Calculus 3 (Math 221) Notes and Study Guide

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Contents

Chapter 1

Parametric Equations and Polar Coordinates

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References.

- Chapter 1, Sections 1–4 of OpenStax Calculus Volume 3.[1]
- Chapter 10 of Calculus, Early Transcendentals by Stewart.
- The Desmos online graphing calculator.[2]

Introduction. Chapter 1 of OpenStax Calculus Volume 3 simply reproduces Chapter 1 of OpenStax Calculus Volume 2[3], and so the corresponding class notes are also reproduced here for convenient review when we see similar topics in Section 3.3, p. 31 and Section 5.5, p. 83 (and to keep the chapter numbering in sync.)

1.1 Parametric Equations

References.

- OpenStax Calculus Volume 3, Section 1.1[1].
- Calculus, Early Transcendentals by Stewart, Section 10.1.

Introduction. Many interesting curves in the plane cannot be described as the graph of a function $y = f(x)$; the circle $x^2 + y^2 = 1$ is a very familiar example.

Relatedly, many curves describe the position of a moving object as a function of time; an object moving around the above circle might have coordinates at time t given by $x(t) = \cos t, y(t) = \sin t$. This is an example of a parametric description of a curve, with the new, "auxilliary" variable t called a parameter.

Note that this parametric form can convey more information that the equation $x^2 + y^2 = 1$ for the curve, because of the information about time and place—this is important for example for computing the velocity of a moving object.

Parametric Equations and Their Graphs. Let us introduce the main new concepts in this chapter:

[1]openstax.org/books/calculus-volume-3/pages/1-introduction
[2]www.desmos.com/calculator
[3]openstax.org/books/calculus-volume-2/pages/7-introduction
[1]openstax.org/books/calculus-volume-3/pages/1-1-parametric-equations

Definition 1.1.1 For two functions F and G defined on a common interval I, the pair of equations

$$x = x(t) = F(t),\ y = y(t) = G(t) \tag{1.1.1}$$

are the parametric equations of a curve.
The set of all points $x(t), y(t)$ for all $t \in I$ is called the graph of these equations; also known as a parametric curve or plane curve, and typically denoted C.
If the domain is a closed interval $I = [a, b]$ then the curve has initial point $(F(a), G(a))$ and terminal point $(F(b), G(b))$; these are are collectively called the endpoints of the curve.
Note however that that the interval could be open or semi-open and so lack one or both endpoints: it can even be infinite, like $(-\infty, \infty)$ or $[0, \infty)$. ◊

The parameter t often has the physical meaning of time, and will informally be referred to as "time" here.

Example 1.1.2 Circles. For any constant $R > 0$ and any point (c, d) in the plane

$$x = c + R\cos t,\ y = d + R\sin t,\ 0 \leq t \leq 2\pi$$

describes a circle of radius R, center (c, d). □

Exercise 1.1.3 Graphing this circle with Desmos. Graph the above parametric curve for the case of radius $R = 1$, center $(2, 3)$, using the Desmos online graphing calculator[2]: input the parametric equations as
(2 + cos(t), 3 + sin(t))
and then edit the limits of the t values (the value π can be entered by typing "pi").
Practice using the mouse/trackpad/finger to move around the graph and to zoom in and out.

Note that in this case, the initial and terminal points are the same; the right-most extremity, $(c + R, d)$:

Definition 1.1.4 A closed curve is one whose initial and final points are the same. ◊

Example 1.1.5 A spiral. The curve

$$x = t\cos t,\ y = t\sin t,\ 0 \leq t < \infty$$

describes one kind of spiral; at time t, the point is at distance t from the origin and as the parameter increases, the position rotates around the origin infinitely often. Its initial point is the origin, but it has no terminal point. □

Exercise 1.1.6 Graph this spiral with Desmos. One catch is that Desmos[3] cannot handle an infinite interval of t values, and anyway the whole spiral is infinitely large; thus, experiment with graphing a couple of turns; say $0 \leq t \leq 4\pi$.

Example 1.1.7 An exponential spiral. The curves

$$x = e^{at}\cos t,\ y = e^{at}\sin t,\ -\infty < t < \infty$$

describe another kind spiral; this time the point is at distance e^{at} from the origin at time t.
This has no initial or terminal point; however it makes sense to say that (for $a > 0$)

$$\lim_{t\to-\infty}(x(t), y(t)) = (0, 0)$$

so informally, it starts at the origin. □

Exercise 1.1.8 Visualizing exponential spirals. The above exponential spiral grows rather fast so to visualize, it is best to keep the parameter a small. Thus, start by look at a case like $x = e^{(t/4)}\cos t$, $y = e^{(t/4)}\sin t$; Desmos[4] input can be done as (exp(t/4) cos(t), exp(t/4) sin(t))
(You can also experiment with inputing exponents, to get the notation $e^{t/4}$.)
Again the infinite interval has to be reduced; start with one turn on either side $t = 0$ with $-2\pi \leq t \leq 2\pi$.

[2]www.desmos.com/calculator
[3]www.desmos.com/calculator

Then larger t intervals can be visualized with the help of zooming in and out.
As a further experiment with Desmos, include the parameter k with the form
(exp(k t) cos(t), exp(k t) sin(t)),
and see how Desmos allows setting up sliders for parameters.

Eliminating the Parameter. Sometimes the parameter can be eliminated, getting back to an equation of the form $y = F(x)$ or $x = G(y)$, or just a more general equation form $F(x, y) = 0$ like the equation for a circle. However, we will soon see that this is not always possible, and even when it is, some useful information can be lost.

Example 1.1.9 Circles again. Consider the parametric equations

$$x = \cos t,\ y = \sin t,\ 0 \leq t \leq 3\pi/2$$

with initial point $(1, 0)$ and terminal point $(0, -1)$. We can use a very familiar trig. identity to get

$$x^2 + y^2 = 1,$$

which looks like the equation of a circle.
However, three things are lost here:

1. Information about where the point is at a give time t,
2. the fact that this only covers three-quarters of the circle, due to the limits on the parameter values, and
3. the "function" form that will allow us to do calculus with curves in the next section, like computing their slopes.

Also, we needed a bit of luck here, with the trig. identity; this strategy often fails, as seen with the example of cycloids below. □

Example 1.1.10 Part of a parabola. For the parametric curve

$$x = \cos^2(t) + 2,\ y = \cos t,\ 0 \leq t \leq 2\pi$$

we can to some extent do better than above, getting a function describing this curve: substituting the second equation into the first gives

$$x = y^2 + 2$$

This equation describes a side-ways parabola, but it hides two facts:

1. The y values are only in the interval $-1 \leq y \leq 1$
2. The curve both starts and ends at the point $(1, 3)$, in between traveling to $(-1, 3)$ and then back-tracking.

□

Exercise 1.1.11 Graph this parametric curve $x = \cos^2(t) + 2$, $y = \cos t$. Note that "$\cos^2 t$" can be typed into Desmos[5] as either "cos^2(t)" or "cos(t)^2".

Cycloids and Other Parametric Curves. A very useful example of a parametric curve are the cycloids

$$x = a(t - \sin t),\ y = a(1 - \cos t) \tag{1.1.2}$$

because this cannot be written as the graph of a function in any useful way, and yet we can answer all kinds of questions about it, like computing the slope at a point on it, the length along the curve between points on this curve, and related areas under the curve.

The origin of this curve is that it describes the trajectory of a point on a wheel of radius a as that wheel rolls along, starting on the ground at point $(0, 0)$ at time $t = 0$.

[4]www.desmos.com/calculator
[5]www.desmos.com/calculator

Note that no domain for t is specified above; this curve can be consider as defined for all time. However it is also use to resrict to a single rotation

$$x = a(t - \sin t),\ y = a(1 - \cos t),\ 0 \le t \le 2\pi \tag{1.1.3}$$

which goes from initial point $(0, 0)$ to terminal point $(2\pi a, 0)$ with $y > 0$ in between, looking like an "arch"; for times before and after that, the curve "repeats" with copies of that arch shifted left and righ by multiples of $2\pi a$.

Exercise 1.1.12 Graph two arches of a cycloid. Set the scale as $a = 1$, so the Desmos[6] input can be (t-sin(t), 1-cos(t)); use interval $0 \le t \le 4\pi$.
Note the special behavior at the points where $y = 0$, and zoom in on the point $(2\pi, 0)$ given by $t = 2\pi$.
If you wish to explore further the capablities of Desmos, use the full form
(a(t-sin(t)), a(1-cos(t))) with parameter a,
and see how it allows setting up a slider for it.

Exercise 1.1.13 Another Desmos experiment: Prolate and Curtoid Cycloids. If you instead look at a point on the edge of the overhanging flange of a train wheel, so that the flange has radius $b > a$, one gets a prolate cycloid

$$x = at - b \sin t,\ y = a - b \cos t \tag{1.1.4}$$

If instead $b < a$, this describes the motion of a point on the wheel that is closer to the center, and is called a curtate cycloid.
Look at these with Desmos[7], using input (a t - b sin(t), a - b cos(t)) and setup sliders for both parameters.
Note the special behavior at the points where $t = 0, 2\pi$, etc.

Study Guide. Study Calculus Volume 3, Section 1.1[8]; in particular

- The Definition of parametric curves and parameters
- everything about Cycloids
- Examples 1 and 2
- Checkpoints 1 and 2
- and one or several exercises from each of the groups: 1–4, 6–9, 51–53. Of these, 6–9 should be done with Desmos[9] or similar software.

1.2 Calculus of Parametric Curves

References.

- OpenStax Calculus Volume 3, Section 1.2[1].
- Calculus, Early Transcendentals by Stewart, Section 10.2.

Derivatives of Parametric Equations. Often we are interested in the slope of a parameteric curve $(x, y) = (F(t), G(t))$, meaning intuitively $m = dy/dx$, but we do not have an explicit formula $y = f(x)$. Fortunately, rather than having to solve for f by eliminating th parameter t, the needed derivative and slope can be computed by implicit differentiation: if $y = G(t) = f(x) = f(F(t))$ then

$$\frac{dy}{dt} = \frac{dG}{dt} = f'(F(t))F'(t) = f'(x)F'(t) = \frac{dy}{dx}\frac{dx}{dt};$$

[6] www.desmos.com/calculator
[7] www.desmos.com/calculator
[8] openstax.org/books/calculus-volume-3/pages/1-1-parametric-equations
[9] www.desmos.com/calculator
[1] openstax.org/books/calculus-volume-2/pages/7-2-calculus-of-parametric-curves

which is the familiar intuitive pattern of the Chain Rule. This can than be solved to get

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \tag{1.2.1}$$

so long as the denominator dx/dt is non-zero.

Note that this will be a formulas in terms of t, not x!

Example 1.2.1 Slope of the spiral $(x, y) = (t\cos(t), t\sin(t))$.
First,

$$dx/dt = \cos(t) - t\sin(t) \text{ and } dy/dt = \sin(t) + t\cos(t),$$

so the slope at the point given by any value of the parameter t is given by

$$\frac{dy}{dx} = \frac{\sin(t) + t\cos(t)}{\cos(t) - t\sin(t)}$$

□

Example 1.2.2 Slope of the cycloid $(x, y) = (t - \sin(t), 1 - \cos(t))$.
$dx/dt = 1 - \cos(t)$ and $dy/dt = \sin(t)$, so the slope at the point given by any value of the parameter t is given by

$$\frac{dy}{dx} = \frac{\sin(t)}{1 - \cos(t)},$$

except where the denominator is zero; that happens when $\cos(t) = 1$, which is for t an integer multiple of 2π; $t = 2n\pi$, where also $\sin(t) = 0$.
This is the points $(x, y) = (2n\pi, 0)$ where the cycloid "touches down", and where the graphs done in Section 1.1, p. 1 suggested that something strange was happening. □

Second-Order Derivatives. Once one has a formula for the first derivative dy/dx (albeit in terms of t), computing second and higher derivatives is relatively strightforward; no furhter implicit differentition is needed. First,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$$

Next, use the above Equation (1.2.1) with y replaced by dy/dx:

$$\frac{d^2y}{dx^2} = \frac{d(dy/dx)/dt}{dx/dt}$$

Since the method above gives dy/dx as a function of t, we can evaluate the two derivatives here.

Exercise 1.2.3 The concavity of the cycloid. Compute the second derivative $\dfrac{d^2y}{dx^2}$ of the above cycloid. Graphs suggest that this curve is always concave down, so check that.

Integrals Involving Parametric Equations: Area Under a Curve. If a curve $y = f(x)$, $a \le x \le b$ also has a parametric form $x = F(t)$, $y = G(t)$, $\alpha \le t \le \beta$, with f an increasing function and $y = g(t) \ge 0$, then it lies over a region $a \le x \le b$ with $x(\alpha) = a$, $x(\beta) = b$, and it makes sense to talk of the area between this curve and the x-axis.

If we could eliminate the parameter and describe the curve as $y = F(x)$, this area would be $A = \int_a^b F(x)\,dx$, but in fact, we do not need to get an explicit formulas for $F(x)$! Instead, use the (inverse) substitution $x = f(t)$ to get

$$A = \int_{x=a}^{b} y\,dx = \int_{x=a}^{b} f(x)\,dx = \int_{t=\alpha}^{\beta} f(x(t))\frac{dx}{dt}dt = \int_{t=\alpha}^{\beta} f(x(t))F'(t)dt = \int_{t=\alpha}^{\beta} y\frac{dx}{dt}dt. \tag{1.2.2}$$

where we use the fact that $y = F(x(t))$ and also $y = g(t)$.

That is, we get the intuitive change of variables

$$A = \int_{x=a}^{b} y\,dx = \int_{t=\alpha}^{\beta} y\frac{dx}{dt}dt. \tag{1.2.3}$$

As always, note how the limits of integration change when the dummy variable is changed by substitution!

Exercise 1.2.4 Compute the area under one arch of the cycloid $(x, y) = (a(t - \sin(t)), a(1 - \cos(t)))$

Arc Length of a Parametric Curve. The formula for the arc length of a curve $y = f(x)$ can be converted to parametric form when $x = F(t)$, as was done for areas:

$$\begin{aligned} L &= \int_{x=a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_{t=\alpha}^{\beta} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \frac{dx}{dt}\, dt \\ &= \int_{t=\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\frac{dx}{dt}\right)^2}\, dt \\ &= \int_{t=\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \end{aligned} \tag{1.2.4}$$

A good intuitive way to see this is that each "infinitesimally" small piece of the curve has length

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{1.2.5}$$

sometimes called the arc length differential; the arc length is then the "sum" or integral of these infinitesimal lengths: $L = \displaystyle\int ds$.

This idea can be used to show that in fact for any parametric curve with $f'(t)$ and $g'(t)$ continuous for $\alpha \le t \le \beta$, the arc length is, as above,

$$L = \int ds = \int_{t=\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{1.2.6}$$

The curve does not have to be in the form of the graph of a function, and in particular, x need not be an increasing function of the parameter.

Exercise 1.2.5 Compute the circumference of a circle or radius R, parameterized as $(x, y) = (R\cos(t), R\sin(t))$

Exercise 1.2.6 Compute the length of one arch of the cycloid $(x, y) = (a(t - \sin(t)), a(1 - \cos(t)))$

Surface Area Generated by a Parametric Curve (Omitted). This topic is not covered in this course, but I include this brief introduction; it is discussed further in Section 7.2 of the OpenStax Calculus text.[2]

The area of the surface produced by rotating a parametric curve about the x-axis can be computed, and the most intuitive way to see the result is to work with a surface area differential dS, much as the arc length differential ds was used above.

[2] openstax.org/books/calculus-volume-2/pages/7-2-calculus-of-parametric-curves

When an infinitesimal part of the parametric curve $x = F(t)$, $y = G(t)$ of arc length ds is rotated about the x-axis, it produces an angled strip of width ds, radius y, circumference $2\pi y$, and thus with infinitesimal area given by the surface area differential $dS = 2\pi y\, ds$.

Thus, with appropriate limits of integration, the surface area is

$$S = \int dS = \int 2\pi y\, ds = \int_{t=\alpha}^{b} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{1.2.7}$$

Exercise 1.2.7 Compute the area of the "football shaped" surface produced by rotating one arch of the cycloid $(x, y) = (a(t - \sin(t)), a(1 - \cos(t)))$ about the x-axis.

Study Guide. Study Calculus Volume 3, Section 1.2[3]; in particular

- Theorems 1 2, 3
- Examples 4, 5, 6, 7, 8
- Checkpoints 4, 5, 6, 7, 8
- and one or several exercises from each of the following groupls: 62–65, 66–70, 71–74, 75–77, 88–90, 104–107 (areas under curves), 108–112 (arclengths: no need to evaluate for 112, just setup the integral).

Note that we omit the final topic of Surface Area Generated by a Parametric Curve

1.3 Polar Coordinates

Revised on March 17.

References.

- OpenStax Calculus Volume 3, Section 1.3[1].
- Calculus, Early Transcendentals by Stewart, Section 10.3.

Defining Polar Coordinates. Polar coordinates describe the location of a point P in the plane in terms of

- the polar distance r from a reference point O, the pole, and
- the polar angle θ, describing the direction of motion from O to P relative to a direction considered horizontal.

In comparison to standard cartesian coordinates (x, y) for P, using the pole as the origin and with the positive x-axis as the horizontal direction, the polar distance is easily described:

$$r = |OP| = \sqrt{x^2 + y^2}. \tag{1.3.1}$$

The angle is measured from the positive x-axis to the ray $\overrightarrow{OP}$, going in the direction towards the positive y-axis (so "anti-clockwise").

It is not so simple to give a formula for it in terms of x and y, so it helps to go the other way first:

Cartesian Coordinate Values from Polar Coordinate Values. The point P with cartesian coordinates (x, y) lines an the circle of radius r, center $(0, 0)$, so the angle θ determines its coordinates to be

$$x = r\cos\theta,\ y = r\sin\theta. \tag{1.3.2}$$

[3] openstax.org/books/calculus-volume-3/pages/1-2-calculus-of-parametric-curves
[1] openstax.org/books/calculus-volume-2/pages/7-3-polar-coordinates

Polar Coordinate Values from Cartesian Coordinate Values. Getting r from x and y was easy. To get θ, first divide, getting $\tan\theta = \dfrac{y}{x}$, if $x \neq 0$

There are two problems: the case $x = 0$ and the multiple angles with the same tangent.

We can of course restrict the allowed θ values to "one rotation"; two favorite choices are

- $-\pi < \theta \leq \pi$ to keep the size of θ small (with a bias to positive values) and
- $0 \leq \theta < 2\pi$ to keep θ positive and still as small as possible.

However that still leaves two possible values for θ, differing by π, and using $\theta = \arctan(y/x)$ does not always give the correct value: it always gives an angle $-\pi/2 < \theta < \pi/2$ and so a point in the right half-plane.

Polar Coordinate Values from Cartesian Coordinate Values: A Solution. Often we prefer the smallest magnitude for θ, and use the value in $(-\pi, \pi]$. (Excluding $\theta = -\pi$ as it would be redundant.) Here is one way to do that:

$$\theta = \begin{cases} \arctan(y/x) & \text{if } x > 0, \text{ so } -\pi/2 < \theta < \pi/2 \\ \arctan(y/x) - \pi & \text{if } x < 0 \text{ and } y < 0, \text{ so } -\pi < \theta < -\pi/2 \\ \arctan(y/x) + \pi & \text{if } x < 0 \text{ and } y \geq 0, \text{ so } \pi/2 < \theta \leq \pi \\ \pi/2 & \text{if } x = 0 \text{ and } y > 0 \\ -\pi/2 & \text{if } x = 0 \text{ and } y < 0 \end{cases} \tag{1.3.3}$$

This still omits the case of the pole, where $x = y = 0$ (so $r = 0$): there the polar angle is ill-defined, but the good news is that any value of θ is acceptable, in that the equations (1.3.2) give the correct cartesian coordinates.

Simpler Equations for Getting from Cartesian to Polar Coordinates. Often we can avoid these complications by working with the following simpler equations whenever possible:

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x} \text{ when } x \neq 0, \tag{1.3.4}$$

with special handling of points with $x = 0$: these are on the y-axis, so we can use

- $\theta = \pi/2$ if $y > 0$
- $\theta = -\pi/2$ or $3\pi/2$ if $y < 0$
- any θ value we want if also $y = 0$, so we are at the origin where the polar angle makes no sense.

Also, since both these equations (1.3.4) and Equations (1.3.2) giving cartesian coordinates in terms of polar coordinates make sense for all real values of r and θ, we sometimes do not restrict to $r \geq 0$, $-\pi < \theta \leq \pi$.

This more flexible approach will help below to produce elegant descriptions of some interesting Polar Curves.

Polar Curves. Many curves have rotational features that make them easiest to describe in terms of polar coordinates, with equations in terms of r and θ, like

$$r = f(\theta).$$

Note well: we will graph them as curves in the cartesian plane, with axes x and y!

The cartesian coordinates are given with the help of the equations (1.3.2) as

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta \tag{1.3.5}$$

so that they are a type of parametric curve, with polar angle θ as the parameter.

The most basic example is the equation $r = C$, for C a positive constant. This is a circle of radius C. Since the equation says nothing about the angle θ, it can take any value, and with the standard range of values

$(-\pi, \pi]$, the angle θ becomes a convenient parameter giving a parametric description of the curve by inserting $r = C$ into the Equations (1.3.2):

$$x = C\cos\theta,\ y = C\sin\theta.$$

Another example is $r = e^\theta$, which is the exponential spiral

$$x = e^\theta \cos\theta, \quad y = e^\theta \sin\theta.$$

Example 1.3.1 The Graph of the Polar Equation $\theta = c$, c a constant. The simple equation $\theta = c$ requires a little more care to graph, and r must be used as the parameter instead of θ.
Equation (1.3.4) gives $\tan\theta = \tan c = y/x$, so $y = (\tan c)x$. This looks like the equation of a straight line through the origin, and even the cases where $\tan c$ does not exist make sense: they give the vertical line $x = 0$.
However, if we restrict to $r \geq 0$, the curve is actually only part of this line: it is the ray starting at the origin and going in the direction specified by the angle c.
The moral is that, as always with graphs and functions, we must specify the domain: do we want to allow all r (and get a line), or $r \geq 0$ (and get a ray)? For example, with $r \geq 0$,

- $\theta = \pi/2$ is the positive y-axis,
- $\theta = -\pi/2$ is the negative y-axis.

□

Tangents to Polar Curves. Since any polar curve given by an equation $r = f(\theta)$ is a parametric curve as in Equation (1.3.5), there is nothing really new here, but it is worth noting the formulas for the tangent slope dy/dx:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Evaluation of this is a good case of the strategy I have been using recently, of evaluating key pieces and then gathering them is stages: the term $dr/d\theta$ should be evaluated and simplified before inserting, to avoid duplicated effort.

Symmetry in Polar Coordinates. (Omitted)

Study Guide. Study Calculus Volume 3, Section 1.3[2]

The main content relevant for us is up to Example 13 and Checkpoint 13 (we skip the topic of symmetry, but I suggest reading it).

Do one or several exercises from each of the ranges 136–141, 142–148, 154–157, and 158–160.

1.4 Area and Arc Length in Polar Coordinates

References.

- OpenStax Calculus Volume 3, Section 1.4[1].
- Calculus, Early Transcendentals by Stewart, Section 10.4.

Areas of Regions Bounded by Polar Curves. A polar curve $r = f(\theta)$ typically encloses a region inside the curve, $r < f(\theta)$ rather than below it. That is, we are often interested in the region between the curve and the pole (origin), rather than between the curve and a horizontal axis.

[2] openstax.org/books/calculus-volume-3/pages/1-3-polar-coordinates
[1] openstax.org/books/calculus-volume-3/pages/1-4-area-and-arc-length-in-polar-coordinates

Thus, the strategy for finding area as the integral of infinitesimal fragment or area differential dA will be based on looking at the thin region that lies between the curve and the pole over a narrow range of polar angle values $d\theta$: a very thin sector of radius $r = f(\theta)$ and angular extent $d\theta$

A circular sector of radius r covering angle θ has area $\frac{1}{2}r^2\theta$; thus this infinitesimal sector has area described by the area differential

$$dA = \frac{1}{2}r^2 d\theta = \frac{1}{2}[f(\theta)]^2 d\theta \tag{1.4.1}$$

If you prefer using approximations with small finite pieces, the range of angles $\theta_i \leq \theta \leq \theta_i + \Delta\theta$ gives a region that is approximately a sector, with area approximation

$$\Delta A_i \approx \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta \quad \text{for some angle } \theta_i^* \text{ in } [\theta_i, \theta_i + \Delta\theta]$$

The transition from finite increments to infinitesimal ones turns this into the exact formulas seen above,

$$dA = \frac{1}{2}r^2 d\theta = \frac{1}{2}[f(\theta)]^2 d\theta.$$

The familiar argument with limits and the FTC then shows that the area of the "sector" inside the curve $r = f(\theta)$, $a \leq \theta \leq b$ is

$$A = \int dA = \frac{1}{2}\int_{\theta=a}^{b} r^2 d\theta = \frac{1}{2}\int_{\theta=a}^{b} [f(\theta)]^2 d\theta. \tag{1.4.2}$$

Arc Length in Polar Curves. The arc length of a polar curve $r = f(\theta)$, $a \leq \theta \leq b$ is given by simply using the results for a parametric curve applied to

$$x = r\cos\theta, \quad y = r\sin\theta.$$

The arc length differential for this polar curve,

$$ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta,$$

simplifies nicely when combined with the formulas for the derivatives

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta + r\sin\theta$$
$$\frac{dr}{d\theta} = \frac{dr}{d\theta}\sin\theta - r\cos\theta$$

to give

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta, \tag{1.4.3}$$

and thus

$$L = \int ds = \int_{\theta=a}^{b} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{1.4.4}$$

Study Guide. Study Calculus Volume 3, Section 1.4[2]; in particular

- Theorems 6 and 7
- Examples 16 and 18

[2]openstax.org/books/calculus-volume-3/pages/1-4-area-and-arc-length-in-polar-coordinates

- Checkpoints 15 and 17
- and one or several exercises from each of the following groups: 188–194, 201–206, 214–217, 218–222.

Chapter Review. When reviewing this chapter, also look at the end of chapter review material in OpenStax Calculus Volume 3, including Key Terms[3], Key Equations[4] and Key Concepts[5].

[3]openstax.org/books/calculus-volume-3/pages/1-key-terms
[4]openstax.org/books/calculus-volume-3/pages/1-key-equations
[5]openstax.org/books/calculus-volume-3/pages/1-key-concepts

Chapter 2

Vectors in Space

References.

- Chapter 2 of OpenStax Calculus Volume 3.[1]
- Chapter 12 of Calculus, Early Transcendentals by Stewart.
- The Desmos online graphing calculator.[2]

2.1 Vectors in the Plane

References.

- OpenStax Calculus Volume 3, Section 2.1[1].
- Calculus, Early Transcendentals by Stewart, Section 12.2.

The earliest meaning of a vector relates to movement from one place to another (like a mosquito as a vector for malaria), and this leads to the geometrical idea of a vector describing the displacement from one location to another.

In this section we will consider the most basic case of vectors in the plane $\mathbb{R}^2$, where for example, points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ determine the two dimensional vector going from point P_1 to point P_2, denoted $\overrightarrow{P_1P_2}$.

In algebraic terms, this is described by the changes in the values of each of the coordinates, so we denote this $\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1 \rangle$. Note that only the change in position or displacement counts, so many different pairs of points represent the same vector. Thus we often name a vector without reference to a pair of endpoints, with an over-arrow like $\vec{a}$, or with boldface like a.

Using vectors as "bundled coordinates". Any vector $\vec{a} = \langle x, y \rangle$ can be considered with the origin $O(0, 0)$ as its starting point so that it ends at $P(x, y)$: $\vec{a} = \overrightarrow{OP}$. Thus vectors give a convenient way to bundle the values of all the coordinates of a point P into a single object, and we can sometimes think of vectors as equivalent to points. But as we shall see, vectors have added features. Thus we call the set of all two dimensional vectors V_2, to distinguish from the set $\mathbb{R}^2$ of points in the plane.

See Example 1 in OSC3 Section 2.1[2].

[1] openstax.org/books/calculus-volume-3/pages/2-introduction
[2] www.desmos.com/calculator
[1] openstax.org/books/calculus-volume-3/pages/2-1-vectors-in-the-plane
[2] openstax.org/books/calculus-volume-3/pages/2-1-vectors-in-the-plane

The length or magnitude of a vector. Vectors share some but not all of the properties of real numbers (which we now often call scalars to distinguish them from vectors) like length which resembles the absolute value of a scalar; addition; and and a form of multiplication. The length or magnitude of a vector $\vec{a} = \langle x, y \rangle$ is the distance between its endpoints, denoted $|\vec{a}|$ or $\|\vec{a}\|$ to mimic the absolute value or magnitude of a scalar. Thus

$$\|\vec{a}\| = \| \langle x, y \rangle \| = \sqrt{x^2 + y^2} \tag{2.1.1}$$

Adding vectors. Since vectors describe displacements or movement from one point to another, one can combine several vectors, by making one move and then the other. If $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$, combining these two displacements changes the first [x] coordinate by $a_1 + b_1$ and so on, so the combined displacement is described by the new vector $\langle a_1 + b_1, a_2 + b_2 \rangle$. This is called the sum of the vectors $\vec{a}$ and $\vec{b}$, and we write

$$\vec{a} + \vec{b} = \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle . \tag{2.1.2}$$

Clearly this is commutative like addition of real numbers:

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \tag{2.1.3}$$

Also, there is a natural zero vector $\vec{0} = \langle 0, 0 \rangle$ representing no change in position, with

$$\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}. \tag{2.1.4}$$

See Example 2.2(b) in OSC3 Section 2.1[3].

Scalar multiples of vectors. Repeated addition of copies of the same vector give natural number multiples of a vector, like

$$2 \langle a_1, a_2 \rangle = \langle a_1, a_2 \rangle + \langle a_1, a_2 \rangle = \langle 2a_1, 2a_2 \rangle .$$

This suggests the natural generalization to defining the scalar-by-vector product for any scalar (real number) c:

$$c\vec{a} = c \langle a_1, a_2 \rangle = \langle ca_1, ca_2 \rangle \tag{2.1.5}$$

Geometrically, $c\vec{a}$ describes a displacement parallel to that described by $\vec{a}$ but of magnitude different by a factor $|c|$, and in the opposite direction if c is negative. It is routine to check that this multiplication is distributive in both factors and associative where it makes sense:

$$(c + d)\vec{a} = c\vec{a} + d\vec{a}, \quad c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}, \quad (cd)\vec{a} = c(d\vec{a}). \tag{2.1.6}$$

The magnitude of products. From the formula above for the length of a vector, it can be checked that the magnitude of a scalar-vector product is the product of the magnitudes:

$$\begin{aligned} \|c\vec{a}\| &= \sqrt{(ca_1)^2 + (ca_2)^2} \\ &= \sqrt{c^2(a_1^2 + a_2^2)} \\ &= \sqrt{c^2}\sqrt{a_1^2 + a_2^2} \\ &= |c|\|\vec{a}\|. \end{aligned} \tag{2.1.7}$$

Vector subtraction and vector-scalar division. Subtraction as always is defined in terms of addition: $\vec{a} - \vec{b}$ must be the vector that satisfies $(\vec{a} - \vec{b}) + \vec{b} = \vec{a}$, and this has to be

$$\vec{a} - \vec{b} = \langle a_1, a_2 \rangle - \langle b_1, b_2 \rangle = \langle a_1 - b_1, a_2 - b_2 \rangle . \tag{2.1.8}$$

Likewise division by a scalar can be defined in terms of multiplication:

$$\vec{a}/c = \frac{1}{c}\vec{a}. \tag{2.1.9}$$

[3] openstax.org/books/calculus-volume-3/pages/2-1-vectors-in-the-plane

See Example 2 in OSC3 Section 2.1[4].

