass to retain coherence in some cases.

Thus beams can be constrained between singular collapse and broad dispersion. This could be potentially useful in controlling and guiding beams of high intensity or in highly nonlinear optical media.

As a final note, the dispersion of the focussing spike seems more or less complete, and so the time scale of the subsequent refocussings is comparable to that of the original: this should be contrasted to the situation seem with dissipative regularizations of the CSE studied in [7] and [9], where the amplitude and phase structure supporting power flux towards the origin survived the collapse of the focus, leading to successive refocusings far more rapid than the original one.

References