Basic Vectors. Two dimensional coordinates were described in terms of getting to a point P with a succession of two moves, parallel to each axis in turn. The displacement described by a vector can be broken up into two such displacements, which can in turn be written as multiples of displacements by distance one:

$$\vec{a} = \langle a_1, a_2 \rangle = \langle a_1, 0 \rangle + \langle 0, a_2 \rangle \; = a_1 \langle 1, 0 \rangle + a_2 \langle 0, 1 \rangle \, .$$

Thus any vector can be written in terms of the two special vectors appearing in the last line: $\vec{a} = a_1 \hat{\imath} + a_2 \hat{\jmath}$ where

$$\hat{\imath} = \vec{\imath} = \langle 1, 0 \rangle \qquad \hat{\jmath} = \vec{\jmath} = \langle 0, 1 \rangle \tag{2.1.10}$$

These are known as the standard basis vectors for the set V_2 of vectors in the plane.

See Examples 4 and 5 in OSC3 Section 2.1[5].

Unit Vectors. Unit vectors are vectors of length one, like the standard basis vectors above. They are often used to indicate a direction of motion, without indicating the magnitude of that motion. For example, the above standard basis vectors indicate the directions ”east” and ”north”. For any non-zero vector $\vec{a}$, there is a unique unit vector $\vec{u}$ with the same direction,

$$\vec{u} = \frac{1}{\|\vec{a}\|} \vec{a} = \frac{\vec{a}}{\|\vec{a}\|}$$

Exercise 2.1.1 Find the unit vector in the direction of the vector $\langle 3, -4 \rangle, = 3\hat{\imath} - 4\hat{\jmath}$.

See Examples 7 and 8 in OSC3 Section 2.1[6].

What is missing? Although we have seen how to do with vectors much of what can be done with real numbers, a few things are missing: we have not defined

- a product of two vectors,
- a quotient of two vectors, or
- the inverse of a vector.

In subsequent sections we will see two versions of the product of vectors, but neither makes quotients or inverses possible.

Study Guide. Study Section 2.1 of Calculus Volume 3[7]; in particular:

- Theorem 2.1.
- Examples 2.1–2.8 and the Checkpoints folowing each.
- One or several exercises from each of the following ranges: 1–8, 11–14, 17–20.

2.2 Vectors in Three Dimensions

References.

- OpenStax Calculus Volume 3, Section 2.2[1].
- Calculus, Early Transcendentals by Stewart, Section 12.2.

[4] openstax.org/books/calculus-volume-3/pages/2-1-vectors-in-the-plane
[5] openstax.org/books/calculus-volume-3/pages/2-1-vectors-in-the-plane
[6] openstax.org/books/calculus-volume-3/pages/2-1-vectors-in-the-plane
[7] openstax.org/books/calculus-volume-3/pages/2-1-vectors-in-the-plane
[1] openstax.org/books/calculus-volume-3/pages/2-2-vectors-in-three-dimensions

Introduction. Much of what we have seen in Section 2.1, p. 12 about vectors in the plane $\mathbb{R}^2$ carries over to three dimensional space, but first we need to look at coordinates, calculating distances, and equations in three dimensions.

2.2.1 Three-Dimensional Coordinate Systems

To specify the location of a point in our three dimensional world, three numbers are needed. For example, the position of an aircraft is specified by its latitude, longitude and altitude, which is to say by measuring distances east or west, north or south and above or below some reference point.

In mathematical terms, we can think of starting with a reference plane (mimicing the surface of the earth in a region small enough that the earth is close enough to flat), marking this plane with a reference point as origin and a grid measuring how far east or west (let us call this number the x-coordinate: positive for points east of the origin, negative for west) and how far north or south (let us call this number the y-coordinate: positive for points north of the origin, negative for south) a point in to plane is from this origin.

In a room with rectangular floor and walls you can specify the location of any point P in the room with three numbers; for example using one corner of the room as the reference point or origin.

First choose a corner as origin O, and name the three edges running from that corner as the x, y and z axes. (A common choice is putting the origin in the bottom-south-west corner and then using x for the edge running east, y for that running north, and z for that running upwards.) Thus the floor connects the x and y axes (the x-y plane), one wall meeting the origin ("south") connects the x and z axes (the x-z plane), and the other wall meeting the origin ("west") connects the y and z axes (the y-z plane).

The distance a horizontally west from the point to the "west wall" or y-z plane, along a line parallel to the x axis, is the x-coordinate; the values b and c of the y- and z-coordinates can be measured similarly.

How the coordinates determine the location of point P. How do these three coordinate values a, b and c determine the point P, and why does the origin have that name? To get to P:

1. start originally at the origin O,
2. move along the x-axis ("east") by a distance a,
3. then move parallel to the y axis ("north") across the x-y plane by a distance b,
4. and finally move parallel to the z-axis ("up") by a distance c.

Thus, a point in space can be described by a triple of numbers.

See Example 11 in OSC3 Section 2.2[2].

The point P with the above coordinate values is usually written (a, b, c), or to indicate the name too, $P(a, b, c)$.

The set of all such triples is denoted as $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ or $\mathbb{R}^3$.

Exercise 2.2.1 What surfaces in the 3D space $\mathbb{R}^3$ are given by the following equations?
(a) $z = 3$
(b) $y = 5$
(c) $x + y = 1$
(Note that it is important to specify that we are in $\mathbb{R}^3$, especially in (b).)

Exercise 2.2.2 $\mathbb{R}^3 y = x$

Exercise 2.2.3 $rC(h, k, l)$

Exercise 2.2.4 Show that $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$ is the equation of a sphere.
Which sphere?

[2] openstax.org/books/calculus-volume-3/pages/2-2-vectors-in-three-dimensions

Solid regions in space described by inequalities. Much as an inequality (or several) in the two variables x and y describe a region within the x-y plane, inequalities in three variables describes solid regions in space. A simple example is that $x^2 + y^2 + z^2 \leq 1$ describes the solid ball of radius 1, center the origin.

Example 2.2.5 Describe the region (in $\mathbb{R}^3$) given by the points whose coordinates satisfy the inequalities

$$1 \leq x^2 + y^2 + z^2 \leq 4, \qquad z \leq 0.$$

□

2.2.2 Vectors in $\mathbb{R}^3$

Much about vectors in $\mathbb{R}^3$ is very similar to what was seen in Section 2.1, p. 12, these notes will be brief, and focus on the more significant differences.

A vector $\vec{v} = \overrightarrow{PQ}$ in three dimensions describes the dispacement between two points $P(p_1, p_2, p_3)$ and $Q(q_1, q_2, q_3)$, so that $\vec{v} = \langle v_1, v_2, v_3 \rangle = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle$, and is the same so long as the displacement is the same; in particular, the above $\vec{v}$ is also $\overrightarrow{OV}$ for $O(0, 0, 0)$ the origin in $\mathbb{R}^3$ and $V(v_1, v_2, v_3)$.

The rules for multiplication by a scalar (number), addition, and thus subtraction are as expected; for any $\vec{v} = \langle v_1, v_2, v_3 \rangle$, $\vec{w} = \langle w_1, w_2, w_3 \rangle$ and scalar $k \in \mathbb{R}$,

$$\begin{aligned} k\vec{v} &= \langle kv_1, kv_2, kv_3 \rangle \\ \vec{v} \pm \vec{w} &= \langle v_1 \pm w_1, v_2 \pm w_2, v_3 \pm w_3 \rangle \end{aligned}$$

and these follow all the expected rules about commutativity, associativity and distributivity of multiplication over addition and subtraction, with the zero vector $\vec{0} = \langle 0, 0, 0 \rangle$.

Also, the length of a vector $\vec{v} = \overrightarrow{PQ}$ is the distance betwen the points P and Q, denoted either $|PQ|$, $\|\vec{v}\|$ or (lazily) $|\vec{v}|$:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

See Example 12 in OSC3 Section 2.2[3].

With this, we can get a unit vector $\vec{u}$ in the same direction as a given non-zero vector:

$$\|\vec{u}\| = \frac{1}{\|\vec{v}\|}\|\vec{v}\|, \text{ also denoted } \hat{u}$$

because unit vectors are sometimes denoted with a "hat" instead of an arrow.

In particular, there are 3D versions of the standard unit vectors seen for vectors in the plane:

$$\begin{aligned} \vec{\imath} &= \hat{\imath} = \langle 1, 0, 0 \rangle \\ \vec{\jmath} &= \hat{\jmath} = \langle 0, 1, 0 \rangle \\ \vec{k} &= \hat{k} = \langle 0, 0, 1 \rangle \end{aligned}$$

. Summary: Properties of Vectors in SpaceLet $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, v_3 \rangle$ be vectors, and let k be a scalar.

- Scalar multiplication: $k\vec{v} = \langle kv_1, kv_2, kv_3 \rangle$
- Vector addition: $\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$
- Vector subtraction: $\vec{v} - \vec{w} = \langle v_1 - w_1, v_2 - w_2, v_3 - w_3 \rangle$
- Vector magnitude: $\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
- Unit vector in the direction of $\vec{v}$: $\hat{v} = \frac{1}{\|\vec{v}\|}\vec{v} = \left\langle \frac{v_1}{\|\vec{v}\|}, \frac{v_2}{\|\vec{v}\|}, \frac{v_3}{\|\vec{v}\|} \right\rangle$, if $\vec{v} \neq \vec{0}$.

[3]openstax.org/books/calculus-volume-3/pages/2-2-vectors-in-three-dimensions

See Example 19 in OSC3 Section 2.2[4].

Study Guide. Study Section 2.2 of Calculus Volume 3[5]; in particular:

- Theorem 2.2.
- Examples 2.11–2.15, 2.18, 2.19 and (as usual) the Checkpoints following each.
- One or several exercises from each of the following ranges: 67–68, 71–74, 75–76, 77–80, 87–90, 91–94.

2.3 The Dot Product (a.k.a. Scalar Product)

Revised on March 14.

References.

- OpenStax Calculus Volume 3, Section 2.3[1].
- Calculus, Early Transcendentals by Stewart, Section 12.3.

Introduction. There are two useful notions of a product of two vectors; in this section we meet the first of them:

Definition 2.3.1 The dot product of two vectors in $\mathbb{R}^3$ is

$$\vec{u} \cdot \vec{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{2.3.1}$$

with the obvious 2D version. This is also known as the scalar product (because the value is a scalar, not another vector) or the inner product. ♢

See Example 21 in OSC3 Section 2.3[2].

Geometric Characterization. Much of the importance of the dot product comes from its geometric properties; in fact the formula can be derived by requiring that

If θ is the angle between the two vectors $\vec{u}$ and $\vec{v}$ then

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta. \tag{2.3.2}$$

See Examples 23 to 25 in OSC3 Section 2.3[3].

along with some basic properties of a product:

- $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$: (Distributive property)
- $(a\vec{u}) \cdot \vec{v} = \vec{u} \cdot (a\vec{v}) = a(\vec{u} \cdot \vec{v})$ (Associative property)

(In fact, (2.3.2) could be used as an equivalent definition.)

The main step is working out what (2.3.2) implies for the basic vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$.

Firstly,

$$\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1$$

because the angle is $\theta = 0$ and the lengths are all 1.

[4] openstax.org/books/calculus-volume-3/pages/2-2-vectors-in-three-dimensions
[5] openstax.org/books/calculus-volume-3/pages/2-2-vectors-in-three-dimensions
[1] openstax.org/books/calculus-volume-3/pages/2-3-the-dot-product
[2] openstax.org/books/calculus-volume-3/pages/2-3-the-dot-product
[3] openstax.org/books/calculus-volume-3/pages/2-3-the-dot-product

Next,

$$\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{\imath} = \hat{\imath} \cdot \hat{k} = \hat{k} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{k} = \hat{k} \cdot \hat{\imath} = 0$$

because now all the angles are $\pi/2$ so with zero cosine. Note that commutativity is not assumed (and will fail for the cross product introduced in the next section); it is instead verified as a consequence of the definition.

Finally

$$\begin{aligned}\vec{u} \cdot \vec{v} &= (u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}) \cdot (v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}) \\ &= (u_1\hat{\imath}) \cdot (v_1\hat{\imath}) + (u_2\hat{\jmath}) \cdot (v_2\hat{\jmath}) + (u_3\hat{k}) \cdot (v_3\hat{k}) + (u_1\hat{\imath}) \cdot (v_2\hat{\jmath}) + \cdots \\ &= u_1v_1\hat{\imath} \cdot \hat{\imath} + u_2v_2\hat{\jmath} \cdot \hat{\jmath} + u_3v_3\hat{k} \cdot \hat{k} + u_1v_2\hat{\imath} \cdot \hat{\jmath} + \cdots \\ &= u_1v_1 + u_2v_2 + u_3v_3\end{aligned}$$

as all the remaining terms involve zero dot products.

Properties. From the formula in definition Definition 2.3.1, p. 17 on can derive some familiar properties (including the one assumed to show the connection with the geometric formula (2.3.2)):

i $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ Commutativity

ii $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ Distributivity

iii $(a\vec{u}) \cdot \vec{v} = \vec{u} \cdot (a\vec{v}) = a(\vec{u} \cdot \vec{v})$ Associativity with scalar multiplication

iv $\vec{u} \cdot \vec{0} = 0$

v $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$

See Example 22 in OSC3 Section 2.3[4].

Direction Angles and Direction Cosines. The direction angles of a non-zero vector $\vec{u}$ are the angles that it makes with the three coordinate axes, typically called α, β and γ. That is, the angles that the vector makes with the three standard basic vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$. The cosines of these are the direction cosines.

$$\cos\alpha = \frac{\vec{u} \cdot \hat{\imath}}{|\vec{u}||\hat{\imath}|}, = \frac{u_1}{|\vec{u}|}, \qquad \cos\beta = \frac{u_2}{|\vec{u}|}, \qquad \cos\gamma = \frac{u_3}{|\vec{u}|}$$

Since the angles are always in $[0, \pi]$, they are unambiguously determined by their cosines; thus is usually enough to know the direction cosines. It can be checked that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1, \quad \vec{u} = |\vec{u}| \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

so the vector $\hat{u} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$ of direction cosines is the unit vector in the direction of $\vec{u}$.

Projections. The vector projection (or just projection) of $\vec{v}$ onto $\vec{u}$ is the vector that is parallel to $\vec{u}$ (a scalar multiple of $\vec{u}$) and with the difference between it and $\vec{v}$ being perpendicular to $\vec{u}$. Calling this vector $\vec{p}$, these conditions are that $\vec{p} = c\vec{u}$ and $(\vec{v} - \vec{p}) \cdot \vec{u} = 0$.

These conditions have a unique solution, denoted $\text{proj}_{\vec{u}}\vec{v}$:

$$\text{proj}_{\vec{u}}\vec{v} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|^2}\vec{u}$$

The scalar projection of $\vec{v}$ onto $\vec{u}$ is the "signed magnitude" of this: the magnitude with sign plus or minus according to whether the projection goes in the same or opposite direction as $\vec{u}$.

This is also called the component of $\vec{v}$ in direction $\vec{u}$, and denoted

$$\text{comp}_{\vec{u}}\vec{v} = \pm|\text{proj}_{\vec{u}}\vec{v}| = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|} = |\vec{v}|\cos\theta, \quad \theta \text{ the angle between } \vec{u} \text{ and } \vec{v}.$$

[4]openstax.org/books/calculus-volume-3/pages/2-3-the-dot-product

See Examples 2.27 and 2.28 in OSC3 Section 2.3[5].

Study Guide. Study Section 2.3 of Calculus Volume 3[6]; in particular:

- The Definitions and Theorems.
- Examples 2.21–2.25, 2.27 and 2.28, and the Checkpoints following each. (Examples 2.29 and 2.30 also show connections to Physics.)
- One or several exercises from each of the following ranges: One or several exercises from each of the ranges 123–126, 131–134, 135–140, 141–144, 147–148, 149–150, 161–164, 167–170, 171–172.

2.4 The Cross Product (a.k.a. Vector Product)

Revised on March 26.

References.

- OpenStax Calculus Volume 3, Section 2.4[1].
- Calculus, Early Transcendentals by Stewart, Section 12.4.

Introduction. There is a special situation in three dimensions that any two non-parallel vectors $\vec{u}$ and $\vec{v}$ determine a plane through the origin, in that all the combinations $a\vec{u} + b\vec{v}$ describe points in a plane, and it is then interesting to find the unique direction perpendicular to that plane. This is done by a special product $\vec{w} = \vec{u} \times \vec{v}$ to be defined soon.

The magnitude of this product is also significant: it will give the area of the triangle determined by the factors $\vec{u}$ and $\vec{v}$ as $\|\vec{u} \times \vec{v}\|/2$. That is, for the points U and V in $\mathbb{R}^3$ such that $\vec{u} = \langle OU \rangle$ and $\vec{v} = \langle OV \rangle$, the triangle $\triangle OUV$ has area $\|\vec{u} \times \vec{v}\|/2$. Note that this means that the parallelogram with corners given by $\vec{0}$, $\vec{u}$, $\vec{v}$ and $\vec{u} + \vec{v}$ has area $\|\vec{u} \times \vec{v}\|$. This will be useful in calcuating areas and volumes, and describing magnetic and torque forces.

The final detail missing is the direction of this product, since for any vector $\vec{w}$ meeting the above conditions, its negation does also. That choice will be resolved by appealing to the ”right-handedness” discussed with 3D coordinate systems, which will be sufficient to determine the value of this new product uniquely: it will be required that the triple $\vec{u}$–$\vec{v}$–$\vec{w}$ be right-handed, in the way that, for example, the triple $\hat{\imath}$–$\hat{\jmath}$–$\hat{k}$ is.

As with the dot product, there is an algebraic formula that gives the product vector; this time, it will be possible to derive this formula from the above list of desired properties with the help of expecting it to be distributive over addition.

Much as with the dot product or scalar product, this is often called the cross product based solely on the notation, but also the vector product, due to its value being a vector.

A warning: this deviates from what you might expect in several ways: this ”product” is neither commutative nor associative.

2.4.1 Definition and a Derivation

The cross product can be defined as

[5]openstax.org/books/calculus-volume-3/pages/2-3-the-dot-product
[6]openstax.org/books/calculus-volume-3/pages/2-3-the-dot-product
[1]openstax.org/books/calculus-volume-3/pages/2-4-the-cross-product

Definition 2.4.1

$$\vec{u} \times \vec{v} = \langle u_1, u_2, u_3 \rangle \times \langle v_1, v_2, v_3 \rangle = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle \tag{2.4.1}$$

This is also known as the vector product, for contrast to the scalar product as in Definition 2.3.1, p. 17. ♢

See Example 31 in OSC3 Section 2.4[2].

One way to see why this has to be the desired formula is to start with the above geometric requirements, see what they say about products of the three standard basis vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$, and then combine by assuming distributivity (which can then be verified in hind-sight.)

The first simple step—which does most of the work—is to see that

$$\hat{\imath} \times \hat{\jmath} = \hat{k} :$$

a To be perpendicular to $\hat{\imath}$ and $\hat{\jmath}$, the product must be a multiple of $\hat{k}$.

b The parallelogram formed by $\hat{\imath}$ and $\hat{\jmath}$ is a square of area 1, so the product has length one, and so is $\pm\hat{k}$.

c The right hand rule then forces the result to be $\hat{k}$, as claimed.

Similarly we get

$$\hat{\jmath} \times \hat{k} = \hat{\imath}, \quad \hat{k} \times \hat{\imath} = \hat{\jmath}$$

The second key step is reversing the order of the factors; right-handedness forces negations:

$$\hat{\jmath} \times \hat{\imath} = -\hat{k}, \quad \hat{k} \times \hat{\jmath} = -\hat{\imath}, \quad \hat{\imath} \times \hat{k} = -\hat{\jmath}$$

So commutativity fails: in fact the pattern of negation seen here is general: the cross-product is anti-commutative.

See Examples 32 and 33 in OSC3 Section 2.4[3].

Finally, what about parallel vectors? They determine a ”parallelogram” of area zero, so

$$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}.$$

Exercise 2.4.2 Assuming that this product is distributive over addition, expand

$$\vec{u} \times \vec{v} = (u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}) \times (v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k})$$

into a sum of nine pieces and use the above nine basic products to verify Equation (2.4.1).

2.4.2 Determinants

To complete the justification of this definition, and for other purposes, it is convenient to introduce the concept of a determinant from linear algebra.

A determinant of order 2 is defined by

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1. \tag{2.4.2}$$

A determinant of order 3 is defined using this as

$$\begin{aligned} \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} &= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}, \\ &= u_1 v_2 w_3 - u_1 v_3 w_2 - u_2 v_1 w_3 + u_2 v_3 w_1 + u_3 v_1 w_2 - u_3 v_2 w_1. \end{aligned} \tag{2.4.3}$$

[2]openstax.org/books/calculus-volume-3/pages/2-4-the-cross-product
[3]openstax.org/books/calculus-volume-3/pages/2-4-the-cross-product

The factors u_i multiplying each order 2 determinant come from the top row; the coefficient of each is the determinant of what you get when you delete from the 3x3 array of numbers the row and column that contain the factor u_i; the signs alternate. (This pattern also describes the order 2 determinant, but then all that is left after deleting one row and one column is a single number.)

See Example 36 in OSC3 Section 2.4[4].

The cross product as a determinant. The coefficients in the cross product are given by order two determinants:

$$\vec{u} \times \vec{v} = \hat{\imath} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}, \tag{2.4.4}$$

This is like the above order 3 determinant except with the components of $\vec{u}$ changed to standard basis vectors, each v_i replaced by u_i and each w_i replaced by v_i. Thus

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{2.4.5}$$

where we allow vectors to appear where previously we had numbers.

See Example 37 in OSC3 Section 2.4[5].

Theorem 2.4.3 The cross product is distributive over addition:

$$(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$$

Proof. This is shown by inserting the definition in all three places and simplifying. ■

Exercise 2.4.4 Show that for any vector $\vec{u}$, $\vec{u} \times \vec{u} = \vec{0}$.
(Think about why this makes inverses impossible with the vector product.)

Exercise 2.4.5 Use the above fact to show that the cross product is anti-commutative: $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$Expand $(\vec{u} + \vec{v}) \times (\vec{u} + \vec{v})$

Theorem 2.4.6 The vector $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

Proof. This is shown by computing $(\vec{u} \times \vec{v}) \cdot \vec{u}$ and simplifying until nothing is left; likewise for $(\vec{u} \times \vec{v}) \cdot \vec{v}$. ■

See Examples 38 and 43 in OSC3 Section 2.4[6].

The Geometrical Meaning of the Cross Product.
Theorem 2.4.7 The Magnitude of the Cross Product. With θ the angle between vectors $\vec{u}$ and $\vec{v}$, their cross product has magnitude

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$$

Proof. See OSC3, Section 2.4[7] ■

Thus, just as $\vec{u} \cdot \vec{v} = 0$ test for two vectors being orthogonal, we have:

Corollary 2.4.8 A Test for Parallel Vectors. Two vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if their cross product vanishes: $\vec{u} \times \vec{v} = \vec{0}$. (This includes the case of them pointing in opposite directions, sometimes called "anti-parallel".)

[4] openstax.org/books/calculus-volume-3/pages/2-4-the-cross-product
[5] openstax.org/books/calculus-volume-3/pages/2-4-the-cross-product
[6] openstax.org/books/calculus-volume-3/pages/2-4-the-cross-product
[7] openstax.org/books/calculus-volume-3/pages/2-4-the-cross-product

2.4.3 Computing Some Areas and Volumes

The Area of a Parallelogram. The area of a parallelogram with two sides given by vectors $\vec{u}$ and $\vec{v}$ is twice the area of the corresponding triangle, so is equal to the length of one side times the perpendicular distance from that side to the other vertex. With the angle between them being θ, that distance is the length of the other side times $\sin\theta$. Combined, the area is the product of the two side lengths and $\sin\theta$: $\|\vec{u}\|\|\vec{v}\|\sin\theta$, and as seen above, this is

$$A = \|\vec{u}\times\vec{v}\|. \tag{2.4.6}$$

If these two vectors lie in the x–y plane, using Equation (??) gives this as

$$A = \left\|\hat{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\right\| = \left|\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right| = |u_1v_2 - u_2v_1|. \tag{2.4.7}$$

See Example 39 in OSC3[8].

The Volume of a Parallelepiped and the Scalar Triple Product. Any three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ determine a parallelopiped with one vertex the origin (now $\vec{0}$), and three edges leaving it along those three vectors to the adjacent vertices. Taking one face to be the parallelogram with vertices $\vec{0}$, $\vec{u}$, $\vec{v}$ and $\vec{u}+\vec{v}$, its volume is the product of the area of this face (which is $\|\vec{u}\times\vec{v}\|$) times the perpendicular distance to the remaining vertex $\vec{w}$. That distance is $\|\vec{w}\|\sin\phi$ where ϕ is the angle between $\vec{w}$ and the $\vec{u}$–$\vec{v}$ plane.

However another way to describe that angle is as the complement of the angle θ between $\vec{w}$ and a vector perpendicular to that plane, [not quite! see the note below] and a vector in that direction is $\vec{u}\times\vec{v}$, so that $\sin\phi = \cos\theta$. Thus, the volume is "area of the u-v plane times $\|\vec{w}\|$ times $\cos\theta$", which is $\|\vec{u}\times\vec{v}\|\|\vec{w}\|\cos\theta$", and from the geometric formula for the dot product in Section 2.3, p. 17, this gives the volume as

$$V = |(\vec{u}\times\vec{v})\cdot\vec{w}|$$

But why is the absolute value needed here? Because we cheated: the angles ϕ and θ are only complementary if $\vec{w}$ is on the same side of the u-v plane as $\vec{u}\times\vec{v}$ which is the case that $\vec{u}$–$\vec{v}$–$\vec{w}$ are in right-handed arrangement and $(\vec{u}\times\vec{v})\cdot\vec{w} \geq 0$ If instead they are "left-handed", this quantity is negative, and the absolute value is needed.

See Example 41 in OSC3[9].

Geometrically, we can swap the roles of the three vectors, and relatedly, one can verify that

$$(\vec{u}\times\vec{v})\cdot\vec{w} = \vec{u}\cdot(\vec{v}\times\vec{w}) \tag{2.4.8}$$

This is called the scalar triple product, and using the formula for the cross product in terms of determinants gives

$$(\vec{u}\times\vec{v})\cdot\vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{2.4.9}$$

See Examples 40 and 42 in OSC3[10].

2.4.4 Algebraic Properties of the Cross Product

To summarize,

i $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$ Anti-commutativity

ii $(\vec{u}+\vec{v})\times\vec{w} = (\vec{u}\times\vec{w}) + (\vec{v}\times\vec{w})$ Distributivity

[8] openstax.org/books/calculus-volume-3/pages/2-4-the-cross-product
[9] openstax.org/books/calculus-volume-3/pages/2-4-the-cross-product
[10] openstax.org/books/calculus-volume-3/pages/2-4-the-cross-product

iii $(a\vec{u}) \times \vec{v} = \vec{u} \times (a\vec{v}) = a(\vec{u} \times \vec{v})$ Associativity with scalar maltiplication

iv $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$:

v $\vec{u} \times \vec{u} = \vec{0}$

vi $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$

See Example 34 in OSC3[11].

Study Guide. Study Section 2.4 of Calculus Volume 3[12]; in particular:

- The definition of the cross product and all the Theorems.
- The description of the right-hand rule.
- Examples 31–43 and the Checkpoints following each. (Example 44 shows an application to physics.)
- One or several exercises from each of the following ranges: 183–186 (part(a) is enough), 189–192, 197–198, 199–200.

2.5 Equations of Lines and Planes in Space

(Slightly revised on February 4.)

References.

- OpenStax Calculus Volume 3, Section 2.5[1].
- Calculus, Early Transcendentals by Stewart, Section 12.5.

Introduction. We have seen that dot and cross products have something to do with parallel lines and planes respectively: we now use these properties to describe lines and planes in three dimensional space. It will be convenient to describe a point P in $\mathbb{R}^3$ by giving its position vector $\vec{p} = \overrightarrow{OP}$, and then to describe lines and planes as collections of position vectors.

Lines in Vector Terms. We can characterize a line L by the property that all vectors between any two points on the line are parallel, and so are multiples of some vector $\vec{v}$, which has a role like the slope of a line in the plane. Starting with any one point $\vec{r}_0$ on the line, the vector $\vec{r} - \vec{r}_0$ from it to any point $\vec{r}$ on the line is a multiple $t\vec{v}$ of $\vec{v}$, so L consists of the points

$$\vec{r} = \vec{r}_0 + t\vec{v}, \quad \text{any real} . \tag{2.5.1}$$

This can be written out in terms of coordinates as three ”scalar” equations: with $\vec{v} = \langle a, b, c\rangle$, $\vec{r}_0 = \langle x_0, y_0, z_0\rangle$, $\vec{r} = \langle x, y, z\rangle$,

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct \tag{2.5.2}$$

which is the general parametric equation for a line in space. These forms are not unique, since we could replace r_0 by any other point on the line, and replace $\vec{v}$ by any non-zero scalar multiple of it. The components a, b and c together are called the direction numbers of line L; any other triple of numbers proportional to these also serve as direction numbers of L.

[11] openstax.org/books/calculus-volume-3/pages/2-4-the-cross-product
[12] openstax.org/books/calculus-volume-3/pages/2-4-the-cross-product
[1] openstax.org/books/calculus-volume-3/pages/2-5-equations-of-lines-and-planes-in-space

The Symmetric Equations for a Line. Each equation in (2.5.2) can be solved for t, and equating the three results gives the symmetric equations of L:

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}. \tag{2.5.3}$$

xml:id='lineparametric' If any of the direction numbers is zero, the equations become even simpler: for example if $a = 0$,

$$x = x_0, \quad \frac{y-y_0}{b} = \frac{z-z_0}{c}. \tag{2.5.4}$$

(But at least one direction number must be non-zero!)

See Example 45 in OSC3 Section 2.5[2].

Planes. The directions of motion possible within a plane can be described two ways: by a single normal vector $\vec{n}$ perpendicular to all vectors $\overrightarrow{PQ}$ joining two points in the plane, or by giving two non-parallel vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ given by three points P, Q and R in the plane. Given a normal vector $\vec{n}$ and point $\vec{r}_0$ in the plane, any vector $\vec{r} - \vec{r}_0$ within the plane is perpendicular to $\vec{n}$, so the plane is given by all solutions $\vec{r}$ of

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0, \quad \text{or} \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0 \tag{2.5.5}$$

Either of these is called a vector equation of the plane.

The scalar equation for a plane. With $\vec{n} = \langle a, b, c \rangle$ [careful: we are recycling those three letters!] and as before $\vec{r} = \langle x, y, z \rangle$,

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0. \tag{2.5.6}$$

This is the scalar equation of the plane through point $P(x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$. Expanding gives the alternate form

$$ax + by + cz + d = 0, \text{ where } d = -(ax_0 + by_0 + cz_0). \tag{2.5.7}$$

Any choice of constants a, b, c (not all zero) and d given a plane. Equation (2.5.7) is the most general linear equation in three unknowns x, y and z, so we now know that the solution of such an equation is always a plane (not a line as the name might suggest, and as is true for a linear equation in two unknowns.)

Three points (usually) determine a plane. Given three points P, Q, and R that do not line on a line, there is a unique plane containing all of them. The two vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel to the plane, so their cross-product $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$ is normal to the plane. Then using $\vec{r_0} = \overrightarrow{OR}$ as a point on the plane give the equation $\vec{n} \cdot (\vec{r} - \vec{r_0})$ as above. \pause This works unless $\vec{n} = \vec{0}$: that is the case where $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel, meaning that the three points are collinear.

See Example 49 in OSC3 Section 2.5[3].

The angle between two planes, and distance to a plane. Two planes are parallel (and so either do not intersect or are identical) if their normals $\vec{n}_1$ and $\vec{n}_2$ are parallel. Otherwise, they intersect along a line, and the angle between them is the acute angle between their normals: the smallest angle θ with

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\| \|\vec{n}_2\|}. \tag{2.5.8}$$

Question 2.5.1
(a) Why is the absolute value here?
(b) What would the angle describe if the absolute value were omitted?

[2] openstax.org/books/calculus-volume-3/pages/2-5-equations-of-lines-and-planes-in-space
[3] openstax.org/books/calculus-volume-3/pages/2-5-equations-of-lines-and-planes-in-space

□

See Example 53 in OSC3 Section 2.5[4].

Also, the shortest distance from a point $P(x_1, y_1, z_1)$ to a point in the plane $ax + by + cz + d = 0$ thrpough a point $Q(x_0, y_0, z_0)$ with normal $\vec{n}$ is

$$D = \frac{\overrightarrow{PQ} \cdot \vec{n}}{\|\vec{n}\|} = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{2.5.9}$$

and for Q, the closest point in the plane, $\overrightarrow{PQ}$ is parallel to the normal vector $\vec{n} = \overrightarrow{a, b, c}$.

See Example 51 in OSC3 Section 2.5[5].

Note that if we use unit normal vectors $\hat{n} = \langle a, b, c \rangle$ and so on, the angle and distance formulas simplify to

$$\cos\theta = |\hat{n}_1 \cdot \hat{n}_2|. \tag{2.5.10}$$

and

$$D = \overrightarrow{PQ} \cdot \hat{n} = |ax_1 + by_1 + cz_1 + d| \tag{2.5.11}$$

Study Guide. Study Section 2.5 of Calculus Volume 3[6]; in particular

- All the Definitions, Theorems, Examples and Checkpoints.
- One or several exercises from each of the following ranges: 243–246; 251–254, 267–270, 271–274 [parts (a) and (b) are enough], 281–282, 289–290.

2.6 Quadric Surfaces: Omitted for now

References.

- OpenStax Calculus Volume 3, Section 2.6[1].
- Calculus, Early Transcendentals by Stewart, Section 12.6.

2.7 Cylindrical and Spherical Coordinates

Revised on March 15.

References.

- OpenStax Calculus Volume 3, Section 2.7[1].
- Calculus, Early Transcendentals by Stewart, Sections 15.7 and 15.8.

Introduction. Coordinate systems for three dimensional space that are convenient for working with domains and functions that have either cylindrical symmetry (depending only on distance from a certain line), or spherical symmetry (depending only on distance from a certain point).

[4] openstax.org/books/calculus-volume-3/pages/2-5-equations-of-lines-and-planes-in-space
[5] openstax.org/books/calculus-volume-3/pages/2-5-equations-of-lines-and-planes-in-space
[6] openstax.org/books/calculus-volume-3/pages/2-5-equations-of-lines-and-planes-in-space
[1] openstax.org/books/calculus-volume-3/pages/2-6-quadric-surfaces
[1] openstax.org/books/calculus-volume-3/pages/2-7-cylindrical-and-spherical-coordinates

2.7.1 Cylindrical Coordinates

One way to describe the location of a point in space is to gives its height z above [or below] the x-y plane as with cartesian coordinates, but then describe the location of the point below [or above] it in that plane with plane polar coordinates, r and θ.

This gives cylindrical coordinates (r, θ, z), (sometimes denoted $(r, \theta, z)_c$ to distinguish from cartesian coordinates), related to cartesian coordinates (x, y, z) by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad [z = z] \tag{2.7.1}$$

To convert from rectangular to cylindrical, one uses

$$r^2 = x^2 + y^2 \text{ (so } r = \sqrt{x^2 + y^2}\text{)}, \quad \tan\theta = y/x \text{ (or } \cos\theta = x/r \text{ or } \sin\theta = y/r\text{)}$$

A unique choice of θ is then determined by using the smallest non-negative value, so that $0 \leq \theta < 2\pi$ (Any such value of θ is allowed on the z-axis $x = y = 0$.)

These coordinates are "cylindrical" because a circular cylinder of radius a around the z-axis has the simple equation $r = a$. Essentially, this is describing the (x, y) part of the location with plane polar coordinates.

See Examples 60 to 62 in OSC3 Section 2.7[2].

2.7.2 Spherical Coordinates

Just as plane polar and cylindrical coordinates are based around horizontal distance r from $x = y = 0$, spherical coordinates start with the distance ρ from the origin $O(0, 0, 0)$ to a point $P(x, y, z)$:

$$\rho^2 = x^2 + y^2 + z^2 \text{ so } \rho = \sqrt{x^2 + y^2 + z^2}. \tag{2.7.2}$$

Thus a sphere of radius a center the original has the simple equation $\rho = a$.

"Longitude" θ and "Latitude" ϕ. To complete the coordinates, we need to describe the location of the point P on the sphere of radius ρ with two numbers.

The familiar case of this is using angles of latitude and longitude to describe position on the earth's surface. Both are done a bit differently here, to avoid negative values.

First, we consider the circle of latitude containing the point: the circle of center $Q(0, 0, z)$ on the z axis at the same "height" as P and passing through P, so that its radius is $r = \sqrt{x^2 + y^2}$.

Position on this circle can be described with polar coordinates, using the angle θ measured from the eastern point $E(r, 0, z)$ of this circle in the direction of positive y, and as above, using values $\theta \in [0, 2\pi)$.

The r and θ here are the same as with cylindrical coordinates, so we still have

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{2.7.3}$$

The position of this circle of latitude on the sphere is specified with a "latitude angle", measured from the North pole $N(0, 0, \rho)$

That is, ϕ is the angle $\angle NOP$, which we can take in the range $0 \leq \phi \leq \pi$, with $\phi = 0$ the North Pole, $\phi = \pi$ the South Pole $S(0, 0, -\rho)$.

Note that the range only has to cover a half circle, not returning to the North pole at $\phi = 2\pi$.

Another way to think of this is that one looks at the half plane of points with a given longitude θ, with cartesian coordinates (z, r) in the place of (x, y) and describe this with a second set of polar coordinates, so now measuring angles from the positive z-axis. Thus the radius is ρ, the polar angle is ϕ, and

$$z = \rho\cos\phi, \quad r = \rho\sin\phi \tag{2.7.4}$$

[2]openstax.org/books/calculus-volume-3/pages/2-7-cylindrical-and-spherical-coordinates

One Difference from Geography. This ”top-down” measurement is used instead of measuring from the equator as in geography; that keeps ϕ non-negative, as was also done above for θ (But arguably it would be slightly better to measure up from the south pole, for reasons that we will see in Section 5.5.)

Equations Connecting these Coordinate Systems. Combining Equations (2.7.3) and (2.7.2) gives

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi \tag{2.7.5}$$

relating spherical coordinates (ρ, θ, ϕ) to cartesian coordinates (x, y, z). (Spherical coordinates are sometimes denoted $(\rho, \theta, \phi)_s$ to distinguish from cylindrical or cartesian coordinates.)

See Examples 63 to 65 in OSC3 Section 2.7[3].

Study Guide. Study Section 2.7 of Calculus Volume 3[4]; in particular

- All the Definitions and Theorems.
- Examples 60–65 and 67, and the Checkpoints following each.
- One or several exercises from each of the following ranges: 363–366, 367–370, 371–378 (graphing not necessary), 379–384, 385–388, 389–392, 393–398, 399–402, 407–410.

Chapter Review. When reviewing this chapter, also look at the end of chapter review material in OpenStax Calculus Volume 3, including Key Terms[5], Key Equations[6] and Key Concepts[7].

[3]openstax.org/books/calculus-volume-3/pages/2-7-cylindrical-and-spherical-coordinates
[4]openstax.org/books/calculus-volume-3/pages/2-7-cylindrical-and-spherical-coordinates
[5]openstax.org/books/calculus-volume-3/pages/2-key-terms
[6]openstax.org/books/calculus-volume-3/pages/2-key-equations
[7]openstax.org/books/calculus-volume-3/pages/2-key-concepts

Chapter 3

Vector-valued Functions

References.

- OpenStax Calculus Volume 3, Chapter 3.[1]
- Calculus, Early Transcendentals by Stewart, Chapter 13.

3.1 Vector-Valued Functions and Space Curves

References.

- OpenStax Calculus Volume 3, Section 3.1[1].
- Calculus, Early Transcendentals by Stewart, Section 13.1.

Introduction. Since a position in space can be described by a vector $\vec{r} = \langle x, y, z \rangle$ and position can be a function of time, it is natural to consider vector functions like

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\vec{k},$$

functions whose value is a vector (or loosely, a vector each of whose components is a function). Many familiar concepts of calculus like limits, continuity, derivatives and integrals extend simply to such vector functions, and they also have the nice geometrical significance of describing curves in space.

See Examples 3.1 and 3.2 in OSC3 Section 3.1[2].

Limits of Vector Functions. A limit of a vector function $\vec{r} = \langle f(t), g(t), h(t) \rangle$ is built up from limits of its components:

$$\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

This limit thus exists if and only if all of the three component limits exist.

See Example 3.3 in OSC3 Section 3.1[3].

[1] openstax.org/books/calculus-volume-3/pages/3-introduction
[1] openstax.org/books/calculus-volume-3/pages/3-1-vector-valued-functions-and-space-curves
[2] openstax.org/books/calculus-volume-3/pages/3-1-vector-valued-functions-and-space-curves
[3] openstax.org/books/calculus-volume-3/pages/3-1-vector-valued-functions-and-space-curves

Continuity. With this notion of a limit, it makes sense to say that vector function $\vec{r}(t)$ is continuous at a if the limit exists there and equals the value

$$\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$$

which is true if and only if all of the component functions are continuous at a.

Space Curves. A continuous vector valued function describes a space curve. A space curve C can be described as the set of points in space whose position vectors are given by the values of a function $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ for values of t in some interval I. The interval can be finite like $I = [a, b]$ or infinite such as $I = (-\infty, \infty)$.

The component equations

$$x = f(t), y = g(t), z = h(t)$$

are parametric equations for C, with t the parameter.

The Visual Meaning of Continuity. Continuity has a familiar geometrical meaning: a vector function $\vec{r}(t)$ is continuous if the space curve it describes has no breaks.

Note: when drawing space curves, arrowheads are typically used on the curve to indicate the direction of motion as the parameter increases.

Study Guide. Study Section 3.1 of Calculus Volume 3[4]; in particular

- All the Definitions, Theorems, Examples and Checkpoints.
- One or several exercises from each of the following ranges: 1, 2, 3, 5–7, 8–13, 15–17, 22–26, 27–32, 33, 34, 35 and 36 (this last four go together).

For the graphing examples and exercises, I suggest trying the (new) Desmos 3D graphing tool[5].

3.2 Calculus of Vector-Valued Functions

References.

- OpenStax Calculus Volume 3, Section 3.2[1].
- Calculus, Early Transcendentals by Stewart, Section 13.2.

Introduction. Here, finally we see some calculus. Much of it is simply extending to 3D what was seen in Calculus 2 for curves in the plane; for example, see OpenStax Calculus Volume 3, Section 1.2[2] and the notes for that section 1.2, p. 4.

Derivatives of Vector-Valued Functions. We can build derivatives of vector functions from derivative of components, but the definition can also be done from first principles, with difference quotients:

Definition 3.2.1 Derivative of $\vec{r}$. The derivative $\vec{r}\,'$ of a vector function of variable t is given by

$$\frac{d\vec{r}}{dt} = \vec{r}\,' = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}. \tag{3.2.1}$$

◊

[4] openstax.org/books/calculus-volume-3/pages/3-1-vector-valued-functions-and-space-curves
[5] www.desmos.com/3d/
[1] openstax.org/books/calculus-volume-3/pages/3-2-calculus-of-vector-valued-functions
[2] openstax.org/books/calculus-volume-3/pages/1-2-calculus-of-parametric-curves

It can be checked that for $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, the derivative (if it exists) is the vector of derivatives of the components:

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\langle f(t), g(t), h(t)\rangle = \langle f'(t), g'(t), h'(t)\rangle\,. \tag{3.2.2}$$

Thus the derivative exists if and only if all of the component derivatives exists.

See Examples 3.4 and 3.5 in OSC3 Section 3.2[3].

Properties of the Derivative of Vector-Valued Functions. The familiar differentiation rules for sums, products and compositions have natural counterparts for vector functions:

Theorem 3.2.2 For $\vec{u}$ and $\vec{v}$ differentiable vector functions, f a differentiable scalar (real-valued) function, and c a scalar constant,

i $\dfrac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t)$

ii $\dfrac{d}{dt}[c\vec{u}(t)] = c\vec{u}'(t)$

iii $\dfrac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$

iv $\dfrac{d}{dt}[\vec{u}(t)\cdot\vec{v}(t)] = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$

v $\dfrac{d}{dt}[\vec{u}(t)\times\vec{v}(t)] = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$

vi $\dfrac{d}{dt}[\vec{u}(f(t))] = \vec{u}'(f(t))f'(t)$

Note: as always with the cross product, the order matters in Item v, p. 30.

See Example 3.6 in OSC3 Section 3.2[4].

Tangent Vectors and the Principle Unit Tangent Vector. For any value of $t = a$, where the derivative vector $\vec{r}'(a)$ exists and is non-zero, it is tangent to the space curve C at point P with position vector $\vec{r}(a)$, and so is called a tangent vector to C at P. The line through P with this tangent vector is the tangent line to C at P with equation

$$\vec{L}(t) = \vec{r}(a) + t\,\vec{r}'(a)$$

It will often be useful to consider the principle unit tangent vector

$$\widehat{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}.$$

and then the tangent line at the point P can be written as

$$\vec{L}(s) = \vec{r}(a) + s\,\widehat{T}(a)$$

Here the parameter s is used because it corresponds to arc-length along this line, as will be discussed in Section 3.3, p. 31.

See Example 3.7 in OSC3 Section 3.2[5].

The existence of a tangent direction given by $\vec{r}'(t)$ and thus of this unit tangent vector is what guarantees that the curve has no ”corners”, as with the graph of a differentiable function, so this important ”niceness” condition has a name:

Definition 3.2.3 A space curve is smooth if it is given by $\vec{r}(t)$ on interval I with both $\vec{r}$ and $\vec{r}'$ continuous, and with $\vec{r}' \neq \vec{0}$ except possibly at the endpoints of I. This is equivalent to the existence of the unit tangent

[3]openstax.org/books/calculus-volume-3/pages/3-2-calculus-of-vector-valued-functions
[4]openstax.org/books/calculus-volume-3/pages/3-2-calculus-of-vector-valued-functions
[5]openstax.org/books/calculus-volume-3/pages/3-2-calculus-of-vector-valued-functions

vector $\widehat{T}(t) = \vec{r}'(t)/\|\vec{r}'(t)\|$.
If the derivative is zero at a finite number of points, the curve is called piecewise smooth. ◊

Integrals of Vector-Valued Functions. Like derivatives, definite integrals of vector functions can be built from first principles with Riemann sums, and one gets the predictable result in terms of integrals of components: For $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$,

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$$

Indefinite integrals work likewise: the one new detail to note is that there is a constant of integration for each component:

$$\int \vec{r}(t)\,dt = \left\langle \int f(t)\,dt + C_1, \int g(t)\,dt + C_2, \int h(t)\,dt + C_3 \right\rangle$$

Equivalently the constant of integration is a vector $\vec{C} = \langle C_1, C_2, C_3\rangle$.

See Example 3.8 in OSC3 Section 3.2[6].

Study Guide. Study Section 3.2 of Calculus Volume 3[7]; in particular

- All the Definitions, Theorems, Examples and Checkpoints.
- One or several exercises from each of the following ranges: 41–50, 51–54, 55–58, 59–61, 62, 63, 63, 64.

3.3 Arc Length and Curvature

Revised on April 2.

References.

- OpenStax Calculus Volume 3, Section 3.3[1].
- Calculus, Early Transcendentals by Stewart, Section 13.3.

3.3.1 Arc Length

The arc length of parameterized curves in the plane was computed in Section 1.2, p. 4 as

$$L = \int_{t=\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}dt, \tag{3.3.1}$$

(see Equation (1.2.6)) and the same procedure can be applied to compute the length of a smooth curve in space as

$$\begin{aligned} L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \\ &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt. \end{aligned} \tag{3.3.2}$$

This has the more compact form $L = \int_a^b \|\vec{r}'(t)\| dt$, valid of two or three dimensions.

[6] openstax.org/books/calculus-volume-3/pages/3-2-calculus-of-vector-valued-functions
[7] openstax.org/books/calculus-volume-3/pages/3-2-calculus-of-vector-valued-functions
[1] openstax.org/books/calculus-volume-3/pages/3-3-arc-length-and-curvature

See Example 9 in OSC3 Section 3.3[2].

Other Parametrizations and the Arc Length Function. A curve can be parameterized in many ways. For example

$$\vec{r}(t) = \left\langle t, t^2, t^3 \right\rangle,\ 1 \leq t \leq 2$$

describes the same curve as

$$\vec{r}(t) = \left\langle e^u, e^{2u}, e^{3u} \right\rangle,\ 0 \leq u \leq \ln 2.$$

One particular useful and elegant parametrization is based on arc length, starting with the arc length function s, given by

$$s(t) = \int_a^t \|\vec{r}'(t)\| dt \tag{3.3.3}$$

which is the length of the part of the curve from the initial point $\vec{r}(a)$ to the point $\vec{r}(t)$. This is equivalent to $\dfrac{ds}{dt} = \|\vec{r}'(t)\|$. If a curve is parameterized as $\vec{r}(s)$ with arclength as its parameter then $s(t) = t - a$.

Parametrizing with Arc Length. Any smooth curve can be expressed with arc length as parameter: $ds/dt = \|\vec{r}'(t)\| > 0$, so $s(t)$ is increasing and has an inverse, $t(s)$, and this inverse gives $\vec{r} = \vec{r}(t(s))$ as the arc length parametrization. However it is often not possible to get an explicit formula for $t(s)$ or for the arc length parametrization.

See Example 10 in OSC3 Section 3.3[3].

3.3.2 Curvature

A smooth curve C described by vector function $\vec{r}$ has a tangent direction at each point (except possibly at its end points) given by the unit tangent vector.

$$\widehat{T}(t) = \frac{d\vec{r}/dt}{\|d\vec{r}/dt\|} = \frac{d\vec{r}/dt}{ds/dt} = \frac{d\vec{r}}{ds} \tag{3.3.4}$$

Change in $\widehat{T}$ as one moves along the curve measures change in direction, so its rate of change measures curvature:

Definition 3.3.1 The curvature of a curve with unit tangent $\widehat{T}(t)$ is

$$\kappa = \kappa(t) = \left\| \frac{d\widehat{T}}{ds} \right\|, = \left\| \frac{d\widehat{T}/dt}{ds/dt} \right\| = \frac{\|\widehat{T}'(t)\|}{\|\vec{r}'(t)\|}.$$

◊

Exercise 3.3.2 Show that the curvature of a circle of radius a is $1/a$ at every point.

See Example 11 in OSC3 Section 3.3[4].

Theorem 3.3.3 The curvature of the curve given by $\vec{r}(t)$ is

$$\kappa(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}, \quad = \|\vec{r}'(s) \times \vec{r}''(s)\|, \tag{3.3.5}$$

where s as usual refers to the arc length parametrization.

[2]openstax.org/books/calculus-volume-3/pages/3-3-arc-length-and-curvature
[3]openstax.org/books/calculus-volume-3/pages/3-3-arc-length-and-curvature
[4]openstax.org/books/calculus-volume-3/pages/3-3-arc-length-and-curvature

Computing the Curvature of Plane Curves. For the special case of a curve in the plane given by $y = f(x)$, the parametric form $\vec{r} = \langle x, f(x), 0\rangle$ gives

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} \tag{3.3.6}$$

so there is a relation to the second derivative, but not as simple as you might have guessed.

Exercise 3.3.4 For
  a the parabola $y = x^2$ and
  b the semi-circle $y = \sqrt{1 - x^2}$,
compute both the curvature and y''.
Discuss why these results suggests that the curvature is a more reasonable geometrical measure than the second derivative.

3.3.3 Normal and Binormal Vectors, and the Osculating Plane

The Principle Unit Normal Vector. With arc-length parametization, $\kappa(s) = \|\vec{r}'(s) \times \vec{r}''(s)\|$ and $\vec{r}'(s) = \widehat{T}(s)$, so

$$\kappa(s) = \left\| \widehat{T}(s) \times \frac{d\widehat{T}}{ds}(s) \right\|.$$

The directions of these three vectors $\widehat{T}(s)$, $\widehat{T}'(s)$ and $\widehat{T}(s) \times \widehat{T}'(s)$ help describe the direction, curvature and "bending" of a curve in more detail.

Firstly, $\widehat{T}'$ is orthogonal (or normal) to $\widehat{T}$ and thus is normal to the curve (unless it is zero). It is natural to normalize it, giving the principal unit normal vector or unit normal.

$$\widehat{N}(t) = \frac{\widehat{T}'(t)}{\|\widehat{T}'(t)\|} \tag{3.3.7}$$

Note: In the special case that the curvature is zero a point, $\widehat{T}' = \vec{0}$ there, and so the unit normal vector and normal direction are undefined.

The Binormal Vector. In the plane, the tangent and normal vectors forrm a basis that can be used to describe all directions in the plane. In space ($\mathbb{R}^3$) instead, there is a whole plane normal to $\widehat{T}$ and thus to the curve, and a third dimension not cover by these two vectors. Then we can get a second normal direction, the binormal vector with

$$\widehat{B}(t) = \widehat{T}(t) \times \widehat{N}(t) \tag{3.3.8}$$

which is automatically a unit vector, normal to both $\widehat{T}(t)$ and $\widehat{N}(t)$.

See Example 12 in OSC3 Section 3.3[5].

Question 3.3.5 Why. □

Note: At a point where the curvature is zero, the binormal vector is also undefined.

The Normal and Osculating Planes of a Curve. At any point P given by $\vec{r}(t)$ on a smooth curve C, the three vectors $\widehat{T}$, $\widehat{N}$, and $\widehat{B}$ form a right-handed set of orthogonal directions, and in some sense specify a natural set of coordinates for looking at the world with P as the origin.

Two planes through the point P are important in this view:

- The plane through P with normal $\widehat{T}$ (and thus containing the two "normal" directions $\widehat{N}$ and $\widehat{B}$) is the normal plane of C at P.

[5] openstax.org/books/calculus-volume-3/pages/3-3-arc-length-and-curvature

- The plane through P containing directions $\widehat{T}$ and $\widehat{N}$, and so with normal $\widehat{B}$, is in some sense the plane that the curve is momentarily moving in, and is called the osculating plane of C at P. In particular, if the curve lines entirely in a plane, $\widehat{B}$ is always normal to that plane, and so the osculating plane is that plane.

Note: At a point where the curvature is zero, the osculating plane is undefined, but the normal plane makes sense at any point on any smooth curve.

See Example 13 in OSC3 Section 3.3[6].

Tangent Lines and Osculating Circles. If we look closely enough at a curve near a point it is approximated by a straight line, the tangent line, determined by $\vec{r}(t)$ and $\widehat{T}(t)$: geometrically, this is the straight line that best fits the curve near that point.

A more precise picture is given by noting the curvature too, and looking instead for the circle that best fits the curve at that point. This circle turns out to have radius $\rho = 1/\kappa$, lying in the osculating plane, to the side of the curve given by the principal normal direction $\widehat{N}(t)$.

Note: At a point where the curvature is zero, this circle effectively becomes a straight line: the tangent line.

Exercise 3.3.6 Find the osculating circle at the origin on the parabola $y = x^2$.

Exercise 3.3.7 Verify that when the curve is a circle, the osculating circle at each point of the curve is that same circle (as it should be!)

For a visualization of the three directions $\widehat{T}$, $\widehat{N}$ and $\widehat{B}$, and the related osculating circle, see this YouTube video:

Figure 3.3.8 The $\widehat{T}$-$\widehat{N}$-$\widehat{B}$ frame and osculating circle

A Relationship Between κ, $\widehat{T}$ and $\widehat{N}$. The formula for curvature in terms of arc-length parameter s is $\kappa(s) = \|\widehat{T}'(s)\|$, and $\widehat{N}(s) = \dfrac{\widehat{T}'(s)}{\left\|\widehat{T}'(s)\right\|}$, so

$$\frac{d\widehat{T}}{ds} = \kappa\widehat{N}. \tag{3.3.9}$$

A geometrical interpretation if this is that the direction of a curve is changing at a rate κ in the principal normal direction.

This perhaps helps to explain why over a short period of time, the curve lies roughly in the osculating plane.

[6] openstax.org/books/calculus-volume-3/pages/3-3-arc-length-and-curvature

Study Guide. Study Section 3.3 of Calculus Volume 3[7]; in particular

- All the Definitions, Theorems, Examples and Checkpoints.
- One or several exercises from each of the following ranges: 102–109, 110, 130, 133–134, and several from 113–126.

3.4 Motion in Space

References.

- OpenStax Calculus Volume 3, Section 3.4[1].
- Calculus, Early Transcendentals by Stewart, Section 13.4.

Introduction. Many of the geometrical ideas from the previous section relate to physical quantities like speed, velocity and acceleration, in the natural case where the value of a function $\vec{r}(t)$ is a position space and the value of its argument t is time.

Velocity, Speed and Acceleration.
Definition 3.4.1 Velocity is the rate of change of position with respect to time as already seen with scalar valued functions, but now velocity is a vector function,

$$\vec{v}(t) = \vec{r}'(t).$$

So the velocity is a tangent vector in the direction of motion. ♢

Definition 3.4.2 Speed is the magnitude of velocity or rate of change of position with respect to time:

$$v = \|\vec{v}(t)\| = \|\vec{r}'(t)\| = \frac{ds}{dt}.$$

♢

See Example 3.14 in OSC3 Section 3.4[2].

Note that arc length s has the natural physical meaning of distance traveled along the curve.

Definition 3.4.3 Acceleration is as always the rate of change of velocity with respect to time, again a vector quantity

$$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t).$$

♢

Force and Motion: Position from Acceleration. Newton's laws relate motion to the force acting on a body of mass m, with the force a vector $\vec{F}$ and determining acceleration through

$$\vec{F} = m\vec{a}.$$

If the force is known as a function of time, along with the position $\vec{r}_0$ and velocity $\vec{v}_0$ at one time t_0, the position at all times can be calculated:

$$\vec{a} = \vec{F}/m, \quad \vec{v}(t) = \vec{v}_0 + \int_{t_0}^{t} \vec{a}(u)du, \quad \vec{r}(t) = \vec{r}_0 + \int_{t_0}^{t} \vec{v}(u)du.$$

[7]openstax.org/books/calculus-volume-3/pages/3-3-arc-length-and-curvature
[1]openstax.org/books/calculus-volume-3/pages/3-4-motion-in-space
[2]openstax.org/books/calculus-volume-3/pages/3-4-motion-in-space

Tangential and Normal Components of Acceleration. With motion along a straight line, acceleration can be thought of as rate of change of speed. With motion in a plane or space, acceleration can also change direction, perhaps with no change is in speed.

These two effects of acceleration can be described in terms of the tangential and normal components of acceleration.

First, compute the speed $v = \|\vec{v}\|$, so that $\widehat{T} = \vec{v}/v$ and

$$\vec{v} = v\widehat{T}.$$

Then

$$\vec{a} = \vec{v}' = \left(v\widehat{T}\right)' = v'\widehat{T} + v\widehat{T}'.$$

Also $\kappa = \|\widehat{T}'\|/\|\vec{r}'\| = \|\widehat{T}'\|/v$, so

$$\|\widehat{T}'\| = \kappa v$$

and $\widehat{N} = T'/\|\widehat{T}'\|$ so

$$\widehat{T}' = \|\widehat{T}'\|\widehat{N} = \kappa v\widehat{N}.$$

All together,

$$\vec{a} = v'\widehat{T} + \kappa v^2\widehat{N}, = a_T\widehat{T} + a_N\widehat{N}. \tag{3.4.1}$$

Here $a_T = v'$ and $a_N = \kappa v^2$ are the tangential and normal components of the acceleration:

- the tangential component a_T measures change of speed;
- the normal component a_N measures change of direction.

Note that there is no "binormal component of acceleration": instantaneously, acceleration acts in the osculating plane. From another perspective, the osculating plane at any time is the plane determined by the current position, velocity and acceleration, in which the motion appears momentarily to lie.

Acceleration Components in Terms of Velocity and Acceleration Vectors. It is often convenient to express everything in terms of the basic derivatives of components of $\vec{r}(t)$, without use of intermediate quantities like $v'(t)$ and $\kappa(t)$ thus avoiding differentiation of square roots. First it can be checked that $\vec{v}\cdot\vec{a} = vv'$, so that

$$a_T = v' = \frac{\vec{v}\cdot\vec{a}}{v} = \frac{\vec{r}'\cdot\vec{r}''}{\|\vec{r}'\|}, \quad = \frac{d\vec{r}}{ds}\cdot\frac{d^2\vec{r}}{ds^2}.$$

Next, recalling from Theorem 3.3.3, p. 32 that $\kappa = \|\vec{r}'(t)\times\vec{r}''(t)\|/\|\vec{r}'(t)\|^3$,

$$a_N = \kappa v^2 = \frac{\|\vec{r}'(t)\times\vec{r}''(t)\|}{\|\vec{r}'(t)\|}, \quad = \left\|\frac{d\vec{r}}{ds}\times\frac{d^2\vec{r}}{ds^2}\right\|.$$

See Example 3.15 in OSC3 Section 3.4[3].

Another Way to Compute $\widehat{N}$. These formulas for a_T and a_N also give a sometimes more convenient way to compute the normal vector:

$$\widehat{N} = \frac{\vec{a} - a_T\widehat{T}}{a_N} = \frac{\vec{r}'' - (a_T/v)\vec{r}'}{a_N} = \frac{v\vec{r}'' - v'\vec{r}'}{\|\vec{r}'(t)\times\vec{r}''(t)\|}$$

Study Guide. Study Section 3.4 of Calculus Volume 3[4]; in particular

[3] openstax.org/books/calculus-volume-3/pages/3-4-motion-in-space
[4] openstax.org/books/calculus-volume-3/pages/3-4-motion-in-space

- The Definitions.
- Theorems 3.7 and 3.8.
- Examples 3.14 and 3.15, and the Checkpoints following each.
- Several exercises from the range 157–162.

Note: The topics after Checkpoint 3.15, Projectile Motion and Kepler's Laws should be interesting reading for some students, but will not be covered on quizzes or tests.

Chapter Review. When reviewing this chapter, also look at the end of chapter review material in OpenStax Calculus Volume 3, including Key Terms[5], Key Equations[6] and Key Concepts[7].

[5]openstax.org/books/calculus-volume-3/pages/3-key-terms
[6]openstax.org/books/calculus-volume-3/pages/3-key-equations
[7]openstax.org/books/calculus-volume-3/pages/3-key-concepts

Chapter 4

Differentiation of Functions of Several Variables

References.

- OpenStax Calculus Volume 3, Chapter 4.[1]
- Calculus, Early Transcendentals by Stewart, Chapter 14.

4.1 Functions of Several Variables

References.

- OpenStax Calculus Volume 3, Section 4.1[1].
- Calculus, Early Transcendentals by Stewart, Section 14.1.

Introduction. Earth's surface temperature (T) at a single time depends on several quantities, the latitude (x) and longitude (y) of the point whose temperature is indicated. Mathematically we can describe temperature as a function with two variables or arguments, $T(x, y)$.

Definition 4.1.1 A real valued function f of two variables is a rule that, for each ordered pair (x, y) in a subset D of $\mathbb{R}^2$, assigns a unique real number denoted by $f(x, y)$. The set D is the domain of f, and the range of f is the set of all the values that arise for $f(x, y)$.

See Example 1 in OSC3 Section 4.1[2].

The rule can be specified with a list of ordered triples (x, y, z) such that in every triple, the initial pair (x, y) is in the set D (the domain of f), and for every ordered pair (x, y) in D, there is exactly one such triple: the z value in that triple is denoted $f(x, y)$. $\diamondsuit$

We often write $z = f(x, y)$: x and y are the independent variables or arguments of f, z is the dependent variable or value of f.

Geometrically, we often think of the domain D as a set in the plane. When a function is specified by a formulas in terms of the independent variables, the domain is to be all pairs for which the formulas makes sense. This is sometimes called the natural domain associated with the formula. (This can be unclear or hard to determine, so I generally recommend stating the domain explicitly if you can.)

Example 4.1.2 The (natural) domain of $f(x, y) = \sqrt{x^2 - y^2}$ is all ordered pairs (x, y) with $|x| \geq |y|$. In the plane this is all points in the two quadrants to the right and left of the two lines $y = x$, $y = -x$. $\square$

[1] openstax.org/books/calculus-volume-3/pages/4-introduction
[1] openstax.org/books/calculus-volume-3/pages/4-1-functions-of-several-variables
[2] openstax.org/books/calculus-volume-3/pages/4-1-functions-of-several-variables

Graphs. The graph of a function f is the corresponding set of points (x, y, z) in $\mathbb{R}^3$ for which $z = f(x, y)$, so it is typically a surface, like the elliptic paraboloid $z = x^2 + (y/2)^2$.

To visualize functons of two variables, I suggest first working out the domain and then examining their graphs with the Desmos 3D Graphing Calculator[3] (or other suitable graphing software that you know about).

The Desmos notation for a function of two variables is "$z = f(x, y)$" or just the formula "$f(x, y)$"; for example, for $f(x, y) = x^2 + y^2$ in Example 4.2(b) of Section 4.1[4], one can type in z = x^2 + y^2 (which gets rendered as $z = x^2 + y^2$) or just x^2 + y^2.

See Examples 2 and 3 in OSC3 Section 4.1[5];

Level Curves and Contour Plots. Such surfaces in three dimensions are sometimes hard to illustrate clearly in a two dimensional drawing, so one alternative method of visualization is drawing level curves, also known as contours, as used in maps to show altitude.

Definition 4.1.3 The level curves of a function f of two variables are the set of curves in the plane given by the solutions of $f(x, y) = k$ for each number k in the range of f. $\diamondsuit$

To plot contours at level $z = k$ with Desmos[6] the notation is "f(x, y) = k" (or replace the parameter k by any letter other than the coordinate names x,y or z). Entering this invites you to add a slider for the parameter k. For example, try $x^2 - y^2 = k$.

Example 4.1.4 For each $k > 0$, the level curve of the function given by the formula $f(x, y) = x^2 + (y/2)^2$ is an ellipse.
The level curve for $k = 0$ is"degenerate": it is the single point $(0, 0)$.
There are no level curves for $k < 0$, as such values are not in the range of f.
Together the level curves fill out the entire plane. $\square$

See Examples 4 and 5 in OSC3 Section 4.1[7];

In general, the collection of all level curves of a function fill out its whole domain, with each point of the domain on exactly one level curve.

Functions of Three or More Variables. It is easy to extend the above ideas to functions with three or more independent variables, denoted $f(x, y, z)$ and so on. These are also often natural: for example, air temperature depends on position specified by the three variables latitude, longitude and altitude, and the total energy of a moving object can depend on six variables: three specifying its position and three its velocity. The domain of a function of three variables is a region in three dimensional space $\mathbb{R}^3$, typically a solid.

Example 4.1.5 The natural domain of the function given by the formula $f(x, y, z) = 1/(x^2 + y^2 + z^2 - 1)$ is every point in space except those on the sphere of radius 1 center the origin. $\square$

Level Surfaces. Level surfaces are the equivalent here of level curves, and are even more important since the graph of a function of three variables is a four dimensional object, ordered quadruples of numbers (x, y, z, w) with $w = f(x, y, z)$, and these are very difficult to graph or visualize. The level surfaces are sets of points in space, rather easier to draw. Note again that different level surfaces for different k values do not intersect, and they divide up the domain up into a nested collection of surfaces, like the layers of a deformed onion or a can of pringles.

Example 4.1.6 The level surfaces of $f(x, y, z) = x + 2y + 4z$ are the various planes $x + 2y + 4z = k$. Note that these are all parallel (they have the same normal $\langle 1, 2, 4\rangle$) and so indeed do not intersect, and together

[3] www.desmos.com/3d
[4] openstax.org/books/calculus-volume-3/pages/4-1-functions-of-several-variables
[5] openstax.org/books/calculus-volume-3/pages/4-1-functions-of-several-variables
[6] www.desmos.com/3d
[7] openstax.org/books/calculus-volume-3/pages/4-1-functions-of-several-variables

fill out all of $\mathbb{R}^3$. □

Even More Variables, and Vector Valued Arguments. One quantity can depend on more than three others; say on n quantities $x_1 \cdots x_n$. We sometimes write such a function with ellipsis as $f(x_1, x_2, \ldots, x_n)$ when n is large.

At times we consider functions of several variables as a function of a single vector argument:

- $f(x, y, z)$ becomes $f(\vec{x})$ with $\vec{x} = \langle x, y, z\rangle$, and
- $f(x, y)$ also becomes $f(\vec{x})$, now with $\vec{x} = \langle x, y\rangle$.

This is particularly convenient with functions of a large number of variables: defining the n-component vector $\vec{x} = \langle x_1, x_2, \ldots, x_n\rangle$, the function $f(x_1, x_2, \ldots, x_n)$ can be expressed in the same concise form $f(\vec{x})$.

See Examples 6 and 7 in OSC3 Section 4.1[8];

Study Guide. Study Section 4.1 of Calculus Volume 3[9]; in particular

- All the Definitions.
- Examples 1.–6, and the Checkpoints following each.
- One or several exercises from each of the following ranges: 5–10, 11–13, 14–29, 30–32.

4.2 Limits and Continuity

References.

- OpenStax Calculus Volume 3, Section 4.2[1].
- Calculus, Early Transcendentals by Stewart, Section 14.2.

Limits and Continuity. To make sense of a statement like "$f(x, y)$ has limit L as the arguments approach the point (a, b)" we need to quantify the idea of a pair of arguments x and y being close to another pair a and b. This is done with the distance between the points (x, y) and (a, b) in the plane.

Thus the precise $\delta - \epsilon$ definition of a limit is

Definition 4.2.1 $f(x, y)$ has limit L as (x, y) approaches (a, b) if for any positive value ϵ, there is a positive value δ such that for any point (x, y) in the domain D of f (other than (a, b) itself), having $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ ensures that $|f(x, y) - L| < \epsilon$.
Note that only points $(x, y) \neq (a, b)$ matter.
If the limit exists and has value L, we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L.$$

◊

It is sometimes nice to consider f as a function of a vector argument $\vec{x} = \langle x, y\rangle$ and the point (a, b) as the vector $\langle a, b\rangle$.

Then the condition is that for any $\vec{x} = \langle x, y\rangle$ in domain D other than $\vec{a} = \langle a, b\rangle$,

$$\|\vec{x} - \vec{a}\| < \delta \text{ ensures that } |f(\vec{x}) - L| < \epsilon.$$

One advantage of this is that it applies equally well for functions on three variables (or even more). In fact it also works for functions on the real numbers: just take $\|\vec{x} - \vec{a}\|$ to mean $|x - a|$ and so on.

[8] openstax.org/books/calculus-volume-3/pages/4-1-functions-of-several-variables
[9] openstax.org/books/calculus-volume-3/pages/4-1-functions-of-several-variables
[1] openstax.org/books/calculus-volume-3/pages/4-2-limits-and-continuity

Limit Laws. All the familiar limits laws apply, for sums, differences, products, quotients and so on.

See Example 8 in OSC3 Section 4.2[2].

Some Cautionary Examples: Limits can Fail to Exist in New Ways. With functions of one variable, it is sometimes convenient to compute limits using the one-sided limits from each side.

It might seem worth trying a similar thing with functions of two variables: approach a point (a, b) along various straight lines passing through it, so that one is really working with a function of one variable: $g(t) = f(a + ct, b + dt)$ with $\langle c, d\rangle$ giving the direction of the line.

However, strange things can happen:

Question 4.2.2 For the functions f and g given by

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2} \quad \text{and} \quad \frac{x^2 - y^2}{x^2 + y^2}$$

with domain everything except the origin $(0, 0)$, consider how the values behave as (x, y) approaches the origin along straight lines.
For each function, does the limit exist as $(x, y) \to (a, b)$ and if so, what is it?Parameterize the lines as $\langle t, mt\rangle$, along with the exceptional case of the vertical line $\langle 0, t\rangle$. □

Question 4.2.3 For the function

$$f(x, y) = \begin{cases} \dfrac{x^2}{y}, & y \neq 0 \\ 0, & y = 0 \end{cases}$$

1. Show that approaching the origin on any straight line gives $f(x, y) \to 0$, but ...
2. ... it does not have a limit at the origin.

Look at the curves $y = kx^2$ □

See Example 9 in OSC3 Section 4.2[3].

Interior Points and Boundary Points. For functions of one variable, the domain is typically an interval like (a, b), $[a, b]$, $(a, b]$ or $[a, b)$ (or a union of these), and in the latter three cases, the domain includes endpoints, a and/or b where limits are handled a bit differently, as one-sided limits.

The issue of the limit at a boundary point $\vec{a}$ is now dealt with more elegantly by simply requiring that $|f(\vec{x}) - L < \epsilon$ for any $\vec{x}$ in the domain of f where $\|\vec{x} - \vec{a}\| < \delta$. That is, we just ignore points $\vec{x}$ outside the domain.

Nevertheless, it is useful to consider what happens at the "edges" of a domain in two (or more) dimensions. The generalization of endpoints is the boundary of a set D in $\mathbb{R}^2$ or $\mathbb{R}^3$, as contrasted to the interior:

Definition 4.2.4 A point $\vec{a}$ in any set D is in the interior of D if it is surrounded by a small enough "disk" or "ball" that lies entirely in D. That is, there is some radius $\delta > 0$ such that all points in the set $B_\delta(\vec{a}) = \{\vec{x} : \|\vec{x} - \vec{a}\| < \delta\}$ of radius δ and center $\vec{a}$ are also in D. ♢

Any such set $B_\delta(\vec{x})$ is called a ball, even though in two dimensions it is a disk. This jargon is sometimes even used in one dimension, where a "ball" is an open interval $(a - \delta, a + \delta)$.

The points of D that are not in its interior are on its boundary. However, the boundary includes points that are "adjacent" to D even if they are not in it: for example the boundary of $B_\delta(\vec{x})$ is the circle in $\mathbb{R}^2$ (or sphere in $\mathbb{R}^3$) where $\|\vec{x} - \vec{a}\| = \delta$, and this is entirely outside that ball.

Definition 4.2.5 A point $\vec{a}$ is in the boundary of D if no ball $B_\delta(\vec{a})$ is either entirely in D or entirely outside it.

[2] openstax.org/books/calculus-volume-3/pages/4-2-limits-and-continuity
[3] openstax.org/books/calculus-volume-3/pages/4-2-limits-and-continuity

That is, every ball around a point in the boundary is partly in the set, partly outside it.
Yet another way of saying this is that one can get as close as you want to such a boundary point both from inside the set and from outside it.
The boundary of a set D is sometimes denoted ∂D. ♢

Question 4.2.6 What is the boundary of each of the four sets (a, b), $[a, b]$,$[a, b)$ and $(a, b]$? □

See Example 10 in OSC3 Section 4.2[4].

Open and Closed Sets. There are two opposite extremes that can make a domain easier to work with:

Definition 4.2.7 A set is open if it consists entirely of interior points: it contains none of its boundary points. ♢

A familiar example is open intervals like (a, b); the balls $B_\delta(\vec{a})$ defined above are another example: hence these are sometimes called open balls.

Definition 4.2.8 A set is closed if it contains all of its boundary points. ♢

A familiar example is closed intervals like $[a, b]$. There are also the closed balls

$$\bar{B}_\delta(\vec{a}) = \{\vec{x} : \|\vec{x} - \vec{a}\| \leq \delta\}$$

(Pay close attention to the inequality sign!)

Remark 4.2.9 Recall that a continuous function on closed and bounded interval always attains a maximum and minimum value; we will see a similar result for functions whose domain is a closed.

Continuity. Continuity is now defined in the familiar way:

Definition 4.2.10 a function f is continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b),$$

and f is continuous if it is continuous at each point (a, b) in its domain. ♢

See Example 11 in OSC3 Section 4.2[5].

The familar properties of continuity apply, for sums, differences, products, compositions (so long as the range of the ”inner” function is within the domain of the ”outer” one) and quotients (where they are defined; away from division by zero).

See Example 12 in OSC3 Section 4.2[6].

Polynomials and Rational Functions.
Definition 4.2.11 A polynomial function in two variables x and y is any sum of constant multiples of terms like $x^n y^m$ with m and n non-negative integers, and a rational function in two variables is a quotient of such polynomials. ♢

As one might guess, polynomials in one variable are continuous at every point, and so are simply continuous, and rational functions are continuous at any point where the denominator is non-zero.

But more is true: since points with zero denominator are not in the domain of the rational function, rational functions are also continuous!

[4] openstax.org/books/calculus-volume-3/pages/4-2-limits-and-continuity
[5] openstax.org/books/calculus-volume-3/pages/4-2-limits-and-continuity
[6] openstax.org/books/calculus-volume-3/pages/4-2-limits-and-continuity

Functions of Three or More Variables. All the above ideas of limits, continuity, polynomials and rational functions extend in an obvious way to functions of three or more variables, like $f(x, y, z)$ or $f(x_1, x_2, \ldots, x_n)$.

The vector notation $\vec{x} = \langle x, y \rangle$, $\vec{x} = \langle x, y, z \rangle$ or $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle$ becomes convenient for dealing with functions of any number of independent variables by using the common notation $f(\vec{x})$.

We have:

Definition 4.2.12 Function f with domain D in $\mathbb{R}^n$ has limit L as $\vec{x}$ approaches $\vec{a}$ if for any positive ϵ, there is a positive δ such that

$$|f(\vec{x}) - L| < \epsilon \text{ whenever } 0 < \|\vec{x} - \vec{a}\| < \delta \text{ and } \vec{x} \text{ is in } D.$$

(Note that only points $\vec{x} \neq \vec{a}$ matter.)
If so, we write

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L.$$

◊

See Example 13 in OSC3 Section 4.2[7].

Study Guide. Study Section 4.2 of Calculus Volume 3[8]; in particular

- All the Definitions, Theorems, Examples and Checkpoints.
- One or several exercises from each of the following ranges: 61, 62, 63–77, 78, 79, 86&87, 88&89, 90–93.

4.3 Partial Derivatives

Revised on March 26.

References.

- OpenStax Calculus Volume 3, Section 4.3[1].
- Calculus, Early Transcendentals by Stewart, Section 14.3.

Introduction. For a differentiable function of one variable, $y = f(x)$, a single number $f'(a)$ describes the rate of change of the function value relative to change in the independent variable x away from $x = a$.

Change in the value of a function of two variables can be more complicated depending on the direction of the change. Consider the simple function $f(x, y) = 2x - y$ for (x, y) near $(0, 0)$. Moving a distance h in the positive x direction changes the value by $2h$; moving a distance h in the positive y direction changes the function value by $-h$; moving any distance in the direction $\langle 1, 2 \rangle$ does not change the value at all.

The reason for this is that there are infinitely many directions in which the values of the independent variables can change. Thus we start by considering the effect of change along the coordinate axes: changing one independent variable while holding the other constant.

4.3.1 Partial Derivatives of $f(x, y)$ at a point

For $f(x, y)$ and a point (a, b) in its domain, consider the function given by allowing just x to vary while the y values is always b: $g(x) = f(x, b)$. Then $g'(a)$ is its rate of change at $x = a$, which is also the rate of change of $f(x, y)$ due to change in x alone at (a, b).

[7] openstax.org/books/calculus-volume-3/pages/4-2-limits-and-continuity
[8] openstax.org/books/calculus-volume-3/pages/4-2-limits-and-continuity
[1] openstax.org/books/calculus-volume-3/pages/4-3-partial-derivatives

This is called the partial derivative of f with respect to x at (a,b), denoted $f_x(a,b)$: using the definition of the derivative in terms of limits,

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} \tag{4.3.1}$$

Similarly, we define the partial derivative of f with respect to y at (a,b) by

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}. \tag{4.3.2}$$

4.3.2 Partial Derivatives as Functions

For a given point (a,b), the partial derivatives at (a,b), $f_x(a,b)$ and $f_y(a,b)$, are numbers. As the point at which these are computed is allowed to vary, this gives functions:

Definition 4.3.1 If f is a function of two variables, its partial derivatives are the functions f_x and f_y with values given by

$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \\ f_y(x,y) &= \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h} \end{aligned}$$

$\diamondsuit$

See Example 14 in OSC3 Section 4.3[2].

That is, to compute f_x, treat y as a constant, so that f is teated as a function of a single variable x and differentiate that. Likewise to compute f_y.

See Example 15 in OSC3 Section 4.3[3].

To distinguish from partial derivatives, the derivative of a function of a single variable, f' or dy/dx, is sometimes called an ordinary derivative.

4.3.3 Notations

A great variety of notations are used in various places for partial derivatives. For $z = f(x,y)$, some are

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \partial_x f = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f \tag{4.3.3}$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \partial_y f = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f \tag{4.3.4}$$

Note the symbol ∂ (a variant of δ, spoken ”partial”) used in place of the d of the Leibniz notation df/dx for derivatives of function of a single variable.

For notation that does not depend on particular variable names, the prime notation f' is now ambiguous, so numerical subscripts are used to indicate the position of the variable in forms like f_1 and $D_2 f$.

4.3.4 Geometrical Meaning

A geometrical interpretation of this is that when computing $f_x(a,b)$, one looks at the intersection of the graph of f with the plane $y = b$.

[2]openstax.org/books/calculus-volume-3/pages/4-3-partial-derivatives
[3]openstax.org/books/calculus-volume-3/pages/4-3-partial-derivatives

This is the curve that is the graph of function $z = g(x) = f(x, b)$, but drawn on plane $y = b$ instead of on the $x - z$ plane.

Then $f_x(a, b) = g'(b)$ which is the tangent slope of the curve at the point $P(a, b, c)$ with $c = f(a, b)$.

Likewise, $f_y(a, b)$ is the tangent slope at this point P to the curve given by the intersection of the graph of f with the plane $x = a$.

See Example 16 in OSC3 Section 4.3[4].

4.3.5 Functions of More than Two Variables

Extensions to functions of three or more variables are fairly intuitive. For example with $u = f(x, y, z)$, we have

$$f_z(x, y, z) = \lim_{h \to 0} \frac{f(x, y, z + h) - f(x, y, z)}{h}$$

When multiple variables are used with notation $u = f(x_1, x_2, \ldots, x_n)$,

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

As above, effectively all variables except one $[x_i]$ are treated as constants in the formula for f, and one then computes as one would the derivative with respect to that one remaining variable.

Other notations include $\dfrac{\partial u}{\partial x_i} = \dfrac{\partial f}{\partial x_i} = \partial_{x_i} f = f_{x_i} = f_i = D_i f$.

See Examples 17 and 18 in OSC3 Section 4.3[5].

4.3.6 Second Partial Derivatives

As with ordinary derivatives one can compute partial derivatives of partial derivatives. The most methodical notations are things like $(f_x)_y$ for the derivative with respect to y of the derivative with respect to x of f.

A variety of more concise notations are used:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} = D_1^2 f = D_x^2 f$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} = D_2 D_1 f = D_y D_x f$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y} = D_1 D_2 f = D_x D_y f$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial x^y} = D_2^2 f = D_y^2 f$$

In the middle two, note the different order of x an y in the different notations, according to whether new derivatives are added to the right or the left.

See Example 19 in OSC3 Section 4.3[6].

4.3.7 Does the Order of Derivatives Matter?

In mixed derivatives like f_{xy} and f_{yx}, the order often does not matter, though this is far from obvious:

[4]openstax.org/books/calculus-volume-3/pages/4-3-partial-derivatives
[5]openstax.org/books/calculus-volume-3/pages/4-3-partial-derivatives
[6]openstax.org/books/calculus-volume-3/pages/4-3-partial-derivatives

Theorem 4.3.2 (Clairaut). If f is defined on an open set D that contains the point (a, b), both partial derivatives f_{xy} and f_{yx} exist, and both are continuous on that disk, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

In particular, if these two mixed derivatives exist and are continuous everywhere, they are equal everywhere.

4.3.8 Higher Derivatives

Differentation can be done repeatedly (so long as the derivatives exist), and with any number of variables, with notations like f_{xyy}, f_{xyzx}, $\dfrac{\partial^3 f}{\partial x \partial y \partial z}$ and so on.

Repeated application of Clairaut's Theorem typically allows reordering of the derivatives, greatly reducing the number of different higher partial derivatives that need to be computed.

4.3.9 Partial Differential Equations

Many physical laws describe a function of several variables in terms of an equation relating its partial derivatives. Three of the most important basic ones for functions of two variables are

The Wave Equation	$u_{tt} = c^2 u_{xx}$
The Heat Equation	$u_t = K u_{xx}$
Laplace's Equation	$u_{xx} + u_{yy} = 0$
Poisson's Equation	$u_{xx} + u_{yy} = f(x,y)$
Schrödinger's Equation	$i\hbar\psi_t = -\dfrac{\hbar^2}{2m}(\psi_{xx} + \psi_{yy} + \psi_{zz}) + V(x,y,z,t)\psi$

See Example 20 in OSC3 Section 4.3[7].

Question 4.3.3 Verify that $u(t,x) = e^{-t}\sin x$ is a solution of the heat equation. □

Study Guide. Study Section 4.3 of Calculus Volume 3[8]; in particular

- The Definitions.
- Clairaut's Theorem.
- All the Examples (and the Checkpoints following each).
- One or several exercises from each of the following ranges: 112–117, 118–128, 129–131, 132–133, 135–140, 145–148.

4.4 Tangent Planes and Linear Approximations

References.

- OpenStax Calculus Volume 3, Section 4.4[1].
- Calculus, Early Transcendentals by Stewart, Section 14.4.

[7] openstax.org/books/calculus-volume-3/pages/4-3-partial-derivatives
[8] openstax.org/books/calculus-volume-3/pages/4-3-partial-derivatives
[1] openstax.org/books/calculus-volume-3/pages/4-4-tangent-planes-and-linear-approximations

Tangent Lines Revisited. Recall that for a differentiable function of one variable $y = f(x)$, the tangent line at point $x = a$ is

$$L(x) = f(a) + f'(a)(x - a).$$

It approximates $f(x)$ well in that the error is

$$E(x) := f(x) - L(x) = \epsilon(x)(x - a) \text{ with } \epsilon(x) \to 0 \text{ as } x \to a, \tag{4.4.1}$$

so that

$$f(x) = f(a) + f'(a)(x - a) + \epsilon(x)(x - a). \tag{4.4.2}$$

In terms of the increments $\Delta x = x - a$ and $\Delta y = f(x) - f(a)$,

$$\Delta y = f'(a)\Delta x + \epsilon\Delta x.$$

Big-O and little-o notation. To discuss the smallness of the errors in such tangent approximations, some notation is useful.

To say that one function $f(x)$ is not "greatly bigger" than another (positive) quantity $g(x)$ in the limit $x \to a$, we use the big-O notation

Definition 4.4.1 $f(x) = O(g(x))$ near a if there is an upper bound M giving $\dfrac{|f(x)|}{g(x)} \leq M$ for $|x - a|$ small enough.
The same is said for vector-valued functions quantities (using vector norms), and for $a = \infty$. ♢

Example 4.4.2 For any quadratic $q(x)$, $q(x) = O(x^2)$ at ∞. □

To say that more, that function $f(x)$ is far smaller than another quantity $g(x)$ in the limit $x \to a$, we use the little-o notation:

Definition 4.4.3 $f(x) = o(g(x))$ as $x \to a$ if

$$\lim_{x\to a} \frac{|f(x)|}{g(x)} = 0$$

(And again the same for vector-valued functions and for $a = \infty$.) ♢

Example 4.4.4 $\ln x = o(1/x)$ as $x \to 0+$. □

With this notation the accuracy of the linear approximation can be stated as

$$f(x) = f(a) + f'(a)(x - a) + o(|x - a|) \tag{4.4.3}$$

for x near a.

In fact Taylor's Theorem says a bit more when f is twice differentiable:

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(\xi_x)}{2}(x - a)^2$$

for some ξ_x between x and a; (see Theorem 6.7 in Section 4.4 of OpenStax Calculus Volume 2.[2])

Stated using big-O notation,

$$f(x) = f(a) + f'(a)(x - a) + O((x - a)^2)$$

[2]openstax.org/books/calculus-volume-2/pages/6-3-taylor-and-maclaurin-series

Tangent Planes. For a function $z = f(x, y)$ whose partial derivatives f_x and f_y exist a point (a, b), the best candidate for a function that approximates f well near (a, b) is

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b), \tag{4.4.4}$$

which in terms of the increments $\Delta x = x - a$ and $\Delta y = y - b$ is

$$L(a + \Delta x, b + \Delta y) = f(a, b) + f_x(a, b)\Delta x + f_y(a, b)\Delta y.$$

This defines the tangent plane $z = L(x, y)$ to f at (a, b).

See Examples 21 and 22 in OSC3 Section 4.4[3].

The Tangent Plane as a Linear Approximation to f. The tangent plane fits as closely as possible to f, at least when one look at the traces in the planes $x = a$ and $y = b$.

The traces of $L(x, y)$ are $z = f(a, b) + f_x(a, b)(x - a)$ in plane $y = b$ and $z = f(a, b) + f_y(a, b)(y - b)$ in plane $x = a$, which are the tangent lines to the traces of f in those planes. Since these two tangent lines determine a unique plane, the plane given by $L(x)$ is the only one that fits this well in these two planes.

See Example 23 in OSC3 Section 4.4[4].

However, as we shall see, this plane is not always a useful approximation is the sense seen in Equation (4.4.1): extra conditions of "continuity nearby" are needed.

Example 4.4.5 Cautionary Example: The Uncomfortable Saddle. Consider the function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & , \quad (x, y) \neq (0, 0) \\ 0 & , \quad x = y = 0 \end{cases}$$

Both partial derivatives exist at the origin $(0, 0)$ because $f(x, 0) = 0$, giving $f_x(x, 0) = 0$ and similarly $f_y(0, y) = 0$. Thus the tangent plane to f at $(0, 0)$ is $z = 0$. However, for nearby points with $y = x$, $f(x, y) = f(x, x) = 1/2$, and with $y = -x$, $f(x, -x) = -1/2$. Thus the value of $L(x, y)$ differs from that of $f(x, y)$ by up to $1/2$ for points arbitrarily close to $(0, 0)$. In fact this shows that f has no limit at the origin, and so is not continuous. $\square$

Notes on the Example. A novelty here is that both partial derivatives exist, yet the function itself is not continuous. And those partial derivatives exist everywhere, not only at $(0, 0)$: for $(x, y) \neq (0, 0)$,

$$f_x(x, y) = \frac{y(x^2 - y^2)}{(x^2 + y^2)^2}, \quad f_y(x, y) = \frac{x(y^2 - x^2)}{(x^2 + y^2)^2}.$$

The problem is that like f itself, neither of these is continuous at $(0, 0)$. For example, $f_x(0, y) = 1/y$ for $y \neq 0$.

Linear Approximations and Differentiable Functions. The approximation behavior that we want from the tangent plane is a 2D version of Equation (4.4.2) or (4.4.3), and this leads to the natural, geometrically based concept of being differentiable:

Definition 4.4.6 Differentiability. A function f of two variables is differentiable at (a, b) if

$$\begin{aligned} f(a + \Delta x, b + \Delta y) &= f(a, b) + f_x(a, b)\Delta x + f_y(a, b)\Delta y && + \epsilon_1 \Delta x + \epsilon_2 \Delta y \\ &= L(a + \Delta x, b + \Delta y) && + \epsilon_1 \Delta x + \epsilon_2 \Delta y \end{aligned}$$

with the functions $\epsilon_i = \epsilon_i(\Delta x, \Delta y)$ both small near (a, b) in that $\epsilon_i(\Delta x, \Delta y) \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

[3] openstax.org/books/calculus-volume-3/pages/4-4-tangent-planes-and-linear-approximations
[4] openstax.org/books/calculus-volume-3/pages/4-4-tangent-planes-and-linear-approximations

With the little-o notation, this can be phrased in the simpler form

$$f(a+\Delta x, b+\Delta y) = f(a,b) + f_x(a,b)\Delta x + f_y(a,b)\Delta y + o\left(\sqrt{(x-a)^2+(y-b)^2}\right) \tag{4.4.5}$$

or with vector notation $\vec{a} = \langle a, b\rangle$, $\vec{x} = \langle x, y\rangle$, $\overrightarrow{\Delta x} = \langle \Delta x, \Delta y\rangle$,

$$f(\vec{a}+\overrightarrow{\Delta x}) = f(\vec{a}) + \langle f_x(\vec{a}), f_y(\vec{a})\rangle \cdot \overrightarrow{\Delta x} + o(\|\overrightarrow{\Delta x}\|), \tag{4.4.6}$$

mimicing Equation (4.4.3).
A function f is differentiable if it is differentiable at each point of its domain. ♢

See Example 24 in OSC3 Section 4.4[5].

A Criterion for Differentiability. Fortunately, there is a simple criterion for differentiability:

Theorem 4.4.7 A function f of two variables is differentiable at (a, b) if its partial derivatives f_x and f_y exist at and near (a, b) and are continuous at (a, b).
That is, the partial derivatives exist on some "ball" $B_\delta(\vec{a})$.

Clearly the continuity requirement excludes the above example.

On the other hand, all polynomial and rational functions are seen to be differentiable on their natural domains, including that example: its natural domain excludes $(0, 0)$ due to division by zero in the formula.

Differentials. By definition, a function $z = f(x, y)$ that is differentiable at point (a, b) is accurately approximated nearby by its tangent plane, and it can be convenient to approximate the change in the value of a function using the notation of differentials as in Section 4.2 of OpenStax Calculus, Volume 1[6] or Section 3.11 of Calculus, Early Transcendentals by Stewart.

For the independent variables x and y of function $z = f(x, y)$, the differentials dx and dy are just any increments in the value of those variables from a starting point (x, y).

For the dependent variable z, the differential is the change in the value of the linear approximation as the independent variables change from (x, y) to $(x + dx, y + dy)$:

$$dz = f_x(x,y)dx + f_y(x,y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

This differs from the increment in the function value, $\Delta z = f(x + dx, y + dy) - f(x, y)$, by the amount

$$\Delta z - dz = \epsilon_1 dx + \epsilon_2 dy.$$

This discrepancy is suitably small when the differentials dx and dy are small.

See Example 25 in OSC3 Section 4.4[7].

Estimating Error with Differentials. One application of this is when the known values x and y of the independent variables represent experimental measurements or other numerical values with some margin of error, so that the exact values are $x + dx$, $y + dy$ and all we know is upper limits on the magnitudes of dx and dy, say $|dx| \le E_x$, $|dy| \le E_y$.

Then we can get an upper limit on the differential dz and use it to estimate the error Δx in using $f(x, y)$ as an approximation of $f(x + dx, y + dy)$:

$$|\Delta z| \approx |dz| \le |f_x(x,y)||dx| + |f_y(x,y)||dy| \le |f_x(x,y)|E_x + |f_y(x,y)|E_y.$$

[5] openstax.org/books/calculus-volume-3/pages/4-4-tangent-planes-and-linear-approximations
[6] openstax.org/books/calculus-volume-1/pages/4-2-linear-approximations-and-differentials
[7] openstax.org/books/calculus-volume-3/pages/4-4-tangent-planes-and-linear-approximations

Functions of Three Variables. All the ideas of this section extend in unsurprising ways to functions of three or more variables. In a nutshell, the linear approximation of $w = f(x, y, x)$ is

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c),$$

the corresponding differential formula is

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz,$$

and a function is differentiable if

$$\begin{aligned} f(a + \Delta x, b + \Delta y, c + \Delta z) = & \quad f(a, b, c) + f_x(a, b, c)\Delta x + f_y(a, b, c)\Delta y + f_z(a, b, c)\Delta z \\ & \quad + \epsilon_1 \Delta x + \epsilon_2 \Delta y + \epsilon_3 \Delta z \\ = & \quad L(a + \Delta x, b + \Delta y, c + \Delta z) \\ & \quad + \epsilon_1 \Delta x + \epsilon_2 \Delta y + \epsilon_3 \Delta z \end{aligned}$$

with $\epsilon_i(x, y, z) \to 0$ as $(x, y, z) \to 0$.

Functions of Many Variables. For higher dimensions, vector notation is convenient:

Definition 4.4.8 For $\vec{x} = \vec{a} + \Delta\vec{x} = \langle a_1, \ldots, a_n\rangle + \langle \Delta x_1, \ldots, \Delta x_n\rangle$, differentiability at $\vec{a}$ means

$$f(\vec{x}) = f(\vec{a}) + \langle f_1(\vec{a}), \ldots, f_n(\vec{a})\rangle \cdot \Delta\vec{x} + \vec{\epsilon} \cdot \Delta\vec{x},$$

$\vec{\epsilon}(\vec{x}) \to 0$ as $\vec{x} \to \vec{a}$. ♢

Study Guide. Study Section 4.4 of Calculus Volume 3[8]; in particular

- All the Definitions and Theorems.
- All the Examples (and the Checkpoints following each).
- The following exercises (in the case of a range, do at least one from the range): 163–164, 170–181, 191, 192, 196, 197.

4.5 The Chain Rule and Implicit Differentiation

References.

- OpenStax Calculus Volume 3, Section 4.5[1].
- Calculus, Early Transcendentals by Stewart, Section 14.5.

Introduction. Since the partial derivatives of any given function can be computed using the rules for ordinary derivatives, we do not really need any new rules for the derivative of compositions.

However, in practice there is a nice pattern, and by stating this as a rule, time and effort can be saved in computing partial derivatives, including an easier version of implicit differentiation.

Theorem 4.5.1 The Chain Rule, Case I. For $z = f(x, y)$ a differentiable function of two variables and $x = g(t)$ and $y = h(t)$ differentiable functions of one variable, the composite function $z = f(g(t), h(t))$ of one variable has [ordinary] derivative

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}, = \frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt}$$

[8] openstax.org/books/calculus-volume-3/pages/4-4-tangent-planes-and-linear-approximations
[1] openstax.org/books/calculus-volume-3/pages/4-5-the-chain-rule

or

$$\frac{d}{dt}f(g(t),h(t)) = f_x(g(t),h(t))g'(t) + f_y(g(t),h(t))h'(t).$$

See Example 26 in OSC3 Section 4.5[2].

The last version emphasizes that the derivatives of f are evaluated with the same arguments as f.

The variables x and y here are independent variables of f but dependent variables of g and h: they are called intermediate variables in this composition, to distinguish from the independent variable t of the composition.

Adding an extra independent variable to the functions g and h simply turns the above ordinary derivative dz/dt into a partial derivative $\partial z/\partial t$ and adds a second partial derivative:

Theorem 4.5.2 The Chain Rule, Case II. For $z = f(x,y)$, $x = g(s,t)$ and $y = h(s,t)$ functions of two variables, all differentiable, the function $z = f(g(s,t),h(s,t))$ is also differentiable, with partial derivatives

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\end{aligned}$$

See Example 27 in OSC3 Section 4.5[3].

The variables x and y here are independent variables of f and dependent variables of g and h: they are called intermediate variables in this composition, to distinguish from the independent variables s and t.

There is nothing special about f having two variables: the above rules extend to $z = f(x_1, x_2, \dots x_n)$.

Also, there is nothing special about these n intermediate variables being functions of one or two variables as in the two cases above. They can instead be functions of m intermediate variables $t_1 \dots t_m$.

The following final version of the Chain Rule includes the above two cases, along with the Chain Rule for functions of a single variable:

Theorem 4.5.3 The Chain Rule, General Version. For $z = f(x_1, x_2, \dots x_n)$ and $x_i = g_i(t_1, \dots, t_m)$, $1 \le i \le n$, all differentiable, the composition $z = f(x_1(t_1, \dots, t_m), \dots, x_n(t_1, \dots, t_m))$ is a function of $t_1, \dots, t_m$ that is also differentiable, with partial derivatives

$$\begin{aligned}\frac{\partial z}{\partial t_i} &= \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \cdots + \frac{\partial z}{\partial x_j}\frac{\partial x_j}{\partial t_i} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_i}\\ &= \sum_{j=1}^{n} \frac{\partial z}{\partial x_j}\frac{\partial x_j}{\partial t_i}\end{aligned}$$

See Examples 28 and 29 in OSC3 Section 4.5[4].

Implicit Differentiation Made Easy. The method of implicit differentiation introduced in an introductory calculus course (see Section 3.6 of Calculus, Early Transcendentals by Stewart or Section 3.8 of OpenStax Calculus Volume 1[5]) can be expedited using the Chain Rule for a function of two variables.

Suppose that a function $y = f(x)$ is specified implicitly, as the solution of an equation in x and y. That equation can be written in terms of a function F of two variables, as $F(x,y) = 0$. Applying the Chain Rule, but now with x as the independent variable, $F(x, f(x)) = 0$ and so $\dfrac{dF}{dx} = 0$, but also the Chain Rule gives

$$0 = \frac{dF}{dx} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}.$$

[2]openstax.org/books/calculus-volume-3/pages/4-5-the-chain-rule
[3]openstax.org/books/calculus-volume-3/pages/4-5-the-chain-rule
[4]openstax.org/books/calculus-volume-3/pages/4-5-the-chain-rule
[5]openstax.org/books/calculus-volume-1/pages/3-8-implicit-differentiation

Solving gives

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{F_x}{F_y}. \tag{4.5.1}$$

Note the difference between $\dfrac{dF}{dx}$ and $\dfrac{\partial F}{\partial x}$.

Example 4.5.4 If $y = f(x)$ is defined by the equation $x^2 + xy^3 + \sin y = 0$, the form $F(x, y) = 0$ has

$$\frac{\partial F}{\partial x} = 2x + y^3, \quad \frac{\partial F}{\partial y} = 3xy^2 + \cos y,$$

so

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x + y^3}{3xy^2 + \cos y}$$

This is the result one would get with implicit differentiation, but all the equation solving has been done in advance in (4.5.1), so no longer needs to be done each time. □

See Example 30 in OSC3 Section 4.5[6].

When Is this Valid? The Implicit Function Theorem. So far it has been assumed that the equation $F(x, y(x)) = 0$ determines a differentiable function $y = f(x)$.

This is not always true; there can be:

1. multiple solutions, describing different parts of the curve $F(x, y) = 0$, and
2. points where that curve has a vertical tangent, so no finite value of dy/dx is possible.

The precise conditions are as follows:

Theorem 4.5.5 The Implicit Function Theorem.

1. If for a point $P(x_0, y_0)$ satisfying $F(x_0, y_0) = k$ (a constant), the function F and its partial derivatives are continuous near P [i.e. on a disk centered at that point] and $F_y(x_0, y_0) \neq 0$, then there is a unique differentiable function $y = f(x)$ defined for values of x near x_0 which solves $F(x, f(x)) = k$ and with $f(x_0) = y_0$; that is, there is a differentiable curve through$P$ which lies in the level set $F(x, y) = k$.
   Further, its derivative is given by (4.5.1):
   $$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{F_x}{F_y}.$$
2. Likewise, if $F_x(x_0, y_0) \neq 0$, there is a unique differentiable function $x = f(y)$ defined for values of y near y_0 which solves $F(f(y), y) = k$ and with $f(y_0) = x_0$, and
   $$\frac{dx}{dy} = -\frac{\partial F/\partial y}{\partial F/\partial x} = -\frac{F_y}{F_x}.$$

Note that in each case, the non-zero derivative condition is merely what is needed for the formulas for the derivatives along the curve to make sense!

A refinement of this will be seen in Theorem 4.6.2, p. 56 of Section 4.6, p. 53.

Implicitly Defined Functions of Several Variables. A function $z = f(x, y)$ of two variables is sometimes specified as the solution of an equation $F(x, y, z) = 0$: for example the function describing a quadric surface, or at least part of such a surface.

The Chain Rule can again be used to compute the partial derivatives of f.

[6] openstax.org/books/calculus-volume-3/pages/4-5-the-chain-rule

(Aside: These partial derivatives help describe the tangent plane at a point on the surface $F(x, y, z) = 0$ by giving components of a normal vector to the tangent plane, which will be useful in studying surfaces defined by equations.)

For such a function f, $u(x, y) = F(x, y, f(x, y)) = 0$ so both partial derivatives of this composition are zero: $u_x = u_y = 0$.

The Chain Rule gives

$$\frac{\partial u}{\partial x} = 0 = \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}.$$

and a similar result for u_y.

A strange thing here is that x and y are the independent variables of u and are also two of the three intermediate variables.

Clearly intermediate variable y does not depend on independent variable x, so $\partial y/\partial x = 0$, and also $\partial x/\partial x = 1$. Thus

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0.$$

This and a similar calculation based on $u_y = 0$ give

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}, \quad \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}. \tag{4.5.2}$$

Again this is valid at a point (x_0, y_0, z_0) if the following version of the Implicit Function Theorem applies:

Theorem 4.5.6 The Implicit Function Theorem for $F(x, y, z) = k$.

1. If for a point $P(x_0, y_0, z_0)$ satisfying $F(x_0, y_0, z_0) = k$, the function F and its partial derivatives are continuous near P [i.e. on a ball centered at that point] and $F_z(x_0, y_0, z_0) \neq 0$, then there is a unique differentiable function $z = f(x, y)$ defined for values of (x, y) near (x_0, y_0) which solves $F(x, y, f(x, y)) = k$ and with $f(x_0, y_0) = z_0$; that is, there is a ”smooth surface” through P which lies in the level set $F(x, y, z) = k$.
   Further, its derivatives are as in (4.5.2).
2. Likewise for the other two possibilities of solving for x or y.

A refinement of this will be seen in Theorem 4.6.4, p. 56.

Study Guide. Study Section 4.5 of OSC3[7]; including

- All Theorems, Examples and Checkpoints.
- One or several exercises from each of the following ranges or groups: 215 or 218; 216, 217, 219 or 220; 227, 230–238, 256.

4.6 Directional Derivatives and the Gradient

Slightly revised on April 18.

References.

- OpenStax Calculus Volume 3, Section 4.6[1].
- Calculus, Early Transcendentals by Stewart, Section 14.6.

[7]openstax.org/books/calculus-volume-3/pages/4-5-the-chain-rule
[1]openstax.org/books/calculus-volume-3/pages/4-6-directional-derivatives-and-the-gradient

Introduction. It is natural to ask about the rate of change of a function $f(x, y)$ as the arguments change in any direction around a point (x_0, y_0) not just along the coordinate axes, and to ask questions like in which direction is change the fastest.

The cautionary Example 4.4.5, p. 48 of the "uncomfortable saddle" in Section 4.4, p. 46 shows that the partial derivatives do not always answer this question.

On the other hand, when a function is differentiableat the point, it is well approximated by the tangent plane there, and then this linear approximation becomes a good candidate for giving information about the rate of change in any direction, from the partial derivatives alone.

Directional Derivatives. A direction of change in the plane can be specified by a unit vector $\vec{u} = \langle u_1, u_2\rangle$, and we can consider how $f(x, y)$ changes in this direction near (x_0, y_0) looking at a "slice" of the function, along the line $\langle x_0, y_0\rangle + t\vec{u}$. The value of the function along this line is $f(x_0 + tu_1, y_0 + tu_2)$ and its rate of change is given by the Chain Rule as

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

This is the directional derivative of f at (x_0, y_0), denoted $D_{\vec{u}}f(x_0, y_0)$, and it does indeed depend only on the partial derivatives at the point.

Note that its value is the same as if one used the linear approximation T at that point in place of f.

See Example 31 in OSC3 Section 4.6[2].

Directional Derivatives Defined With Limits. Equivalently, the directional derivative can be defined in terms of limits as

$$D_{\vec{u}}f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

This exists when f is differentiable at the point (x_0, y_0).

To summarize, for any unit vector $\vec{u} = \langle u_1, u_2\rangle$ (or indeed any non-zero vector) and any function f differentiable at (x_0, y_0), the directional derivative of f at (x_0, y_0) in direction $\vec{u}$ is

$$D_{\vec{u}}f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 = \langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle \cdot \vec{u}$$

See Example 32 in OSC3 Section 4.6[3].

The Gradient Vector. The vector $\langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle$ appearing in the formula for the directional derivative encapsulates all information about directional and partial derivatives of f at (x_0, y_0) in a way that has a nice geometrical meaning.

It is called the gradient vector of f at (x_0, y_0), and denoted grad$f$ or ∇f, the latter sometimes pronounced "del f".

Considered as a function of position,

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}.$$

Note: this is our first example of a vector valued function of several variables.

See Example 33 in OSC3 Section 4.6[4].

[2] openstax.org/books/calculus-volume-3/pages/4-6-directional-derivatives-and-the-gradient
[3] openstax.org/books/calculus-volume-3/pages/4-6-directional-derivatives-and-the-gradient
[4] openstax.org/books/calculus-volume-3/pages/4-6-directional-derivatives-and-the-gradient

Directional Derivatives in Terms of The Gradient Vector. The directional derivative above can be written as

$$D_{\vec{u}} f(x, y) = \vec{u} \cdot \nabla f(x, y),$$

so the gradient contains all information about directional derivatives.

Tangent Lines to Level Curves. If $\dfrac{\partial F}{\partial y} \neq 0$ at point (x_0, y_0), the implicit function theorem says that the level curve C of $F(x, y)$ for value $k = F(x_0, y_0)$ is described nearby by a function $y = f(x)$.

Then Chain Rule differentiation of $k = F(x, f(x))$ gives

$$0 = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

so

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}$$

Thus the tangent line to the level curve through this point has this slope, and $\left\langle -\frac{\partial F}{\partial y}(x_0, y_0), \frac{\partial F}{\partial x}(x_0, y_0)\right\rangle$ is a tangent vector to the curve. This is perpendicular to the gradient vector $\left\langle \frac{\partial F}{\partial x}(x_0, y_0), \frac{\partial F}{\partial y}(x_0, y_0)\right\rangle$, so the gradient at such a point on the curve is normal to the tangent line to the level curve at that point, and can be considered normal to the level curve C.

Similarly, if $\partial F/\partial x \neq 0$ at any point, swapping x and y above shows that part of the described by an implicit function $x = g(y)$, and again the gradient vector is normal to the level curve. Hence,

Theorem 4.6.1 The Gradient is Normal to Level Curves. At any point (x_0, y_0) where $F(x, y)$ is differentiable and its gradient $\nabla F(x_0, y_0)$ is non-zero, this gradient vector is normal to the level curve of F through that point, in that it is normal to the line tangent to the level curve at that point.

See Example 35 in OSC3 Section 4.6[5].

Maximizing the Directional Derivative. The gradient gives the direction in which the directional derivative is greatest, and is thus the direction of most rapid increase of the value of the function.

One physical interpretation is that if the function value is altitude, the gradient vector indicates the direction "straight up-hill". To see this, recall that if the angle between ∇F at (x_0, y_0) and a unit vector $\vec{u}$ is θ,

$$D_{\vec{u}} F(x_0, y_0) = \nabla F \cdot \vec{u} = |\nabla F||\vec{u}| \cos\theta,$$

and this has maximum value when $\cos\theta = 1$, which is when $\theta = 0$, so $\vec{u}$ is a positive multiple of ∇F. Thus $\vec{u} = \nabla F/|\nabla F|$ gives the direction in which $D_{\vec{u}} F$ is greatest and so F is increasing the fastest. Further, the maximum value of the directional derivative is then $|\nabla F|$, because $|\vec{u}| = 1$.

See Example 34 in OSC3 Section 4.6[6].

Aside: the Directions of Fastest Decrease and of No Change. The opposite direction $-\nabla F$ gives fastest decrease of F.

In between, moving perpendicular to ∇F is moving tangent to the level curve, which is what happens at that point if movement is along the level curve. Thus moving perpendicular to the direction of fastest increase or decrease in F is moving in the "direction of no change in F", given by moving along a level curve.

The Implicit Function Theorem in Terms of The Gradient.

[5]openstax.org/books/calculus-volume-3/pages/4-6-directional-derivatives-and-the-gradient
[6]openstax.org/books/calculus-volume-3/pages/4-6-directional-derivatives-and-the-gradient

Theorem 4.6.2 If at a point $P(x_0, y_0)$ on a level set $F(x, y) = k$ the gradient is non-zero ($\nabla f(x_0, y_0) \neq \vec{0}$) then there is a differentiable parametric curve $x = f(t), y = g(t)$ which passes through the point (x_0, y_0) and lies in the level set $F(x, y) = k$: $F(f(t), g(t)) = k$.
Also, the parameter can be chosen to be one of x or y and so one can ensure than $\frac{d}{dt}\langle x(t), y(t)\rangle$ is a multiple of $\langle F_y, -F_x\rangle$ (a normal to ∇F), ensuring that this part of the level set near the point P is a smooth curve. Thus, if the gradient of F is non-zero everywhere on this level set, the whole level set is a collection of smooth curves. Even if ∇F is zero at some points, the rest of the level set is a collection of smooth curves.

Proof. (Sketch)
At least one of the components $F_x(x_0, y_0)$ and $F_y(x_0, y_0)$ of the gradient is non-zero, so at least one option in Theorem 4.5.5, p. 52 in Section 4.5, p. 50 applies, and the parameter t can always be chosen to be one of x or y. Then either $dx/dt = 1$ or $dy/dt = 1$, so the velocity is non-zero, ensuring smoothness. ■

Exercise 4.6.3 Consider the level sets $F(x, y) = x^2 - y^2 = k$. Break it up into the three cases

1. $k > 0$
2. $k < 0$
3. and the exceptional case $k = 0$

Functions of Three Variables. For a differentiable function f of three variables one can likewise define the directional derivative of in direction $\vec{u} = \langle u_1, u_2, u_3\rangle$

$$D_{\vec{u}} f(x, y, z) = \lim_{h\to 0} \frac{f(x + u_1 h, y + u_2 h, z + u_3 h) - f(x, y, z)}{h}$$

and the gradient

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle .$$

See Example 36 in OSC3 Section 4.6[7].

Again

$$D_{\vec{u}} f(x, y, z) = \vec{u} \cdot \nabla f(x, y, z),$$

and again the directional derivative is greatest in the direction of the gradient.

See Example 37 in OSC3 Section 4.6[8].

Theorem 4.6.4 The Implicit Function Theorem in 3D. If at a point $P(x_0, y_0, z_0)$ on a level set $F(x, y, z) = k$ the gradient is non-zero then there is a differentiable parametric surface $x = f(t, s), y = g(t, s), z = h(t, s)$ which passes through the point (x_0, y_0, z_0) and lies in the level set $F(x, y, z) = k$: $F(f(t, s), g(t, s), h(t, s)) = k$. Further, the parameters can be chosen to be one of the pairs (x, y), (x, z) or (y, z), which ensures that this part of the level set near the point P is a smooth surface approximated well by a tangent plane.

Tangent Planes to Level Surfaces. For a function F, consider the level surface S given by $F(x, y, z) = k$ through a point $P(x_0, y_0, z_0)$, consider and any curve $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ that lies in this level surface and passes through this point when $t = t_0$. The composition of F with these three component functions give the constant function $F(x(t), y(t), z(t)) = k$, and by the Chain Rule, its zero derivative is also

$$0 = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = \nabla F \cdot \vec{r}'(t)$$

In particular $\nabla F(x_0, y_0, z_0) \cdot \vec{r}'(t_0) = 0$.

The possible directions $\vec{r}'(t_0)$ for a curve passing through this point and staying in the level curve are thus all orthogonal to $\nabla F(x_0, y_0, z_0)$ and thus are directions in the plane:

$$\begin{aligned}&\nabla F(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0\rangle \\ &\quad = F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0\end{aligned}$$

[7] openstax.org/books/calculus-volume-3/pages/4-6-directional-derivatives-and-the-gradient
[8] openstax.org/books/calculus-volume-3/pages/4-6-directional-derivatives-and-the-gradient

Thus it is natural to call this plane the tangent plane to this level surface S at point P and to say that the direction of $\nabla F(x_0, y_0, z_0)$ is normal to S at P.

Normal Lines to Level Surfaces. The line through this point normal to the surface is called the normal line to S through P, and has symmetric equations

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}.$$

Tangent Planes to Graphs. An important special case is the tangent plane to the graph of a function of two variables, $z = f(x, y)$. This is given as a level surface by $F(x, y, z) = f(x, y) - z = 0$, so $F_z(x, y, z) = -1$, leading to the equation

$$f_x(x_0, y_0)(x - x_0) + f_x(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

Using $z_0 = f(x_0, y_0)$, this is

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_x(x_0, y_0)(y - y_0),$$

which is the equation of the tangent plane to $z = f(x, y)$ at (x_0, y_0) seen in Section 4.4, p. 46.

Study Guide. Study Section 4.6 of Calculus Volume 3[9]; in particular

- All Theorems, Examples and Checkpoints.
- One or several exercises from each of the following ranges or groups: 263–273, 274–279, 280–283, 299–301, 302–305.

4.7 Maxima/Minima Problems

References.

- OpenStax Calculus Volume 3, Section 4.7[1].
- Calculus, Early Transcendentals by Stewart, Section 14.7.

Introduction. We have seen the significance of the gradient ∇f when it is non-zero: it gives the direction of fastest increase (and decrease) of the value of f.

What happens at a point where the gradient vector vanishes? All directional derivatives vanish a such a point, as $D_{\vec{u}} f = \nabla f \cdot \vec{u} = 0$, so loosely, the function does not increase or decrease in any direction.

This suggests that f has a local maximum, local minimum or other type of critical point, just as with the case where the derivative of a function of one variable is zero. We shall see that this is so, but with some extra geometrical possibilities at critical points not seen with functions of a single variable.

Definition 4.7.1 A function f of two variables has a local maximum at point (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b) (in the sense that this is true for all points within some disk with center (a, b)). The number $f(a, b)$ is a local maximum value. A local minimum at (a, b) and the local minimum value there are defined similarly in terms of $f(x, y) \geq f(a, b)$. A local extremum is either a local maximum or a local minimum. ♢

Theorem 4.7.2 If f has a local extremum at point (a, b), both first order partial derivatives exist there, and the domain of f includes some disk (or ball) around the point, then $f_x(a, b) = f_y(a, b) = 0$. That is, $\nabla f(a, b) = 0$.

The "disk" requirement is for the same reason that a function of one variable can have a local extremum at an endpoint of its domain even though $f' \neq 0$ there.

[9] openstax.org/books/calculus-volume-3/pages/4-6-directional-derivatives-and-the-gradient
[1] openstax.org/books/calculus-volume-3/pages/4-7-maxima-minima-problems

Caution at the Boundary of the Domain of f. This theorem says nothing about points at the boundary of the domain of f, since at such points, at least one partial derivative fails to exist. That is because the definitions

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}, \quad f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$$

require $f(a+h,b)$ and $f(a,b+h)$ to exist for all h values near 0, both positive and negative, so the domain of f needs to extend a bit to the "left", "right", "above" and "below" (a,b). Thus we will consider boundary points separately.

Definition 4.7.3 A point (a,b) in the domain of f is a critical point (or stationary point) if $f_x(a,b) = f_y(a,b) = 0$, or if at least one of these partial derivatives does not exist. Strictly, this includes any point on the boundary of the domain, because strictly one or both derivatives do not exist at such a point. $\Diamond$

Exercise 4.7.4 Check that the functions

$$f_1(x,y) = x^2+y^2, f_2(x,y) = -x^2-y^2,$$
$$f_3(x,y) = x^2-y^2, f_4(x,y) = xy$$

each have a single critical point, at $(0,0)$. Which have local minima, which have local maxima, and which have neither? What about the critical points of $f_5(x,y) = x^2$?

See Example 38 in OSC3 Section 4.7[2].

Note that if we freeze one variable ($x=0$ or $y=0$), each of the resulting functions of a single variable has a local extremum, and the second derivative test tells us which it is. However, something new happens in the last two cases, and a modified second derivative test helps:

Theorem 4.7.5 The Second Derivatives Test. Suppose that $f_x(a,b) = f_y(a,b) = 0$ and also the quantity

$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

is not zero. Then

- If $D > 0$ and $f_{xx}(a,b) > 0$ then f has a local minimum at (a,b).
- If $D > 0$ and $f_{xx}(a,b) < 0$ then f has a local maximum at (a,b).
- If $D < 0$ then f does not have a local extremum at (a,b).
- If $D = 0$, no conclusion!

See Example 39 in OSC3 Section 4.7[3].

Notes on the Second Derivatives Test.

1. When $D < 0$, the point is called a saddle point, as in the last two of the four examples above.
2. To see why there is no conclusion when $D = 0$, consider examples like $f(x,y) = \pm x^4 \pm y^4$.
3. D is given by the determinant formula

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

Absolute Maximum and Minimum Values. With a continuous function of one variable on a closed interval $[a,b]$, absolute maximum and minimum values are attained, and occur at points where $f' = 0$, or f' does not exist, or at at endpoints. Thus typically, finding the absolute extrema only involves comparing the values of f at a finite collection of such points. With functions of two variables, closed intervals are replaced by closed, bounded domains, which have an infinite number of points at their boundary where extrema can occur, regardless of how the partial derivatives behave.

[2]openstax.org/books/calculus-volume-3/pages/4-7-maxima-minima-problems
[3]openstax.org/books/calculus-volume-3/pages/4-7-maxima-minima-problems

Definition 4.7.6 A bounded set in $\mathbb{R}^2$ or $\mathbb{R}^3$ is one that lies inside some disk or ball.
(This also works in $\mathbb{R}$, where a ”ball” is an open interval $\{x : |x - a| < \delta\}$.) ♢

And to recap some definitions from Section 4.2, p. 40:

Definition 4.7.7 A boundary point for a set is one such that every disk/ball centered at that point is partly in that set, partly outside it, and the boundary of a set is the set of all its boundary points. ♢

Note that a boundary point might or might not be in the set itself:

the two set of points $\{(x, y) : x \geq 0\}$ and $\{(x, y) : x > 0\}$ both have the boundary points $\{(x, y) : x = 0\}$, but the first set contains all these boundary points, while the second contains none.

Definition 4.7.8 A closed set is one that contains all its boundary points. A set is open if it contains none of its boundary points. Many sets are neither open nor closed. ♢

Theorem 4.7.9 Extreme Value Theorem for Functions of Two Variables. If function f is continuous on a closed, bounded set D in $\mathbb{R}^2$ then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points in D.

Thus there a basic strategy for finding these absolute extrema:

1. Find the critical points of f in D, ignoring points on the boundary of D.
2. Find the values of f at each such point.
3. Find the extreme values of f on boundary of D.
4. Find the largest and smallest of all these values.

However, for now, Item 3, p. 59 might rely on case-by-case stategies, depending on the form of the boundary, since unlike the situation with end-points of an interval, there are usually infinitely many boundary points. Section 4.8, p. 59 introduces a powerful method for finding those extrema on the boundary of a set.

See Example 40 in OSC3 Section 4.7[4].

Study Guide. Study Section 4.7 of Calculus Volume 3[5]; in particular

- All the Definitions, Theorems, Examples and Checkpoints.
- Both the Problem Solving Strategies.
- One or several exercises from each of the following ranges: 310–313, 314–317, 318–340, 345–347, 348–357.

4.8 Lagrange Multipliers

References.

- OpenStax Calculus Volume 3, Section 4.8[1].
- Calculus, Early Transcendentals by Stewart, Section 14.8.

Introduction. We often wish to find the optimum value of some quantity (like designing a car of minimum weight, or maximum fuel efficiency) subject to various constraints (like sufficient strength and passenger capacity).

For example, we might wish to design a rectangular box of volume one cubic metre with minimum surface area. This involves choosing values of the three dimensions x, y, and z of the box such that

[4] openstax.org/books/calculus-volume-3/pages/4-7-maxima-minima-problems
[5] openstax.org/books/calculus-volume-3/pages/4-7-maxima-minima-problems
[1] openstax.org/books/calculus-volume-3/pages/4-8-lagrange-multipliers

- $V(x, y, z) = xyz = 1$ and
- $A(x, y, z) = 2xy + 2xz + 2yz$ is minimized.

There is also the restriction that x, y and z must be positive, which determines the domain of functions A and V.

4.8.1 Finding the Extrema Values on a Curve of a Function of Two Variables

There is a geometrical approach to such problems, most easily seen in two dimensions.

The problem then is to find the maximum of minimum of function $F(x, y)$ over points (x, y) lying on a level curve of a function G: $G(x, y) = k$, for some given constant k. At a typical point (x_0, y_0) of this level curve, the level curve $F(x, y) = F(x_0, y_0)$ that passes through the same point crosses the level set of G.

Thus, moving along the set $G(x, y) = k$, one can get to either side of this level curve of F, and the two sides given higher and lower values of F.

Therefore, the value of F is not at a maximum or minimum at such a point (x_0, y_0), amongst points with $G(x, y) = k$.

By elimination, an extremum of F on the level curve $G(x, y) = k$ can only occur at a point where the two level curves do not cross, but are "tangent", and this can only happen when they have the same tangent direction, and thus the same normal direction. The normal vectors are given by the gradients, so the necessary condition for an extremum of F under the constraint that $G(x_0, y_0) = 0$ is

$$\vec{\nabla} F(x_0, y_0) = \lambda \vec{\nabla} G(x_0, y_0) \quad \text{for some value of } \lambda. \tag{4.8.1}$$

The factor λ is the Lagrange Multiplier, which gives this method its name. The same result can be derived purely with calculus, and in a form that also works with functions of any number of variables. Note: it is typical to fold the constant k into function G so that the constraint is $G = 0$, but it is nicer in some examples to leave in the k, so I do that.

Deriving Necessary Condition $\vec{\nabla} F = \lambda \vec{\nabla} G$ from Calculus. The necessary condition for a local extremum of F on the set $G(x, y) = k$ can also be derived by calculus.

Suppose a local extremum occurs at point (x_0, y_0) and consider the generic case that neither $G_x(x_0, y_0)$ nor $G_y(x_0, y_0)$ is zero there.

By the Implicit Function Theorem Theorem 4.5.5, p. 52, there is a solution $y = g(x)$ of $G(x, y) = k$ near a point where F has a local maximum [minimum].

Points $(x, g(x))$ satisfy the constraint, so $F(x_0, y_0)$ being a local maximum [minimum] means that $f(x) = F(x, g(x))$ has a local maximum [minimum] at x_0, and so has a critical point.

The Chain Rule and implicit differentiation give

$$0 = \frac{df}{dx} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = F_x + F_y \frac{-G_x}{G_y}$$

so

$$F_x G_y = F_y G_x \quad \text{or} \quad \frac{F_x}{G_x} = \frac{F_y}{G_y}, \text{ at } .$$

Thus at (x_0, y_0), $\vec{\nabla} F = \langle F_x, F_y \rangle$ is a multiple of $\vec{\nabla} G = \langle G_x, G_y \rangle$. The special case where one of $G_x(x_0, y_0)$ or $G_y(x_0, y_0)$ is zero can also be handled: one can always get $F_x G_y = F_y G_x$, which says that the two vectors are parallel.

The Augmented Function $F+\lambda(k-G)$. For differentiable functions F and G, the above conditions $\vec{\nabla} F = \lambda \vec{\nabla} G$ and $G = k$ are exactly the conditions for a critical point of the augmented function $H = F + \lambda(k-G)$; that is, solutions of $\vec{\nabla} H(x, y, \lambda) = 0$.

Aside: If the constraint is in form $G = 0$, the augmented function is $H = F - \lambda G$.

To see this, note that the critical points of H are given by

$$\frac{\partial H}{\partial x} = \frac{\partial F}{\partial x} - \lambda \frac{\partial G}{\partial x} = 0, \quad \frac{\partial H}{\partial y} = \frac{\partial F}{\partial y} - \lambda \frac{\partial G}{\partial y} = 0,$$

$$\frac{\partial H}{\partial y} = k - G = 0.$$

The first two equations are equivalent to $\vec{\nabla} F - \lambda \vec{\nabla} G = 0$; the last is $G(x, y)(x, y, z) = k$.

The System of Equations to Solve. Although the form $\vec{\nabla}[F + \lambda(k-G)] = 0$ can be easier to remember, and redefining G so that $k = 0$ gives the even simpler form

$$\vec{\nabla}(F - \lambda G) = 0.$$

In practice one can work with the component equations

$$F_x(x, y) = \lambda G_x(x, y),\ F_y(x, y) = \lambda G_y(x, y),\ G(x, y) = k. \tag{4.8.2}$$

Since in the end only the value of x and y are needed, one approach is to eliminate λ by multiplying the first two equations by $G_y(x, y)$ and $G_x(x, y)$ respectively, getting

$$F_x(x, y)G_y(x, y) = \lambda G_x(x, y)G_y(x, y) = F_y(x, y)G_x(x, y)$$

so that

$$F_x(x, y)G_y(x, y) - F_y(x, y)G_x(x, y) = \begin{vmatrix} F_x(x, y) & F_y(x, y) \\ G_x(x, y) & G_y(x, y) \end{vmatrix} = 0. \tag{4.8.3}$$

This single equation involving a 2-by-2 determinant is then combined with the constraint equation $G(x, y) = k$ to get two equations in the two relevant unknowns x and y.

Another way to see this result is to think of the gradients as the vectors $\vec{\nabla} F = F_x\hat{\imath} + F_y\hat{\jmath}$ and $\vec{\nabla} G = G_x\hat{\imath} + G_y\hat{\jmath}$ in $\mathbb{R}^3$; then the condition of them being parallel is that they have zero cross product, and that multiplication gives

$$(F_x\hat{\imath} + F_y\hat{\jmath}) \times (G_x\hat{\imath} + G_y\hat{\jmath}) = \begin{vmatrix} F_x(x, y) & F_y(x, y) \\ G_x(x, y) & G_y(x, y) \end{vmatrix} \hat{k} = 0.$$

See Example 42 and 43 in OSC3 Section 4.8[2].

4.8.2 Finding the Extrema Values on a Surface of a Function of Three Variables

A very similar argument using the implicit function theorem shows that for differentiable functions F and G of any number of variables, F can only have its maximum or minimum values over the set defined by equation $G(x, y) = k$ at points where $\vec{\nabla} F$ and $\vec{\nabla} G$ are parallel, which can again be expressed as

$$\vec{\nabla} F = \lambda \vec{\nabla} G \tag{4.8.4}$$

Example 4.8.1 The box of smallest area for given volume is a cube. Consider the problem of designing a rectangular box of specified volume $V = xyz$ and minimum surface area $A = 2xy + 2xz + 2yz$:
$F(x, y, z) = 2(xy + xz + yz)$ is to be minimized under the constraint $G(x, y, z) = xyz - V = 0$.

[2] openstax.org/books/calculus-volume-3/pages/4-8-lagrange-multipliers

The conditions on the three components of the gradients are

$$2(y+z) = \lambda yz,\ 2(x+z) = \lambda xz,\ 2(x+y) = \lambda xy$$

Multiplying these by x, y and z respectively gives

$$V = xyz = \lambda 2x(y+z) = \lambda 2y(x+z) = \lambda 2z(x+y). \tag{4.8.5}$$

Eliminating the Lagrange multiplier λ as soon as possible is usually a good idea, since it is not part of the final answer, and we can do that now: from (4.8.5) and the fact that $V \neq 0$, we see that $\lambda \neq 0$, giving

$$xy + xz = yx + yz = zx + zy.$$

The first equality gives $xz = yz$ and since $z \neq 0$, this gives $x = y$; similarly, $y = z$, so $x = y = z$: a cube, as claimed.
Finally, the constraint $xyz = V$ gives the dimensions as $x = y = z = \sqrt[3]{V}$, and so the minimum area is $A = 2xy + 2xz + 2yz = 6(\sqrt[3]{V})^2 = 6V^{2/3}$. This is the only possible extremum, so must give the desired minimum of area: for any size, the optimal shape is a cube. □

Example 4.8.2 The open-topped box of minimum surface area for a given volume is a half cube (half as high as wide). Consider the problem of designing a rectangular open topped box of width x, depth y, and height z that minimizes surface area $A = xy + 2xz + 2yz$ while having a prescribed volume $V = xyz$, say $V = 4$: $A = F(x, y, z) = xy + 2xz + 2yz$ is to be minimized under the constraint $G(x, y, z) = xyz = V$, V constant.
The condition on the the gradients is $\vec{\nabla} A = \lambda \vec{\nabla} G$, or

$$y + 2z = \lambda yz,\ x + 2z = \lambda xz,\ 2x + 2y = \lambda xy.$$

Multiplying these by x, y, and z respectively gives

$$\lambda xyz = xy + 2xz = xy + 2yz = 2xz + 2yz. \tag{4.8.6}$$

Again, eliminate the Lagrange multiplier λ as soon as possible; using the last two of the equations above:

$$xy + 2xz = xy + 2yz \tag{4.8.7}$$
$$xy + 2xz = 2xz + 2yz. \tag{4.8.8}$$

To avoid division by zero, note that $xyz = V > 0$, so none of x, y or z can be zero.
From Equation (4.8.7) we get $x = y$: the bottom is square.
Then Equation (4.8.8) becomes $x^2 + 2xz = 4xz$, so $x = 2z$: the height is half the horizontal dimensions; a half-cube.
Finally, the volume constraint $xyz = V = 4$ becomes $x^3/2 = V = 4$, so $x = y = \sqrt[3]{2V} = 2$, $z = x/2 = 1$. □

Example 4.8.3 The open-topped box of greatest volume for a given surface area is again a half cube. Consider now the related problem of designing a rectangular open topped box of width x, depth y, and height z, but this time specifying the value of the surface area $A = xy + 2xz + 2yz$ and maximizing the volume $V = xyz$: $F(x, y, z) = V = xyz$ is to be maximized under the constraint $G(x, y, z) = xy + 2xz + 2yz = A$, A constant (say $A = 12$).
The condition on the the gradients is $\vec{\nabla} V = \lambda \vec{\nabla} G$, or

$$yz = \lambda(y + 2z),\ xz = \lambda(x + 2z),\ xy = \lambda(2x + 2y).$$

Multiplying these by x, y and z respectively gives

$$V = xyz = \lambda(xy + 2xz) = \lambda(xy + 2yz) = \lambda(2xz + 2yz).$$

Again we can and should eliminating the Lagrange multiplier λ now. Division by zero is avoided because $V \neq 0$ ensures that none of λ, x, y, and z is zero, so

$$xy + 2xz = xy + 2yz = 2xz + 2yz,$$

just as in Equation (4.8.6). just as in Equation (4.8.6). Thus, just as in that example, we get the half-cube shape, $x = y$, $z = x/2$.

- From $xy + 2xz = xy + 2yz$ we get $x = y$: the bottom is square.
- Then $xy + 2xz = 2xz + 2yz$ becomes $x^2 = 2xz = 4xz$, so $x = 2z$: the height is half the horizontal dimensions; a half-cube.
- Finally, the area constraint $xy + 2xz + 2yz = A$ becomes $3x^2 = A$, so $x = y = \sqrt{A/3}$, $z = x/2$.

□

See Example 44 in OSC3 Section 4.8[3].

Dual Problems. The fact that the optimum shape is the same for these two problems is not a coincidence: Swapping the function to be extremized with the constraint function and swapping between maximizing and minimizing always gives the same solutions like this, through getting the same equations once we have eliminated that Lagrage multiplier λ.

All that happens is that the factor λ moves from one gradient to the other, which is like changing λ to $1/\lambda$.

Pairs of optimization problems related in this way are called Dual Problems.

An alternate strategy for solving makes this duality even clearer, by eliminating the Lagrange multiplier λ from the beginning and treating the functions F and G equally. This is to note that the condition (4.8.4) is again that the two gradient vectors are parallel and so their cross product is zero:

$$\begin{aligned}
&\vec{\nabla} F \times \vec{\nabla} G \\
&= \begin{vmatrix} F_y(x,y,z) & F_z(x,y,z) \\ G_y(x,y,z) & G_z(x,y,z) \end{vmatrix} \hat{\imath} + \begin{vmatrix} F_z(x,y,z) & F_x(x,y,z) \\ G_z(x,y,z) & G_x(x,y,z) \end{vmatrix} \hat{\jmath} + \begin{vmatrix} F_x(x,y,z) & F_y(x,y,z) \\ G_x(x,y,z) & G_y(x,y,z) \end{vmatrix} \hat{k} \\
&= \vec{0}.
\end{aligned} \tag{4.8.9}$$

This now gives three equations (each of these 2-by-2 determinants must be zero), but in fact any two holding ensures that the third one does also, so only any two need be solved; along with the constraint equation $G(x, y, z) = k$ this gives a total of three equations in the three unknowns, as expected.

4.8.3 Finding Extreme Values under Several Constraints

A function $F(x, y, z)$ can have it domain restricted by two constraints, $G(x, y, z) = k_1$, $H(x, y, z) = k_2$, which generically limits the points (x, y, z) to a curve in space. Arguments like those above show that extrema can only occur at solutions of

$$\vec{\nabla} F = \lambda \vec{\nabla} G + \mu \vec{\nabla} H, \tag{4.8.10}$$

where there are now two Lagrange multipliers, λ and μ.

In practice, one can proceed by solving the system of five equations

$$F_x = \lambda G_x + \mu H_x,\ F_y = \lambda G_y + \mu H_y,\ F_z = \lambda G_z + \mu H_z,$$

$$G(x, y, z) = k_1,\ H(x, y, z) = k_2.$$

As noted above, it is often best to to try to eliminate λ and μ as soon as possible, since their values are unneeded.

Example 4.8.4 Find the points in $\mathbb{R}^3$ of the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $x + y + z = 1$ that are (a) closest to and (b) furthest from the origin. (This intersection is an ellipse.)Rather than extremizing the distance, use its square: $F(z, y, z) = x^2 + y^2 + z^2$. □

Yet again, one strategy for eliminating the two Lagrange multipliers is to note that the condition is that the three vectors $\vec{\nabla} F(x, y, z)$, $\vec{\nabla} G(x, y, z)$ and $\vec{\nabla} H(x, y, z)$ lie in a plane, and so the parallelepiped with these

[3] openstax.org/books/calculus-volume-3/pages/4-8-lagrange-multipliers

three vectors as its edges has zero volume, or equivalently, these vectors have zero scalar triple product, as discussed in Subsection 2.4.3, p. 22. Thus the condition is

$$\vec{\nabla} F(x,y,z) \cdot (\vec{\nabla} G(x,y,z) \times \vec{\nabla} H(x,y,z)) = 0$$

which expands to the determinant equation

$$\begin{vmatrix} F_x(x,y,z) & F_y(x,y,z) & F_z(x,y,z) \\ G_x(x,y,z) & G_y(x,y,z) & G_z(x,y,z) \\ H_x(x,y,z) & H_y(x,y,z) & H_z(x,y,z) \end{vmatrix} = 0 \tag{4.8.11}$$

This single equation plus the two constraint equations gives three equations in just the three relevant unknowns x, y and z.

See Example 45 in OSC3 Section 4.8[4].

Study Guide. Study Section 4.8 of Calculus Volume 3[5]; in particular

- The Problem Solving Strategy.
- Theorem 20, and its implied extension to the case of two constraints.
- All Examples and Checkpoints.
- Several exercises from the range 358–368, including at least one with a function of three variables $f(x,y,z)$.
- Several of the "modelling" exercises from the range 380–390.

Chapter Review. When reviewing this chapter, also look at the end of chapter review material in OpenStax Calculus Volume 3, including Key Terms[6], Key Equations[7] and Key Concepts[8].

[4] openstax.org/books/calculus-volume-3/pages/4-8-lagrange-multipliers
[5] openstax.org/books/calculus-volume-3/pages/4-8-lagrange-multipliers
[6] openstax.org/books/calculus-volume-3/pages/4-key-terms
[7] openstax.org/books/calculus-volume-3/pages/4-key-equations
[8] openstax.org/books/calculus-volume-3/pages/4-key-concepts

Chapter 5

Multiple Integration

References.

- OpenStax Calculus Volume 3, Chapter 5.[1]
- Calculus, Early Transcendentals by Stewart, Chapter 15.

Introduction. This Chapter is under construction.

5.1 Double Integrals over Rectangular Regions, and Iterated Integrals

Revised on March 17.

References.

- Section 5.1 of OpenStax Calculus Volume 3[1].
- Sections 15.1 and 15.2 of Calculus, Early Transcendentals by Stewart.

Introduction. The simplest extension of the idea of the definite integral $\displaystyle\int_a^b f(x)\ dx$ over interval $a \leq x \leq y$ is to try to "combine" or "integrate" all values of a function $f(x, y)$ for arguments $a \leq x \leq b$, $c \leq y \leq d$. That is, we consider the function on a rectangular domain

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\ c \leq y \leq d\}$$

which has the same role as the interval $I = [a, b]$ for the definite integral $\int_a^b f(x)\ dx$.

5.1.1 Double Integrals over Rectangles

The simplest extension of the idea of the definite integral $\displaystyle\int_a^b f(x)\ dx$ over interval $a \leq x \leq y$ is to try to "combine" or "integrate" all values of a function $f(x, y)$ for arguments $a \leq x \leq b$, $c \leq y \leq d$. That is, we consider the function on a rectangular domain

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\ c \leq y \leq d\}$$

which has the same role as the interval $I = [a, b]$ for the definite integral $\int_a^b f(x)\ dx$.

[1]openstax.org/books/calculus-volume-3/pages/5-introduction
[1]openstax.org/books/calculus-volume-3/pages/5-1-double-integrals-over-rectangular-regions

The Definite Integral and Area Under the Graph of $f(x)$. Recall the definition

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where the points x_i^* each lie in one of the n subintervals of interval $[a,b]$ of width $\Delta x = (b-a)/n$, and recall that this is motivated as a calculation of the area between the curve $y = f(x)$ and the x-axis over the interval $[a,b]$, at least if $f(x) > 0$, got from approximations with the sums of the area of many thin rectangles.

Likewise we will motivate the definition of the definite integral of $f(x,y)$ over region R as the volume of the solid region between the surface $z = f(x,y)$ and the x-y plane over this rectangle, and start with approximations with the sums of the volume of many thin thin rectangular boxes (”matchsticks”), each with base a small rectangle and height given by f.

Double Integrals and Volume under the Graph of $f(x,y)$. As before, the interval $a \le x \le b$ is divided into n equal subintervals $[x_{i-1}, x_i]$, $x_i = a + i\Delta x$, $\Delta x = (b-a)/n$. Likewise we divide interval $c \le y \le d$ is divided into m equal subintervals $[y_{j-1}, y_j]$, $y_j = a + j\Delta y$, $\Delta y = (d-c)/m$. This divides the region R into nm little rectangles each of area $\Delta A = \Delta x \Delta y$, and in each of these we choose a point (x_{ij}^*, y_{ij}^*) where f will be evaluated, to give the height of a thin rectangular box.

The volume of each matchstick is $f(x_{ij}^*, y_{ij}^*)\Delta A$, giving approximate volume for the whole solid as

$$V \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

See Example 1 in OSC3 Section 5.1[2].

The Midpoint Rule. Perhaps most natural choice of these points is to put each one at the middle of the rectangle:

$$x_{ij}^* = \bar{x}_i = \frac{x_{i-1} + x_i}{2} = a + (i - 1/2)\Delta x,$$
$$y_{ij}^* = \bar{y}_j = \frac{y_{j-1} + y_j}{2} = c + (j - 1/2)\Delta y,$$

so that

$$V \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f(\bar{x}_i, \bar{y}_j)\Delta A.$$

This is indeed useful if you wish to approximate volumes numerically.

Exact Volume as a Limit. Given the above approximation of the volume, it is natural to define the exact volume as the limit when the rectangles get ever smaller:

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

The Double Integral. The above idea is also useful in other contexts where the value of f need not be non-negative, so more generally we use:

Definition 5.1.1 For function f continuous on rectangle R, the double integral of f over rectangle R is

$$\iint_R f(x,y)\ dA = \lim_{m,n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*)\Delta A$$

[2]openstax.org/books/calculus-volume-3/pages/5-1-double-integrals-over-rectangular-regions

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This appears to depend on the choice of the values x_{ij}^* and y_{ij}^* for each choice of n and m. However, it can be proven that due to continuity, the result is the same for any choice.

In fact more is true: the spacing of the points x_i and y_j does not have to be equal, and the evaluation points do not need to be the midpoints, they just need to lie in the relevant small rectangle: $(x_{ij}^*, y_{ij}^*) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$.

What still matters is that all the intervals shrink to length zero: with

$$\begin{aligned} \Delta x_i &= x_i - x_{i-1}, & \Delta x &= \max_i \Delta x_i \\ \Delta y_j &= y_j - y_{j-1}, & \Delta y &= \max_j \Delta y_j \\ \Delta A_{ij} &= \Delta x_i \Delta y_j \end{aligned}$$

one can say that so when $\Delta x \to 0$ and $\Delta y \to 0$,

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij} \to \iint_R f(x,y)\ dA.$$

The sum appearing in this definition,

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$$

is called a double Riemann sum, and includes the volume approximation above as an example. Thus it is natural to define the volume of the solid over rectangle R and under the graph of a positive-valued function f to be the value of this double integral.

We will see in Subsection 5.1.3, p. 67 how such integrals can sometimes be evaluated using results we already know for integrals of functions of one variable.

5.1.2 Properties of Double Integrals Over Rectangles

Before learning how to evaluate such integrals, we note a few rather intuitive and familiar properties about sums, constant multiples and comparisons.

$$\iint_R f(x,y) + g(x,y)\ dA = \iint_R f(x,y)\ dA + \iint_R g(x,y)\ dA. \tag{5.1.1}$$

$$\iint_R kf(x,y))\ dA = k \iint_R f(x,y)\ dA \text{ for any } k. \tag{5.1.2}$$

$$\text{If } f(x,y) \le g(x,y) \text{ then } \iint_R f(x,y))\ dA \le \iint_R g(x,y)\ dA \tag{5.1.3}$$

when the first inequality holds for all points (x, y) in R.

Once we define double integrals over more general domains $D \subset \mathbb{R}^2$ in Section 5.2, p. 70 it will be noted in Subsection 5.2.4, p. 71 that these properties all still hold.

5.1.3 Iterated Integrals

The essential idea of this section is one formula, which in a sense does for double integrals what the Fundamental Theorems of Calculus did for definite integrals, by allowing evaluation using anti-derivatives.

Theorem 5.1.2 (Fubini). If f is continuous on the rectangle $R = [a, b] \times [c, d]$ then

$$\iint\limits_R f(x, y)\ dA = \int_{x=a}^{b} \left[\int_{y=c}^{d} f(x, y)\ dy \right] dx = \int_{y=c}^{d} \left[\int_{x=a}^{b} f(x, y)\ dx \right] dy$$

In particular, all the integrals here exist.
More generally, if f is bounded on R and is continuous except except on a finite number of smooth curves, all these integrals exist and are equal.

(Aside: Indicating the variable name for each integral with "$\int_{x=a}^{b}$" and "$\int_{y=c}^{d}$" is not mandatory, but it can help with clarity, so I recommend it. It also follows the pattern of summation notation, "$\sum_{i=1}^{n}$".)

The allowance for being discontinuous along curves will be important in the next section!

See Examples 2–7 and 9 in OSC3 Section 5.1[3].

The full proof of Fubini's Theorem is not given here; it is dealt with in a more advanced calculus course.

Such as Math 311: Advanced Calculus

However, it is fairly easy to verify in the special case of $\iint_R f(x)g(y)dA$, as will be seen soon.

The main objective here is to explain iterated integrals and see how to evaluate them using familiar techniques for definite integrals.

An iterated integral is one like the middle expression above:

$$\int_{x=a}^{b} \left[\int_{y=c}^{d} f(x, y)\ dy \right] dx$$

(which can be shortened as $\int_a^b \int_c^d f(x, y)\ dy\ dx$).

Evaluation of this iterated integral starts with the inside integral

$$\int_{y=c}^{d} f(x, y)\ dy$$

which is what is sometimes called a partial integral: like a partial derivative, we focus on the variable y that appears in the differential dy (and sometime in the limits of integration, as a reminder!), and treat x as a constant.

Evaluating Partial Integrals. For any given x value (in the range $a \le x \le b$) this definite integral gives a number, but since that number depends on x, the overall result is a function of x: $\displaystyle\int_{y=c}^{d} f(x, y)\ dy = G(x)$, and so

$$\int_{x=a}^{b} \int_{y=c}^{d} f(x, y)\ dy\ dx = \int_{a}^{b} G(x)\ dx$$

So once the partial integral is evaluated, what remains is a familiar definite integral in variable x.

This then gives a number (not a function of x or y), which by Fubini's Theorem is also the value of the double integral.

A Product Rule. One useful special case is when the integrand is a product of two functions, one in each variable:

$$\iint\limits_R f(x)g(y)\ dA = \int_{x=a}^{b} \int_{y=c}^{d} f(x)g(y)\ dy\ dx$$

[3] openstax.org/books/calculus-volume-3/pages/5-1-double-integrals-over-rectangular-regions

In the inner integral, x is effectively a constant, so $f(x)$ is a constant factor, and can be moved outside of this inner, y integral:

$$\int_{x=a}^{b}\int_{y=c}^{d} f(x)g(y)\ dy\ dx = \int_{x=a}^{b} f(x)\left[\int_{y=c}^{d} g(y)\ dy\right] dx$$

Now, the whole integral $\int_{y=c}^{d} g(y)\ dy$ gives a constant, so can be moved outside the outer x integral:

$$\int_{x=a}^{b} f(x)\left[\int_{y=c}^{d} g(y)\ dy\right] dx = \int_{y=c}^{d} g(y)\ dy \int_{x=a}^{b} f(x)\ dx$$

All together, the double integral of a product has been broken into a product of ”single” integrals:

Proposition 5.1.3 A Product Rule.

$$\iint\limits_{R} f(x)g(y)\ dA = \int_{x=a}^{b}\int_{y=c}^{d} f(x)g(y)\ dy\ dx = \int_{a}^{b} f(x)\ dx \int_{c}^{d} g(y)\ dy \tag{5.1.4}$$

Note well: this is about the only situation where the integral of a product is a product of integrals! Intuitively, it relies on the differential dA itself being like a product $dA = dx\,dy$, so that it can also be factored into a differential for use in each of the two one dimensional integrals at right.

Note that the double integral of any polynomial in two variables can be broken up into a sum of product integrals like this.

Note 5.1.4 At this point, OpenStax Calculus Volume 3, Section 5.1[4] defines the average value of a function over a rectangle; these notes instead postpone that to Equation (5.2.14) in Section 5.2, p. 70, where the idea can be applied to the average value over more general regions in the plane.

Iterated Integrals Over Other Regions. In an iterated integral, the dummy variable of the outer integral is fixed in the inner integral: that is, the inner integral is evaluated once for each value of the ”outer” variable.

Thus, it makes sense for the limits of the inner integral to depend on the that outer variable. This leads to integral of the forms

$$\int_{x=a}^{x=b}\int_{y=B(x)}^{y=T(x)} f(x,y)\ dy\ dx \quad \text{and} \quad \int_{y=c}^{y=d}\int_{x=L(y)}^{x=R(y)} f(x,y)\ dx\ dy.$$

In the first of these, the set of all x and y values in $\mathbb{R}^2$ involved are those between the vertical lines $x = a$ and $x = b$, bounded above and below by a top curve $y = T(x)$ and a bottom curve $y = B(x)$, and the inner integral gives

$$G(x) = \int_{y=B(x)}^{y=T(x)} f(x,y)\ dy.$$

Similarly, in the second, the set of all x and y values are those between the horizontal lines $y = c$ and $y = d$, bounded at the sides by a left curve $x = L(y)$ and a right curve $x = R(y)$.

Such integrals will be part of our strategy for evaluating integrals over more general regions.

Study Guide. Study Section 5.1 of OpenStax Calculus Volume 3[5]; in particular

- All the Definitions and Theorems.

[4]openstax.org/books/calculus-volume-3/pages/5-1-double-integrals-over-rectangular-regions
[5]openstax.org/books/calculus-volume-3/pages/5-1-double-integrals-over-rectangular-regions

- Iterated integrals, and Fubini's Theorem, which is the key to how most integral will actually be evaluated.
- Examples 2 to 9 (and the corresponding Checkpoints).
- One or several exercises from each of the the groups 11 & 12, 13–20, 21–34.

5.2 Double Integrals over General Regions

Revised on March 17.

References.

- Section 5.2 of OpenStax Calculus Volume 3[1].
- Section 15.3 of Calculus, Early Transcendentals by Stewart.

5.2.1 The Double Integral Over a Bounded Domain

Consider the problem of calculating the volume of the solid that sits above the bounded region D in the plane, and under the surface $z = f(x, y)$. Since the region is bounded, one can surround it with a rectangle R, and then extend the solid with a zero volume sheet covering the part of R outside D.

The resulting solid is the one over R and under the graph of the extended function

$$F(x,y) = \begin{cases} f(x,y) & \text{for } (x,y) \in D, \\ 0 & \text{for } (x,y) \text{ in } R \text{ but not in } D. \end{cases}$$

Thus it seems reasonable to define the volume of this solid as $\iint_R F(x,y)\ dA$, and to define the double integral of f over D as the integral of this extended function over the extended domain R:

Definition 5.2.1 Integral over a bounded domain D. For any rectangle $R = [a,b] \times [c,d]$ containing bounded region D

$$\iint\limits_D f(x,y)\ dA := \iint\limits_R F(x,y)\ dA \tag{5.2.1}$$

$\Diamond$

(Note the notation "$x := y$" for "x is defined to be y".)

Does This Integral Exist? However, at the boundary of the domain, F is likely to be discontinuous, jumping to value zero outside D, so does this integral even exist?

Fubini's Theorem gives an answer: so long as the boundary of D is made up of a finite set of smooth curves, F is continuous everywhere except on those curves, and that is enough for its double integral to exist (and to be computable using iterated integrals.)

Theorem 5.2.2 If f is continuous on a bounded domain D except on a finite number of smooth curves, and the boundary of D is made up of a finite number of smooth curves, then $\iint_D f(x,y)\ dA$ as defined in Equation (5.2.1) above exists.

5.2.2 Iterated Integrals over Non-rectangular Regions

Type I Regions. Double integrals are easiest when the domain can be described by inequalities such as a domain of Type I (which I also like to call "Type $dy\ dx$", because as we will see, that indicates the order in which the integrations are done):

$$D = \{(x,y) | a \le x \le b, B(x) \le y \le T(x)\}$$

[1] openstax.org/books/calculus-volume-3/pages/5-2-double-integrals-over-general-regions

This region is bounded above and below by curves $y = T(x)$, $y = B(x)$, and at left and right by vertical lines $x = a$ and $x = b$.

Thus the domain divides into a collection of vertical line segments, and the integral can be done first along each such line (integral in y with fixed x) and then integrating the resulting value over x:

$$\begin{aligned}\iint\limits_D f(x,y)\ dA &= \int\limits_a^b \int\limits_{B(x)}^{T(x)} f(x,y)\ dy\ dx \\ &= \int\limits_{x=a}^{b} \left[\int\limits_{y=B(x)}^{T(x)} f(x,y)\ dy \right] dx \end{aligned} \tag{5.2.2}$$

Note: I suggest labeling the variable in the limits of integration, as with $\int_{x=a}^{b}$, even when not strictly necessary; this can provide more clarity in some later calculations.

Type II Regions. Reversing the roles of x and y gives a domain of Type II (which I also call "Type $dx\ dy$"):

$$D = \{(x,y) | c \le y \le d, L(y) \le x \le R(y)\}$$

This is the collection of horizontal lines bounded below by $y = c$, above by $y = d$, to the left by curve $x = L(y)$,and to the right by $x = R(y)$, and the integral can be evaluated as

$$\iint\limits_D f(x,y) dA = \int\limits_{y=c}^{d} \left[\int\limits_{x=L(y)}^{R(y)} f(x,y)\ dx \right] dy \tag{5.2.3}$$

See Example 11 in OSC3 Section 5.2[2].

Theorem 5.2.3 (Fubini's Theorem, Strong Form). The above formulas (5.2.2) and (5.2.3) are valid for regions of Type I and Type II respectively.

See Examples 12 and 13 in OSC3 Section 5.2[3].

5.2.3 A Strategy for Evaluating Double Integrals

In setting up double integrals as iterated integrals, the first question to ask is:

Can the minimum and maximum allowed values of one variable be specified as functions of the other?

If so, the integral is done over the variable whose values are so described, and then over the other variable.

The integral done last must have numerical lower and upper limits, not depending on another variable.

Note well: If the upper or lower limit on one of the iterated integrals depends on another dummy variable of integration, this integral must be inside the integral over that other variable, because that other dummy variable only has a specific variable inside the integral over that variable.

5.2.4 Properties of Double Integrals

The three properties stated in Subsection 5.1.2, p. 67 for double integrals over a rectangle apply to all double integrals; they are worth updating here for the more general case, and adding some more.

1.
$$\iint\limits_D f(x,y) + g(x,y)\ dA = \iint\limits_D f(x,y)\ dA + \iint\limits_D g(x,y)\ dA. \tag{5.2.4}$$

[2]openstax.org/books/calculus-volume-3/pages/5-2-double-integrals-over-general-regions
[3]openstax.org/books/calculus-volume-3/pages/5-2-double-integrals-over-general-regions

2.
$$\iint_D kf(x,y))\ dA = k \iint_D f(x,y)\ dA \text{ for any } k. \tag{5.2.5}$$

3.
$$\text{If } f(x,y) \le g(x,y) \text{ then } \iint_D f(x,y))\ dA \le \iint_D g(x,y)\ dA \tag{5.2.6}$$

when the first inequality holds for all points (x,y) in D.

4. Further, a domain can be cut into pieces and the integrals over the pieces added:

   Proposition 5.2.4 If domain D is the union of two domains D_1 and D_2 with no overlap except possibly on their boundaries, then

$$\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA. \tag{5.2.7}$$

There is now a natural definition of the area of a region D as a double integral, by identifying it as the volume of the cylinder of height 1 over that region:

Definition 5.2.5 The area of region D is

$$A(D) = \iint_D 1\ dA \tag{5.2.8}$$

$\Diamond$

Exercise 5.2.6 Verify that this gives the area of the rectangular region $R = [a,b] \times [c,d]$ to be $A(R) = (b-a)(d-c)$.
The easy way to do this is via a simple iterated integral, but as a further exercise one can also calculate this using the original definition in terms of limits of sums, Definition 5.1.1, p. 66.

Exercise 5.2.7
1. Sketch the triangular region between the lines $y = 0$, $x = 3$ and $y = 2x$
2. Show that it is both of Type I and Type II, and set up iterated integral expressions for its area in the corresponding "$dx\ dy$" and "$dy\ dx$" forms.
3. Use one or both of these integrals to calculate its area.
4. Check your answer with geometry.

See Example 8 in OSC3 Section 5.1[4] and Example 18 in OSC3 Section 5.2[5].

Combined with the inequality of Double Integral Property 3, p. 72 in (5.2.6),

Proposition 5.2.8 if $m \le f(x,y) \le M$ for all points in D,

$$m\ A(D) \le \iint_D f(x,y)dA \le M\ A(D) \tag{5.2.9}$$

5.2.5 Domains of More Complicated Shapes: Divide and Conquer

Not all domains fit the types seen so far.

For other domain shapes, often the best hope is to seek a way to divide the domain into several pieces, each of one of the types above. For a region whose boundary consists of one or more smooth curves, vertical cuts through each point on the boundary with a vertical tangent will (usually) divide it into a collection of Type

[4]openstax.org/books/calculus-volume-3/pages/5-1-double-integrals-over-rectangular-regions
[5]openstax.org/books/calculus-volume-3/pages/5-2-double-integrals-over-general-regions

I regions. (Likewise cutting on each horizontal tangent will divide it into Type II regions, and sometimes a mixture of both can give simpler pieces.)

Then the total integral is the sum of the integral over these pieces, using Double Integral Property 4, p. 72 as in Equation (5.2.7).

Sometimes, the formulas for the curves at the top and bottom (and/or at left and right) change at certain points; then a few extra vertical (and/or horizontal) cuts at these transition points give smooth curves and a single formula for the limits on each piece.

This strategy is not quite universal —it assumes for example that there are only a finite number of boundary points with vertical (resp. horizontal) tangents— but it is widely useful.

Exercise 5.2.9 Draw some regions with "meandering boundaries" (not of either Type I or Type II) and see what these vertical and/or horizontal tangent line cuts do.

See Example 14 in OSC3 Section 5.2[6].

5.2.6 Changing the Order of Integration

Unlike with double integrals over rectangles, one cannot simply swap the order of the integrals in an iterated integral when the limits of the inner integral depend on the variable in the outer one. For example, the upper half of the unit disk is of both Type I and Type II and an integral over it is given by

$$\int_{x=-1}^{1} \int_{y=0}^{\sqrt{1-x^2}} dy\ dx \tag{5.2.10}$$

but swapping those limits gives

$$\int_{y=0}^{\sqrt{1-x^2}} \int_{x=-1}^{1} dx\ dy$$

which is nonsensical, because the limits on the outer integral cannot depend on the "inner", dummy variable x.

Instead, swapping requires several steps:

1. Check that the domain is of both Type I and Type II; if not, divide it into pieces tht are.
2. Work out the full range of possible values that can be attained by the inner variable over all possible values of the outer variable, by looking at the extreme possible values, which correspond to values on the boundary of the domain: these become the limits of the new outer integral.
3. For each value of this new outer integral variable, solve for the allowable values of the new inner one: these will now (in general) depend on the new outer variable, and become the limits of the new inner integral.

For this, it helps to express the domain in terms of inequalities, and then solve the corresponding equations to descibe the boundary of the domain: those equations describe the relation between x and y that applies at the two limits of the inner integral.

It can also be very helpful to sketch the domain!

Example 5.2.10 For an integral over the unit disk, the Type I form is as above in (5.2.10), and the domain is

$$D = \{(x, y)| -1 \leq x \leq 1, 0 \leq y \leq \sqrt{1-x^2}\}$$

(Note that the variables are always listed "from the outside inward".)
Considering all allowable value for x the highest allowable value of y is 1, occuring for $x = \pm 1$ and the lowest value is simply 0.
Thus $0 \leq y \leq 1$ is the overall range, giving the outer integral $\int_{y=0}^{1} \dots dy$

[6]openstax.org/books/calculus-volume-3/pages/5-2-double-integrals-over-general-regions

Next, the boundary consists of all points at the limits of the inner integral:

$$y = 0 \text{ and } y = \sqrt{1-x^2}$$

Each of these can be solved for x:

1. $y = 0$ adds no constraints on x other than the overall restriction $-1 \le x \le 1$.
2. $y = \sqrt{1-x^2}$ gives the extremities $x^2 = 1 - y^2$ and thus the solutions $x = -\sqrt{1-y^2}$ and $x = \sqrt{1-y^2}$, limiting points in the domain to $-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$.
3. The combination of these restriction is just the latter, so for a given value of y, the values of x in the domain are $-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$
4. That is, the domain is

$$D = \{(x,y) | -1 \le y \le 1, -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}\}$$

which means that the Type II iterated integral form (slicing the half-disk horizontally) is

$$\int_{y=0}^{1} \int_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\ dy \tag{5.2.11}$$

□

Exercise 5.2.11 Make sketches to illustrate this: first the vertical slicing of Type I and then the horizontal slicing of Type II.

Example 5.2.12 The iterated integral

$$\int_{y=0}^{1} \int_{x=y}^{1} \sin(x^2)\ dx\ dy \tag{5.2.12}$$

can not be evaluated in this order in terms of elementary functions, because there is no elementary anti-derivative of $\sin(x^2)$. However, it can be evaluated by first changing the order of integration.

Firstly, $y \le x \le 1$ and $0 \le y \le 1$ gives the range of all possible values of x as $0 \le x \le 1$.

Then for a given value of x, the extremes of y come from $y \le x$ and $y = 0$: that is, $0 \le y \le x$, giving the new iterated integral form

$$\int_{x=0}^{1} \int_{y=0}^{x} \sin(x^2)\ dy\ dx, = \int_{x=0}^{1} \left[ ] \int_{y=0}^{x} dy \right] \sin(x^2)\ dx. \tag{5.2.13}$$

The inner integral is now easy:

$$\int_{y=0}^{x} dy = \Big[y \Big]_{y=0}^{x} = [x - 0] = x,$$

so the iterated integral (5.2.13) gives

$$\int_{x=0}^{1} x \sin(x^2)\ dx$$

Finally, the substitution $u = x^2$, $du = 2xdx$ gives

$$\int_{u=0}^{1} \sin(u) \frac{du}{2} = \left[\frac{-\cos(u)}{2} \right]_{u=0}^{1} = \frac{-\cos(1) - (-\cos(0))}{2} = \frac{1 - \cos(1)}{2}.$$

□

See Examples 15 and 16 in OSC3 Section 5.2[7].

[7] openstax.org/books/calculus-volume-3/pages/5-2-double-integrals-over-general-regions

5.2.7 The Average Value of a Function over a Region

The average value of a function $f(x)$ over interval $[a,b]$ was defined as the definite integral over that interval divided by the "size" of the interval, measured by its length $b-a$. This is the natural choice of size, because it ensures that the average of a constant function equals its value.

Similarly, it makes sense to define the average value of a function over region R as the integral divided by the area $A(R)$ of the region:

$$f_{ave} = \overline{f} = \frac{\iint\limits_R f(x,y)\ dA}{A(R)} = \frac{\iint\limits_R f(x,y)\ dA}{\iint\limits_R\ dA} \tag{5.2.14}$$

For the case of rectangle $R=[a,b]\times[c,d]$, $A(R)=(b-a)(d-c)$, so

$$f_{ave} = \overline{f} = \frac{\iint\limits_R f(x,y)\ dA}{(b-a)(d-c)}.$$

Now that we have some practice describing domains in both Type I and Type II styles, also review Examples 17, 18 and 19 in OSC3 Section 5.2[8].

5.2.8 Optional Topic: Improper Double Integrals

We can make sense of an integral over an infinite "corner" region like

$$D = \{(x,y)|a \le x \le \infty, c \le y \le \infty\}$$

as a double limit in both the upper limits:

$$\iint\limits_D f(x,y)\ dA = \lim_{(b,d)\to(\infty,\infty)} \int\limits_{x=a}^{b}\int\limits_{y=c}^{d} f(x,y)\ dy\ dx \tag{5.2.15}$$

$$= \lim_{(b,d)\to(\infty,\infty)} \int\limits_{y=c}^{d}\int\limits_{x=a}^{b} f(x,y)\ dx\ dy \tag{5.2.16}$$

Example 5.2.13 For domain $D=[0,\infty)\times[0,\infty)$

$$\iint_D xye^{-(x^2+y^2)}dA = \lim_{(b,d)\to(\infty,\infty)} \int_{x=0}^{b}\int_{y=0}^{d} xye^{-(x^2+y^2)}\ dy\ dx$$

$$= \left(\lim_{b\to\infty}\int_0^b xe^{-x^2}\ dx\right)\left(\lim_{d\to\infty}\int_0^d ye^{-y^2}\ dy\right) = \left(\lim_{b\to\infty}\int_0^b xe^{-x^2}\ dx\right)^2$$

The single integral can be evaluated using the substitution $u=x^2$, $du=2xdx$ as

$$\frac{1}{2}\lim_{B\to\infty}\int_0^B e^{-u}\ du = \frac{1}{2}\lim_{B\to\infty}[-e^{-u}]_{u=0}^{B} = \frac{1}{2}\lim_{B\to\infty}(-e^{-B}+e^{-0}) = 1/2,$$

so the double integral is $\displaystyle\iint_D xye^{-(x^2+y^2)}dA = 1/4.$ □

Study Guide. Study Section 5.2 of OpenStax Calculus Volume 3[9] up to the subsection on Calculating Volumes, Areas, and Average Values; that is, omitting the final topic of Improper Double Integrals (However, study it if interested: such integrals come up in physics and mathematical probability.) In particular

[8]openstax.org/books/calculus-volume-3/pages/5-2-double-integrals-over-general-regions
[9]openstax.org/books/calculus-volume-3/pages/5-2-double-integrals-over-general-regions

- Theorems 3 to 7.
- The Definitions of Type I and Type II regions, and how to visualize them.
- The strategy for changing the order of integration, by converting between a ”Type I description” and a ”Type II description”.
- Examples 11–18, and the Checkpoints following each.
- Exercises 61–63 (They go together.)
- Exercises 66 and 67. (They go together.)
- Exercise 72 or 73.
- One or more of Exercises 74 to 77.
- Exercise 78 and/or 79.
- One or more of Exercises 80 to 85.

5.3 Double Integrals in Polar Coordinates

References.

- Section 5.3 of OpenStax Calculus Volume 3[1].
- Calculus, Early Transcendentals by Stewart, Section 15.4.

Introduction. After rectangles, perhaps the single most common shape for a domain in two dimensions is a disk, and other domains with some radial symmetry are common too.

Calculations are often easiest if one uses an approach that emphasizes the symmetry of the domain: using polar coordinates based at the center of the disk. It is simple enough to chose coordinates with origin at the center of the disk, so we work with cartesian coordinates x and y and polar coordinates r and θ as seen in Section 1.3, p. 7:

$$x = r\cos\theta, y = r\sin\theta, \quad \text{so that} \quad r^2 = x^2 + y^2, \tan\theta = y/x.$$

5.3.1 Disks, Annuli, Sectors, and Polar Rectangles

The disk of radius R, $\{(x, y) : x^2 + y^2 \leq R^2\}$, is described in polar coordinates as

$$D = \{(r, \theta) | 0 \leq r \leq R, 0 \leq \theta \leq 2\pi\},$$

which is a type of polar rectangle.

In general, a polar rectangle is a set

$$\{(r, \theta) | a \leq r \leq b, \alpha \leq \theta \leq \beta\}, \ , \ , \ , \text{constants.} \tag{5.3.1}$$

See Example 24 in OSC3 Section 5.3[2].

This includes annuli and sectors of disks:

- An annulus is a polar rectangle covering all angles: $\alpha = 0$, $\beta = 2\pi$.
- A sector is a polar rectangle that includes the origin: $a = 0$.

[1]openstax.org/books/calculus-volume-3/pages/5-3-double-integrals-in-polar-coordinates
[2]openstax.org/books/calculus-volume-3/pages/5-3-double-integrals-in-polar-coordinates

5.3.2 Integration Over a Polar Rectangle

Integration over a polar rectangle can be done by starting again from approximations, as in Section 5.1, p. 65. (In Section 5.7, p. 85 we will see another method: changing coordinates, as with substitution.)

Consider approximations using the midpoint rule.

First subdivide the r and θ values into

$$a = r_1 < r_2 < \cdots < r_n = b \text{ and } \alpha = \theta_1 < \theta_2 < \cdots < \theta_m = \beta$$

with spacing $r_i - r_{i-1} = \Delta r$, $\theta_i - \theta_{i-1} = \Delta\theta$.

Then find the midpoints $r_i^* = (r_{i-1} + r_i)/2$, $\theta_i^* = (\theta_{i-1} + \theta_i)/2$: this divides the domain into many small polar rectangles

$$R_{ij} = \{(r, \theta) | r_{i-1} \le r \le r_i, \theta_{j-1} \le \theta \le \theta_j\},$$

each with a "midpoint" (r_i^*, θ_j^*).

Next, we can approximate the integral of $f(r, \theta)$ over each small polar rectangle as $f(r_i^*, \theta_j^*)A(R_{ij})$, with $A(R_{ij})$ the area of each polar rectangle.

The area of a sector is $\{(r, \theta)|0 \le r \le b, \alpha \le \theta \le \beta\}$ is $(\beta - \alpha)b^2/2$, so by subtraction,

$$A(R_{ij}) = (\theta_i - \theta_{i-1})(r_i^2 - r_{i-1}^2)/2 = \Delta\theta r_i^* \Delta r$$

Summing all these approximations and taking the limit leads to

$$\iint_D f(r, \theta) dA = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(r_i^*, \theta_j^*) r_i^* \, \Delta r \, \Delta\theta.$$

This is simply the double integral in variables r and θ of function $f(r, \theta)r$:

$$\iint_D f(r, \theta) dA = \int_{\theta=\alpha}^{\beta} \int_{r=a}^{b} f(r, \theta) \, r \, dr \, d\theta \tag{5.3.2}$$

In effect, the infinitesimal area is now expressed as $dA = r\, dr\, d\theta$ with polar coordinates, in place of $dA = dx\, dy$ with cartesian coordinates.

See Examples 25, 26 and 27 in OSC3 Section 5.3[3].

5.3.3 Integrals in Polar Coordinates Over Other Domains

Polar coordinates can be useful for integrals over domains of other shapes, like annuli and sectors. Just as with the Type I and Type II domains seen in Section 5.2, p. 70 with cartesian coordinates, there are two special cases where it is worth converting to polar coordinates:

- For a domain $D = \{(r, \theta)|\alpha \le \theta \le \beta, r_1(\theta) \le r \le r_2(\theta)\}$,

$$\iint_D f(x, y) \, dA = \int_{\theta=\alpha}^{\beta} \int_{r=r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) \, r \, dr \, d\theta \tag{5.3.3}$$

- For a domain $D = \{(r, \theta)|a \le r \le b, \theta_1(r) \le \theta \le \theta_2(r)\}$,

$$\iint_D f(x, y) \, dA = \int_{r=a}^{b} \int_{\theta=\theta_1(r)}^{\theta_2(r)} f(r\cos\theta, r\sin\theta) \, d\theta \, r \, dr \tag{5.3.4}$$

[3] openstax.org/books/calculus-volume-3/pages/5-3-double-integrals-in-polar-coordinates

See Example 28 in OSC3 Section 5.3[4].

The former is more common in practice.

5.3.4 Calculating Areas and Volumes using Polar Coordinates

One basic use of double integrals is computing areas, and this sometimes involved polar integrals: if a region D is a polar rectangle as in Equation (5.3.1), or a more general polar analogue of a Type I or Type II region as in Equations (5.3.3) and (5.3.4), The area is $A(D) = \iint_D dA$ is no given by the appropriate choice of

$$A(D) = \int_{\theta=\alpha}^{\beta} \int_{r=r_1(\theta)}^{r_2(\theta)} r\ dr\ d\theta$$

or

$$A(D) = \int_{r=r_1}^{r_2} \int_{\theta=\theta_1(r)}^{\theta_2(r)} d\theta\ r\ dr$$

For area calcuations, see Examples 33 and 34 in OSC3 Section 5.3[5].

Also, the volume over such a region between the x-y plane and the surface $z = f(x, y)$ is as usual given by the corresponding polar integral of function f.

For volume calcuations, see Examples 29 to 32 in OSC3 Section 5.3[6].

5.3.5 Optional Topic: Improper Double Integrals Using Polar Coordinates

Improper double integrals over the whole plane of radially symmetric functions (and even some others with a still simple polar form) are often best evaluated using polar coordinates. This requires a variant of the ”expanding rectangles” approach seen in Subsection 5.2.8, p. 75, instead using expanding discs to cover the whole plane: with $B_R = B_R((0, 0))$ the disc of radius R centered at the origin

$$\iint\limits_{\mathbb{R}^2} f(x, y) dA = \lim_{R\to\infty} \iint_{B_R} f(x, y) dA = \lim_{R\to\infty} \int_{r=0}^{R} \int_{\theta=0}^{2\pi} f(x, y)\ r\ dr\ d\theta \tag{5.3.5}$$

One interesting application of this is evaluating the integral associated with the Gaussian distribution,

$$I = \int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi} \tag{5.3.6}$$

First,

$$I = \int_{-\infty}^{\infty} e^{-x^2}\ dx, = \int_{-\infty}^{\infty} e^{-y^2}\ dy$$

Multiplying these two forms and using the product rule backwards:

$$\begin{aligned} I^2 &= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} e^{-x^2} e^{-y^2}\ dy\ dx \\ &= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\ dA \\ &= \int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} e^{-r^2} r\ dr\ d\theta \\ &= \int_{r=0}^{\infty} e^{-r^2} r\ dr\ \int_{\theta=0}^{2\pi} d\theta \end{aligned}$$

[4] openstax.org/books/calculus-volume-3/pages/5-3-double-integrals-in-polar-coordinates
[5] openstax.org/books/calculus-volume-3/pages/5-3-double-integrals-in-polar-coordinates
[6] openstax.org/books/calculus-volume-3/pages/5-3-double-integrals-in-polar-coordinates

The θ integral is simple: $\int_{\theta=0}^{2\pi} d\theta = 2\pi$. The improper r integral is

$$\begin{aligned}
\int_{r=0}^{\infty} e^{-r^2} r\ dr &= \lim_{R\to\infty} \int_{r=0}^{R} e^{-r^2} r\ dr \\
&= \lim_{R\to\infty} \int_{u=0}^{R^2} e^{-u} \frac{du}{2} \quad \text{using } u = r^2,\ du = 2rdr \\
&= \lim_{R\to\infty} \left[\frac{-e^{-u}}{2} \right]_{u=0}^{R^2} \\
&= \lim_{R\to\infty} \left(\frac{-e^{-R^2} + e^0}{2} \right) = \frac{1}{2}
\end{aligned}$$

Thus $I^2 = \frac{1}{2} 2\pi = \pi$, confirming Equation (5.3.6).

An aside on the normal (Gaussian) distribution of statistics. A change of variables gives the form usually seen in statistics:

$$\int_{-\infty}^{\infty} e^{-x^2/2}\ dx = \sqrt{2\pi} \tag{5.3.7}$$

so that

$$\int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = 1 \tag{5.3.8}$$

meaning that $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is a probability distribution: a non-negative function with integral over its whole domain being one.

In fact this is the standard normal distribution, or unit normal distribution; the reason for the factor of 1/2 can be verified in this (somewhat challenging) exercise.

Exercise 5.3.1 The standard deviation of the standard normal distribution is one. Confirm that the standard normal distribution $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ has variance 1 and therefore its standard deviation (the square root of the variance) is also 1. That is, verify

$$\int_{-\infty}^{\infty} x^2 e^{-x^2/2}\ dx = \sqrt{2\pi}$$

The mean of the standard normal distribution,

$$\mu := \frac{\int_{-\infty}^{\infty} x\phi(x)\ dx}{\int_{-\infty}^{\infty} \phi(x)\ dx} = \int_{-\infty}^{\infty} x\phi(x)\ dx = \frac{\int_{-\infty}^{\infty} xe^{-x^2/2}\ dx}{\sqrt{2\pi}}$$

is zero due to the oddness of the integrand, and then a change of variables shows that the more general distribution

$$\phi_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-((x-\mu)/\sigma)^2/2} \tag{5.3.9}$$

has integral one and so it is also probability density, with mean

$$\int_{-\infty}^{\infty} x\phi_{\mu,\sigma}(x)\ dx = \mu$$

and standard deviation

$$\int_{-\infty}^{\infty} x^2\phi_{\mu,\sigma}(x)\ dx = \sigma$$

This is the Normal Distribution of mean μ and standard deviation σ, sometimes denoted $\mathcal{N}(\mu, \sigma^2)$; it is probably the most important distribution in mathematical statistics.

Study Guide. Study Section 5.3 of Calculus Volume 3[7]; in particular

[7] openstax.org/books/calculus-volume-3/pages/5-3-double-integrals-in-polar-coordinates

- The definition of a double integral in polar coordinates.
- Theorem 5.8, expressing these double integrals as iterated integrals.
- Examples 24–32 and the Checkpoints following each.
- One or several exercises from each of the following ranges: 122–127, 128–131, 144–147, 148–151, 153–157, 158–159.

It might help to review polar coordinates in Section 7.3 of Section 5.3 of Calculus Volume 2[8], and their application to computing areas in the following Section 5.4[9].

5.4 Triple Integrals

Revised on March 17.

References.

- Section 5.4 of OpenStax Calculus Volume 3[1].
- Section 15.6 of Calculus, Early Transcendentals by Stewart.

Introduction. The ideas used to define double integrals and then evaluate then in terms of iterated integrals can be extended to triple integrals for functions of three variables: as usual, getting from one dimension to two has taken care of most the hard work and new ideas.

5.4.1 Triple Integrals over a Box

In place of rectangles in the plane, we start with integrals over boxes in space:

$$B = \{(x, y, z) | a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\} = [a, b] \times [c, d] \times [r, s]$$

Then we divide this into many small boxes, by first dividing the ranges of x, y and z values into sub-intervals.

Using l x-subintervals of length $\Delta x = (b - a)/l$, m y-subintervals of length $\Delta y = (d - c)/m$, n z-subintervals of length $\Delta z = (s - r)/n$, the boxes are

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

with

$$x_i = a + i\Delta x, y_j = c + j\Delta y, z_k = r + k\Delta z.$$

Choosing a point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in each box, the integral is approximated by the triple Riemann sum for function f on box B:

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f((x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V, \quad \Delta V = \Delta x \Delta y \Delta z.$$

One natural choice of the evaluation points is again the mid-points

$$(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) = (\bar{x}_i, \bar{y}_j, \bar{z}_k) = \left(\frac{x_{i-1} + x_i}{2}, \frac{y_{j-1} + y_j}{2}, \frac{z_{k-1} + z_k}{2} \right).$$

[8] openstax.org/books/calculus-volume-2/pages/7-3-polar-coordinates
[9] openstax.org/books/calculus-volume-2/pages/7-3-polar-coordinates
[1] openstax.org/books/calculus-volume-3/pages/5-4-triple-integrals

Definition 5.4.1 Triple Integral over a Box. The triple integral of f over box B is

$$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f((x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V,$$

if this limit exists. ◊

As you can probably guess, triple integrals can be expressed in terms of three nested integrals, over the three variables:

Theorem 5.4.2 Fubini's Theorem for Triple Integrals. If f is continuous on the rectangular box $[a,b]\times[c,d]\times[r,s]$ then

$$\iiint_B f(x,y,z)\,dV = \int_{x=a}^{b}\int_{y=c}^{d}\int_{z=r}^{s} f(x,y,z)\ dz\ dy\ dx \tag{5.4.1}$$

In particular, this triple integral exists, as do all the integrals on the right-hand side. Also, the order in which the variables of f are listed does not matter, so the same result is given by any of the six possible orderings of the three integrals.

For most practical purposes, this iterated integral could be used as the definition of the triple integral over a rectangular box.

See Examples 36 and 37 in OSC3 Section 5.4[2].
Note that Example 37 shows the natural extension of the product rule to triple integrals. Also, Example 36 can be broken up into several applications of that product rule, as can any integral of a polynomial.

5.4.2 Triple Integrals over Bounded Regions

For a region E in $\mathbb{R}^3$ that is bounded, and with its boundary a finite collection of smooth surfaces, we can define the triple integral with the same method as in Subsection 5.2.1, p. 70: define the function outside region E to have value zero, and integrate this extended function over a box containing E.

In practice, the most manageable regions are ones akin to the Type I and Type II regions in the plane. Just as Type I regions were described as "Type *dy-dx*" in Subsection 5.2.2, p. 70, indicating the order in which the iterated integrals can easily be done, this jargon can be extended to three dimensional regions as "Type *dz-dy-dx*" (as with the order in Equation (5.4.1)) and so on.

That is, a type *dz-dy-dx* region E is one of the form

$$E = \{(x,y,z)|a \le x \le b, g_1(x) \le y \le g_2(x), u_1(x,y) \le z \le u_2(x,y)\} \tag{5.4.2}$$

(As described in the text, this is a type I region in space, sitting over a type I region D in the plane.)

5.4.3 Iterated Integral Form for Type *dz-dy-dx* Regions in Space

The triple integral over such a region is given by

$$\iiint_E f(x,y,z)\,dV = \int_{x=a}^{b}\int_{y=g_1(x)}^{g_2(x)}\int_{z=u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

See Examples 38 and 39 in OSC3 Section 5.4[3].

Example 5.4.3 Integration over a Ball. The ball $E = \{(x,y,z)|x^2+y^2+z^2 \le R^2\}$, can be described by the inequalities $-R \le x \le R$, $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$, $-\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}$.

[2]openstax.org/books/calculus-volume-3/pages/5-4-triple-integrals
[3]openstax.org/books/calculus-volume-3/pages/5-4-triple-integrals

Thus, the integral over such a ball is

$$\iiint\limits_{x^2+y^2+z^2\le R^2} f(x,y,z)\,dV = \int\limits_{x=-R}^{R} \int\limits_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int\limits_{z=-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$

(or any of five similar reorderings).

However, we will soon see a better approach in Section 5.5, p. 83, explointng the radial symmetry.

□

5.4.4 Changing the Order of Integration

With iterated triple integrals there are six possible order for the integrals, and just as seen with Type I and Type II regions for double integrals, not all are possible for all domains. Also, some choices of order might make the differnce between an easier integral, a harder one, or one that cannot not be evaluated using known antiderivatives.

Thus, it can be useful to seek to change the order of integrals, and again one key is solving equations for the boundary of the domain, which then give the inequalities describing the limits of each integral.

An important point is that, as in Equation (5.4.2) and in Equations (5.2.2) and (5.2.3) fot Type I and Type II double integrals, the limits of integration of each integral might depend on the variable of an integral ”outside” it (integral sign to the left) but cannot refer to the variable in another integral ”inside it” (integral sign to the right).

See Examples 40 and 41 in OSC3 Section 5.4[4].

5.4.5 Volumes and Averages

Just as area can be computed in terms of a double integral, a bounded region E in $\mathbb{R}^3$ has volume

$$V(E) = \iiint\limits_{E} 1\ dV, \text{ often abbreviated } \iiint\limits_{E}\ dV$$

and it should be no surprise that the average of a function $f(x,y,z)$ over region E is defined as

$$\bar{f} = \frac{\iiint_E f(x,y,z)\,dV}{V(E)}.$$

See Example 42 in OSC3 Section 5.4[5].

Study Guide. Study Section 5.4 of Calculus Volume 3[6]; in particular

- The definition of a triple integral.
- Fubini’s Theorem for triple Integrals (Theorem 5.9), expressing triple integrals as iterated integrals.
- Theorem 5.10, expressing a triple integral over a general region in terms of iterated integrals.
- Examples 36–39 and above all, Examples 40 and 41. (Plus as usual the Checkpoints following each).
- One or several exercises from each of the following ranges: 181–184, 185–188, 191–194, 195–198, and 211 and/or 212.

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[4] openstax.org/books/calculus-volume-3/pages/5-4-triple-integrals
[5] openstax.org/books/calculus-volume-3/pages/5-4-triple-integrals
[6] openstax.org/books/calculus-volume-3/pages/5-4-triple-integrals

5.5 Triple Integrals in Cylindrical and Spherical Coordinates

Revised on March 18.

References.

- Section 5.5 of OpenStax Calculus Volume 3[1].
- Sections 15.7 and 15.8 of Calculus, Early Transcendentals by Stewart.

5.5.1 Preview: Double Integrals in Polar Coordinates Revisited

To evaluate double integrals in cartesian coordinates x,y and in plane polar coordinates r,θ, we use the iterated integral forms

$$\iint\limits_D f\,dA = \iint\limits_D f(x,y)\,dx\,dy = \iint\limits_D f(r\cos\theta, r\sin\theta)r\,dr\,d\theta$$

The basic geometrical idea is embodied in the formula

$$dA = dx\,dy = r\,dr\,d\theta \tag{5.5.1}$$

which intuitively speaking describes the area of an infinitesimal roughly rectangular region in terms of the infinitesimal ranges of the coordinates over that region.

Let us re-derive these formulas for the infinitesimal area dA in a heuristic, geometrical way, to prepare for the 3D versions.

For cartesian coordinates, it is clear that increasing x by up to dx and y by up to dy sweeps out a rectangle of sides dx and dy and thus of area $dA = dx\,dy$.

For plane polar coordinates, changing r by dr (without changing θ) moves to a point a distance dr away, along a line through the origin, but instead changing the angle θ by amount $d\theta$ (without changing r) moves a distance $r\,d\theta$ along a circle around the origin, so at right angles to the first movement.

Thus increasing r and θ by amounts of up to dr and $d\theta$ respectively sweeps out a roughly rectangular region of sides dr and $rd\theta$, and so of area $dA = r\,dr\,d\theta$.

5.5.2 Triple Integrals in Cylindrical Coordinates

Recall cylindrical coordinates, introduced in Subsection 2.7.1, p. 26, and in particular the change of coordinates formulas (2.7.1)

To express triple integrals in terms of three iterated integrals in these coordinates r, θ and z, we need to describe the infinitesimal volume dV in terms of those coordinates and their differentials dr, $d\theta$ and dx.

This is easy using the above results for plane polar coordinates, because varying the r and θ coordinates by amounts of up to dr and $d\theta$ again sweep out a horizontal area of $r\ dr\ d\theta$, and varying z by up to dz creates a "cylinder" of height dz over this region, thus having volume

$$dV = r\ dr\ d\theta\ dz. \tag{5.5.2}$$

For a domain E of "Type dr-$d\theta$-dz" — meaning one described by inequalities $a \le z \le b$, $f_1(z) \le \theta \le f_2(z)$, $g_1(\theta, z) \le r \le g_2(\theta, z)$ — the triple integral in cylindrical coordinates is given by "summing" over these pieces:

$$\iiint\limits_E f(x,y,z)dV =$$

[1]openstax.org/books/calculus-volume-3/pages/5-5-triple-integrals-in-cylindrical-and-spherical-coordinates

$$\int_{z=a}^{b} \int_{\theta=f_1(z)}^{f_2(z)} \int_{r=g_1(\theta,z)}^{g_2(\theta,z)} f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz. \tag{5.5.3}$$

See Example 43 in OSC3 Section 5.5[2].

Other domain shapes can be handled with other orders of the integrals. For the rest of this section details of domains and the limits of the three iterated integrals are omitted, and dV is loosely replaced by the form in Equation (5.5.2), or any of the five other versions got by reordering the differentials. For example,

$$\iiint\limits_E f(x, y, z) dV = \iiint\limits_E f(r\cos\theta, r\sin\theta, z)\ dz\ rdr\ d\theta.$$

See Example 44 in OSC3 Section 5.5[3].

See Examples 45 and 46 in OSC3 Section 5.5[4].
Note that in each of these volume calculations there is another way to do things, due to the radial symmetry: factoring out the θ integral first to get a factor of 2π, leaving just a double integral.

5.5.3 Triple Integrals in Spherical Coordinates

Next we revisit spherical coordinates, introduced in Subsection 2.7.2, p. 26, and in particular the change of coordinates formulas (2.7.4) and (2.7.5)

First, consider the volume of the small region near the point with spherical coordinates (ρ, ϕ, θ) that is swept out by varying the coordinates by up to $d\rho$, $d\phi$ and $d\theta$ respectively.

- Changing ρ by $d\rho$ moves a distance $d\rho$ along a line through the origin, perpendicular to the sphere of radius ρ.
- Changing ϕ by $d\phi$ moves on that sphere "southward" along a curve of longitude, of radius ρ, so the distance moved is $\rho d\phi$.

  This movement is perpendicular to the above movement due to changing ρ.
- Changing θ by $d\theta$ moves on that sphere "eastward" along a circle of latitude of radius $r = \rho\sin\phi$, so the distance moved is $r\, d\theta = \rho\sin\phi\, d\theta$.

  This movement is perpendicular to both of the previous movements.

Altogether, changes in the three coordinates sweep our a region that is roughly a rectangular box of dimensions $d\rho$ by $\rho d\phi$ by $\rho\sin\phi\, d\theta$, and of infinitesimal volume

$$dV = d\rho\, (\rho\, d\phi)\, (\rho\sin\phi\, d\theta) = \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta \tag{5.5.4}$$

and so

$$\begin{aligned} &\iiint\limits_E f(x, y, z)\ dV = \\ &\iiint\limits_E f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\ d\rho\ d\theta\ d\phi \end{aligned} \tag{5.5.5}$$

Again there can be reordering of the three differentials to get various iterated integrals appropriate to various shapes of domain.

See Examples 47 to 51 in OSC3 Section 5.5[5].

Study Guide. Study Section 5.5 of OpenStax Calculus Volume 3[6]; in particular

[2]openstax.org/books/calculus-volume-3/pages/5-5-triple-integrals-in-cylindrical-and-spherical-coordinates
[3]openstax.org/books/calculus-volume-3/pages/5-5-triple-integrals-in-cylindrical-and-spherical-coordinates
[4]openstax.org/books/calculus-volume-3/pages/5-5-triple-integrals-in-cylindrical-and-spherical-coordinates
[5]openstax.org/books/calculus-volume-3/pages/5-5-triple-integrals-in-cylindrical-and-spherical-coordinates
[6]openstax.org/books/calculus-volume-3/pages/5-5-triple-integrals-in-cylindrical-and-spherical-coordinates

- The two new forms of Fubini's Theorem.
- All the Examples (and the Checkpoints following each).
- One or several exercises from each of the following ranges and pairs: 241–246, 249–252, 253–256, 267 & 268, 269–272.

5.6 Calculating Centers of Mass and Moments of Inertia (Omitted)

References.

- Section 5.5 of OpenStax Calculus Volume 3[1].
- Section 15.6. of Calculus, Early Transcendentals by Stewart.

Physics: Mass and Center of Mass. Two related physical quantities are the mass and center of mass of an object occupying the region E with density $\rho(x, y, z)$. The mass is

$$m = \iiint\limits_E \rho(x, y, z)\, dV$$

The center of mass is the point $(\bar{x}, \bar{y}, \bar{z})$ which is the "density weighted average" location of the mass of a body, with components given by

$$\bar{x} = \frac{\iiint_E x\rho(x, y, z)\, dV}{m} = \frac{\iiint_E x\rho(x, y, z)\, dV}{\iiint_E \rho(x, y, z)\, dV}, \quad \bar{y} = \frac{\iiint_E y\rho(x, y, z)\ dV}{m}, \quad \bar{z} = \frac{\iiint_E z\rho(x, y, z)\, dV}{m}.$$

Physics: Moments of Inertia. Coming later.

5.7 Change of Variables in Multiple Integrals

Revised on March 20.

References.

- Section 5.7 of OpenStax Calculus Volume 3[1].
- Section 15.9 of Calculus, Early Transcendentals by Stewart.

Introduction. The main new challenge in multiple integrals is dealing with domains of shapes other that rectangles.

Iterated integrals with the limits of "inner" integrals depending on "outer" variables help in some cases, as do polar, cylindrical and spherical coordinates, but we now develop a more general strategy.

As usual we deal with two-dimensional integrals befoer going on to thre dimensions, but in fact we first revisit one-dimensional integrals with modified notation that fits better with what happens in multiple dimensions.

5.7.1 Changing Variables in 1D Integrals

Inverse Substitution. To extend changes of variables in multiple integrals beyond those seen for polar, cylindrical and spherical coordinates, let us first review and rephrase the 1D version.

[1] openstax.org/books/calculus-volume-3/pages/5-6-calculating-centers-of-mass-and-moments-of-inertia
[1] openstax.org/books/calculus-volume-3/pages/5-7-change-of-variables-in-multiple-integrals

Consider a function $f(x)$ defined on an interval $[a, b]$. The idea is to change from the original variable x to a new variable u related by a function $x = g(u)$ where g is one-to-one on the interval $[a, b]$, so that g is either increasing or decreasing there.

Note: we express the "old" variable as a function of the "new" one, the inverse substitution approach seen with the use of trigonometric substitutions like $x = a \sin\theta$ for integrals in involving $\sqrt{a^2 - x^2}$; see Section 3.3 of OpenStax Calculus Volume 2[2].

Let $F(x)$ be an anti-derivative of $f(x)$, so that $F'(x) = f(x)$. Then

$$\frac{d}{du}F(g(u)) = F'(g(u))g'(u) = f(g(u))g'(u).$$

Integrating with respect to u gives (in reversed order)

$$\int f(g(u))g'(u)\,du = F(g(u)) + C = F(x) + C$$

The last expression is the indefinite integral of f, so reversing order again

$$\int f(x)\,dx = \int f(g(u))g'(u)\,du. \tag{5.7.1}$$

Changing the Domain in 1D Definite Integrals. When one changes variables in a definite integral, the domain is changed. In the 1D case, this is just changing the upper and lower limits of integration:

$$\int_{x=a}^{b} f(x)\,dx = \int_{u=c}^{d} f(g(u))g'(u)\,du, \quad a = g(c),\ b = g(d) \tag{5.7.2}$$

One catch here is that given the original domain $a \le b$, the endpoints of the new domain must be found by equation solving or with inverses: $c = g^{-1}(a)$, $d = g^{-1}(b)$. Usually this will not be a problem because the change of variables will be chosen precisely for the sake of getting a known, simple domain for the new integral. (This idea was irrelevant with functions one variable, where the domain of integration is always equally simple: an interval.)

Integrals over Intervals Considered as Domains. To mimic the notation for multiple integrals over domains, it is convenient to express an definite interval as being over an interval $I = [a, b]$. Thus,

$$\int_{[a,b]} f(x)\,dx \text{ means } \int_a^b f(x)\,dx, \text{ where we must have } a \le b.$$

If function g is increasing, the Substitution Rule can be written as

$$\int_{[a,b]} f(x)\,dx = \int_{[c,d]} f(g(u))g'(u)\,du. \tag{5.7.3}$$

or

$$\int_I f(x)\,dx = \int_J f(g(u))g'(u)\,du. \tag{5.7.4}$$

where the domain for u is $J = [c, d]$.

[2]openstax.org/books/calculus-volume-2/pages/3-3-trigonometric-substitution

Integrals over Intervals with Decreasing g ($g' < 0$). However, if function g is decreasing, $a < b$ leads to $c > d$, so the new interval is $J = [d, c]$, not $[c, d]$. Thus

$$\int_{[a,b]} f(x)\,dx = \int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du = -\int_d^c f(g(u))g'(u)\,du = -\int_{[d,c]} f(g(u))g'(u)\,du$$

Since g is decreasing, $g'(u) < 0$ and so $-g'(u) = |g'(u)|$, giving

$$\int_{[a,b]} f(x)\,dx = \int_{[d,c]} f(g(u))|g'(u)|\,du. \tag{5.7.5}$$

The Substitution Rule for Intervals as Domains. Equation (5.7.5) for $x = g(u)$ decreasing and the version in Equation (5.7.2) for $x = g(u)$ increasing take the common form

$$\int_I f(x)\ dx = \int_J f(g(u))|g'(u)|\,du, = \int_J f(x)\left|\frac{dx}{du}\right|\,du \tag{5.7.6}$$

with the appropriate versions of interval J.

This is the form that we will now mimic for multiple integrals: finding the replacement for $g'(u) = \dfrac{dx}{du}$ is the key step.

5.7.2 Transformations: Changes of Coordinates in 2D (and then 3D)

We have seen how a double integral is changed by one change of variables, from cartesian coordinates to polar coordinates as described by

$$(x, y) = T(r, \theta) = (r\cos\theta, r\sin\theta) \tag{5.7.7}$$

A function like T that takes points in the plane $\mathbb{R}^2$ to other points in the plane is called a transformation. We now consider more general transformations to new variables u and v,

$$(x, y) = T(u, v) = (g(u, v), h(u, v)) \quad \text{or} \quad x = g(u, v), y = h(u, v).$$

where we want both g and h to be differentiable, (so all four first partial derivatives exist and are continuous): these are called C^1 transformations.

In this notation "C" refers to being continuous, and the "1" means that the first derivatives also exist and are continuous.
Similarly, C^0 is sometimes used to denote continuous functions, and C^2 means functions with all the first and second derivatives existing and being continuous.

Images and One-to-One Transformations. If $T(u_1, v_1) = (x_1, y_1)$, the point (x_1, y_1) is called the image of (u_1, v_1). The set of all such images is the range, also called the image of the domain of the transformation.

With polar coordinates, we tried to have unique values of r and θ for each point (x, y) in the plane.

A transformation that achieves this uniqueness is one-to-one: no two points in its domain have the same image. For example, the transformation from polar coordinates to cartesian coordinates is one-to-one on any domain on which $r > 0$ and $-\pi < \theta \le \pi$, which is the same as excluding the origin.

See Examples 65 and 66 in OSC3 Section 5.7[3].

(To include the origin too, one could allow only $\theta = 0$ when $r = 0$.)

[3] openstax.org/books/calculus-volume-3/pages/5-5-triple-integrals-in-cylindrical-and-spherical-coordinates

The Inverse of a One-to-One Transformation. If a transformation is one-to-one, then for any point (x, y) in its range, there is only only point (u, v) with $T(u, v) = (x, y)$, so this point (u, v) is the inverse of (x, y) under T, and there is an inverse transformation T^{-1},

$$T^{-1}(x, y) = (u, v).$$

For T the transformation in Equation (5.7.7) from polar to cartesian coordinates, and on a domain with $r > 0$ and $-\pi < \theta \le \pi$,

$$T^{-1}(x, y) = (r(x, y), \theta(x, y)),$$

is given by the equations (1.3.3) in Section 1.3, p. 7.

5.7.3 Transformations and Double Integals

Consider a double integral $\iint\limits_R f(x, y)\, dA$ and a transformation $(x, y) = T(u, v)$ such that the domain R in the (x, y) plane is the image of a domain S in coordinates (u, v). This is sometimes expressed as $R = T(S)$. One example we have seen is when the new coordinates are polar coordinates r and θ, R is a polar rectangle, so S is the rectangular domain $a \le r \le b$, $\alpha \le \theta \le \beta$.

We want to get an expression for this integral as an integral in the new variables u and v, such as

$$\iint\limits_D f(x, y) dA = \int_a^b \int_\alpha^\beta f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr \tag{5.7.8}$$

for that case.

The key to deriving this formula for polar coordinates was looking at a small rectangle of values of r and θ, of dimensions Δr and $\Delta\theta$, and computing the area of the corresponding region of (x, y) values.

Look at a small rectangle S of (u, v) values with one corner (u_0, v_0) and adjacent corners $(u_0 + \Delta u, v_0)$ and $(u_0, v_0 + \Delta v)$. Its image under transformation T is roughly quadrilateral, with one corner A of coordinates $(x_0, y_0) = T(u_0, v_0) = (g(u_0, v_0), h(u_0, v_0))$.

The image B of the adjacent corner $(u_0 + \Delta u, v_0)$ has coordinates $(g(u_0 + \Delta u, v_0), h(u_0 + \Delta u, v_0))$ and these are given approximately by linearizations of g and h respectively:

$$\begin{aligned}
g(u_0 + \Delta u, v_0) &\approx g(u_0, v_0) + g_u(u_0, v_0) \cdot \Delta u + g_v(u_0, v_0) \cdot 0, \\
h(u_0 + \Delta u, v_0) &\approx h(u_0, v_0) + h_u(u_0, v_0) \cdot \Delta u + h_v(u_0, v_0) \cdot 0,
\end{aligned}$$

since the change in the v argument is 0.

Thus, this edge of this region in the (x, y) plane is described by vector

$$\overrightarrow{AB} \approx g_u(u_0, v_0)\Delta u\, \hat\imath + h_u(u_0, v_0)\Delta u\, \hat\jmath = [g_u(u_0, v_0)\, \hat\imath + h_u(u_0, v_0)\, \hat\jmath]\Delta u$$

Similarly, the image C of the other adjacent corner $(u_0, v_0 + \Delta v)$ leads to

$$\overrightarrow{AC} \approx (g_v(u_0, v_0)\, \hat\imath + h_v(u_0, v_0)\, \hat\jmath\,)\Delta v$$

and the final corner D opposite A is reached by edge

$$\begin{aligned}
\overrightarrow{CD} &\approx g_u(u_0, v_0 + \Delta v)\, \hat\imath\, \Delta u + h_u(u_0, v_0 + \Delta v)\, \hat\jmath\, \Delta u \\
&\approx [g_u(u_0, v_0) + (g_u)_v(u_0, v_0)\Delta v]\Delta u\, \hat\imath + [h_u(u_0, v_0) + (h_u)_v(u_0, v_0)\Delta v]\Delta u\, \hat\jmath \\
&= [g_u(u_0, v_0)\, \hat\imath + h_u(u_0, v_0)\hat\jmath]\Delta u \\
&+ [g_{uv}(u_0, v_0)\hat\imath + h_{uv}(u_0, v_0)\hat\jmath]\Delta v\Delta u \\
&\approx [g_u(u_0, v_0)\hat\imath + h_u(u_0, v_0)\hat\jmath]\Delta u
\end{aligned}$$

$$\approx \overrightarrow{AB}$$

where we have ignored far smaller terms proportional to $\Delta u \Delta v$.

So the image R of a small rectangle S is roughly a parallelogram.

The Area Approximation and the Jacobian. The area ΔA of small region R can thus be approximated by the cross product formula for parallelogram area:

$$\begin{aligned}\Delta A &\approx \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| \\ &\approx \left|[g_u(u_0, v_0)h_v(u_0, v_0) - g_v(u_0, v_0)h_u(u_0, v_0)]\Delta u \Delta v \hat{k}\right| \\ &= |g_u(u_0, v_0)h_v(u_0, v_0) - g_v(u_0, v_0)h_u(u_0, v_0)| \, \Delta u \Delta v\end{aligned}$$

The quantity inside the absolute value is the determinant of a 2×2 matrix, and is known as the Jacobian of the transformation given by $x = g(u, v)$, $y = h(u, v)$, denoted

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}. \tag{5.7.9}$$

Beware the unfortunate double meaning of vertical bars: absolute value and determinant!

With this notation the image $R = T(S)$ of small rectangle S is approximately a parallelogram of area

$$\Delta A \approx \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta u \Delta v \tag{5.7.10}$$

(c.f. Equation (2.4.7) for the area of a parallelogram in Subsection 2.4.3, p. 22), and for the "infinitesimal rectangle" of dimensions du by dv appearing in double integrals we get the change of coordinates formula using

$$dA = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\ dv. \tag{5.7.11}$$

Note: The vertical bar notation is used here in two different ways: for the determinant in Equation (5.7.9) defining the Jacobian, and the for the absolute value of that in $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$. To avoid possible confusion, the expression is sometimes written as

$$dA = \left|\det \begin{bmatrix} g_u & g_v \\ h_u & h_v \end{bmatrix}\right| du\ dv.$$

That is, with appropriate approximations and limits as seen in Section 5.3, p. 76 for polar coordinates, we get:

Theorem 5.7.1 Change of Coordinates in Double Integrals. For T a one-to-one C^1 transformation and $R = T(S)$,

$$\iint\limits_R f(x, y) dA = \iint\limits_S f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv. \tag{5.7.12}$$

The Jacobian does here what $g'(u) = \dfrac{dx}{du}$ does in the one-dimensional case of Equation (5.7.6).

Example 5.7.2 Let us test this for the case of polar coordinates.
The Jacobian is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}$$

$$\begin{aligned}
&= \frac{\partial(r\cos\theta)}{\partial r}\frac{\partial(r\sin\theta)}{\partial\theta} - \frac{\partial(r\cos\theta)}{\partial\theta}\frac{\partial(r\sin\theta)}{\partial r} \\
&= \cos\theta\, r\cos\theta - (-r\sin\theta)\sin\theta \\
&= r(\cos^2\theta + \sin^2\theta) \\
&= r
\end{aligned}$$

so $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$, giving the factor of r seen in Equation (5.7.8). $\square$

See also Example 69 in OSC3 Section 5.7[4].

Note: using the order (θ, r) gives Jacobian

$$\frac{\partial(x,y)}{\partial(\theta,r)} = -r,$$

so the absolute value is necessary to get the same final result for the integral.

See Examples 70 and 71 in OSC3 Section 5.7[5].

5.7.4 Triple Integrals

The ideas above extend fairly routinely to transformations in three variables and triple integrals.

Definition 5.7.3 A C^1 transformation $(x,y,z) = T(u,v,w)$ is a function of the form

$$x = g(u,v,w), \quad y = h(u,v,w), \quad z = k(u,v,w)$$

with all three functions differentiable. $\Diamond$

The Jacobian of such a transformation is the determinant

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

Theorem 5.7.4 For T a one-to-one C^1 transformation and R the image of region S under T (i.e. $R = T(S)$),

$$\begin{aligned}
&\iiint\limits_R f(x,y,z)\, dV \\
&= \iiint\limits_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw
\end{aligned} \tag{5.7.13}$$

Exercise 5.7.5 Cylindrical and Spherical Coordinates.

See also Example 72 in OSC3 Section 5.7[6].

Use the above formula to derive the formulas (5.5.3) and (5.5.5) for integrals in cylindrical and spherical coordinates seen in Section 5.5, p. 83.
Note the importance of the absolute value in Equation (5.7.13) for the case of spherical coordinates $(\rho, \theta, \phi)_s$: this is because they are left-handed whereas the caretesion coordinates $(x,y,z)_s$ are right-handed.

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[4] openstax.org/books/calculus-volume-3/pages/5-5-triple-integrals-in-cylindrical-and-spherical-coordinates
[5] openstax.org/books/calculus-volume-3/pages/5-5-triple-integrals-in-cylindrical-and-spherical-coordinates
[6] openstax.org/books/calculus-volume-3/pages/5-5-triple-integrals-in-cylindrical-and-spherical-coordinates

See Example 73 in OSC3 Section 5.7[7].

Study Guide. Study Section 5.7 of OpenStax Calculus Volume 3[8], including

- All the Definitions, Theorems, and Examples (and Checkpoints).
- One or several exercises from each of the following ranges and pairs: 365–361, 362–367, 368–373, 374–377, 378–387, 388 & 389, 390–393, 399 & 400, 404 & 405.

Chapter Review. When reviewing this chapter, also look at the end of chapter review material in OpenStax Calculus Volume 3, including Key Terms[9], Key Equations[10] and Key Concepts[11].

[7] openstax.org/books/calculus-volume-3/pages/5-5-triple-integrals-in-cylindrical-and-spherical-coordinates
[8] openstax.org/books/calculus-volume-3/pages/5-7-change-of-variables-in-multiple-integrals
[9] openstax.org/books/calculus-volume-3/pages/5-key-terms
[10] openstax.org/books/calculus-volume-3/pages/5-key-equations
[11] openstax.org/books/calculus-volume-3/pages/5-key-concepts

Chapter 6

Vector Calculus

Revised on April 21.

References.

- Chapter 6 of OpenStax Calculus Volume 3.[1]
- Chapter 16 of Calculus, Early Transcendentals by Stewart.

6.1 Vector Fields

Revised on March 27.

References.

- OpenStax Calculus Volume 3, Section 6.1[1].
- Calculus, Early Transcendentals by Stewart, Section 16.1.

6.1.1 Definitions

Many physical quantities are described by vectors (like velocity, force) and depend on position in space or on a surface. The mathematical description of these is a vector field:

Definition 6.1.1 2D Vector Field. For D a set in $\mathbb{R}^2$ (a plane region), a vector field on $\mathbb{R}^2$ is a function $\vec{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\vec{F}(x, y)$.
Such a vector field can be described with component functions as $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$; that is

$$\vec{F}(x, y) = P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath} = \langle P(x, y), Q(x, y)\rangle$$

$\Diamond$

See Examples 1 to 6 in Section 6.1 of OSC3[2].

Scalar valued functions of several variables like P and Q are sometimes called scalar fields to distinguish them from vector fields.

There is an unsurprising 3D version:

[1]openstax.org/books/calculus-volume-3/pages/6-introduction
[1]openstax.org/books/calculus-volume-3/pages/6-1-vector-fields
[2]openstax.org/books/calculus-volume-3/pages/6-1-vector-fields

Definition 6.1.2 3D Vector Field. For E a set in $\mathbb{R}^3$, a vector field on $\mathbb{R}^3$ is a function $\vec{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\vec{F}(x, y, z)$.
Again such a vector field can be described with component functions, so

$$\vec{F}(x, y, z) = P(x, y, z)\hat{\imath} + Q(x, y, z)\hat{\jmath} + R(x, y, z)\hat{\jmath}.$$

◊

See Examples 7 and 8 in Section 6.1 of OSC3[3].

6.1.2 Gradient Vector Fields, or Conservative Vector Fields

Vector fields often arise as the gradient of a scalar function: for a function f of two variables the gradient

$$\mathrm{Grad} f(x, y) = \langle f_x, f_y\rangle = f_x(x, y)\hat{\imath} + f_y(x, y)\hat{\jmath} \tag{6.1.1}$$

which is a vector field on $\mathbb{R}^2$, and likewise for f a function of three variables, its gradient is a vector field on $\mathbb{R}^3$:

$$\mathrm{Grad} f(x, y, z) = \langle f_x, f_y, f_z\rangle = f_x(x, y, z)\hat{\imath} + f_y(x, y, z)\hat{\jmath} + f_z(x, y, z)\hat{k}. \tag{6.1.2}$$

Two common short-hands for either of these are

$$\mathrm{Grad} f = \nabla f = \vec{\nabla} f \tag{6.1.3}$$

where the symbol ∇ or $\vec{\nabla}$ is the fake vector

$$\nabla f = \vec{\nabla} f = \langle \partial_x, \partial_y\rangle \tag{6.1.4}$$

in the plane, and

$$\nabla f = \vec{\nabla} f = \langle \partial_x, \partial_y, \partial_z\rangle \tag{6.1.5}$$

in space.

The second "vector" form of this notation will be more important later, from Section 6.5, p. 109, Divergence and Curl onward.

Definition 6.1.3 A vector field $\vec{F}$ is called a gradient vector field or conservative vector field if it is the gradient of some scalar function; that is, if there is a function f such that $\nabla f = \vec{F}$.
Function f is then called a potential function for $\vec{F}$. ◊

See Examples 9 to 11 in Section 6.1 of OSC3[4].

6.1.3 The Cross-Partial Property and Non-Conservative Vector Fields

For a conservative 2D vector field $\vec{F} = \langle P, Q\rangle\,, = \langle f_x, f_y\rangle$, look at the cross partial derivatives:

$$P_y = (f_x)_y = (f_y)_x = Q_x$$

(recalling Clairaut's Theorem 4.3.2, p. 46 from Section 4.3, p. 43)

This can be extended to 3D vector fields, and can be written in a suggestive mock-determinant notation that we will see more of in Section 6.5, p. 109:

Theorem 6.1.4 The Cross-Partial Property of Conservative Vector Fields. For a 2D conservative vector field $\vec{F} = \langle P, Q\rangle$,

$$Q_x - P_y = \begin{vmatrix} \partial_x & \partial_y \\ P & Q \end{vmatrix} = 0 \tag{6.1.6}$$

[3] openstax.org/books/calculus-volume-3/pages/6-1-vector-fields
[4] openstax.org/books/calculus-volume-3/pages/6-1-vector-fields

For a 3D conservative vector field $\vec{F} = \langle P, Q, R \rangle$,

$$Q_x - P_y = R_x - P_z = R_y - Q_z = 0, \tag{6.1.7}$$

which has the menomonic mock-determinant form

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} \tag{6.1.8}$$

Note: Here I start using the alternative compact notation for partial derivatives introduced in (4.3.3) and (4.3.4) in Subsection 4.3.3, p. 44.

Exercise 6.1.5 A non-conservative vector field. Verify that $\vec{F}(x,y) = \langle -y, x \rangle$ is not conservative.

See Example 12 and 13 in Section 6.1 of OSC3[5].

Exercise 6.1.6 A non-conservative vector field that has the cross-partial property. Verify that the vector field

$$\vec{F}(x,y) = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

has the cross-partial property (6.1.6).
However, it will be seen in Section 6.3, p. 99 that is not a gradient vector field on its natural domain of all points in $\mathbb{R}^2$ except for the hole at the origin.

Study Guide. Study Section 6.1 of Calculus Volume 3[6]; in particular

- The Definition and Theorems.
- Examples 1–5 and 10–13, and the Checkpoints following each.
- The following exercises (for ranges, do at least one in each range): 1, 2, 15–20, 22–24.

6.2 Line Integrals

Revised on April 2.

References.

- Section 6.2 of OpenStax Calculus Volume 3[1].
- Section 16.2 of Calculus, Early Transcendentals by Stewart.

Introduction. It is often useful to ”sum” (integrate) a quantity along a curve. One example is computating the total change along a curve from a variable charge density. Another physical example is the summation of increments of work done by or against a force to get the total work done as an object moves along a curve.

These ”charge” and ”work” examples are different in an important way: the first involves integrating a scalar quantity with respect to increments in arc length, ds, whereas the second involves a vector quantity, the force, acting in relation to increments in spatial coordinates like dx.

[5] openstax.org/books/calculus-volume-3/pages/6-1-vector-fields
[6] openstax.org/books/calculus-volume-3/pages/6-1-vector-fields
[1] openstax.org/books/calculus-volume-3/pages/6-2-line-integrals

6.2.1 Scalar Line Integrals: Integrating With Respect to Arc Length Along a Curve

6.2.1.1 Scalar Line Integrals in The Plane

If a plane curve C is given parametrically as

$$x = x(t),\ y = y(t), \quad a \le t \le b \tag{6.2.1}$$

or in vector form $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$, then one can start by considering sums of the form

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

where the interval $[a, b]$ of t values is divided into n subintervals $[t_{i-1}, t_i]$, $1 \le i \le n$, each point $(x_i^*, y_i^*) = (x(t_i^*), y(t_i^*))$ is given by some t_i^* in $[t_{i-1}, t_i]$ and so lies on the i-the sub-arc, and Δs_i is the length of that sub-arc.

A now familiar limit process turns these approximations into an integral along the curve:

Definition 6.2.1 If f is defined at each point of a smooth curve C given by Equation (6.2.1) then the line integral of f along C is

$$\int_C f(x, y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i \tag{6.2.2}$$

This is also called the line integral of f along C with respect to arc-length. ◊

Next, we can express this in terms of an integral in variable t over the interval $[a, b]$.

For $f(x, y) = 1$, the limit in Equation (6.2.2) is the definition of arc-length as in Section 3.3, p. 31 and it was seen there that in effect,

$$\begin{aligned} ds &= \sqrt{dx^2 + dy^2} \\ &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt, && (6.2.3) \\ &= \left\|\frac{d\vec{r}}{dt}\right\| dt, \quad \text{with } \vec{r} = \langle x, y\rangle && (6.2.4) \end{aligned}$$

The same argument here gives:

$$\int_C f(x, y)\, ds = \int_{t=a}^{b} f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{6.2.5}$$

See Example 14 and Checkpoint 13 in Section 6.2 of OSC3[2].

6.2.1.2 Scalar Line Integrals in Space

The above concepts extend easily to space curves C, given parametrically as

$$x = x(t),\ y = y(t),\ z = z(t), \quad a \le t \le b \tag{6.2.6}$$

or in vector form, $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$.

We again define the line integral of f along C with respect to arc-length as

$$\int_C f(x, y, z)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s_i$$

[2]openstax.org/books/calculus-volume-3/pages/6-2-line-integrals

where in the sum, the interval $[a, b]$ is divided into n (equally long) subintervals with Δs_i the distance between the endpoints of the corresponding part of the curve, and (x_i^*, y_i^*, z_i^*) some point in that part of the curve.

Also as above, we use the Fundamental theorem of Calculus to turn this limit of a sum into a definite integral over interval $[a, b]$:

$$\begin{aligned}\int_C f(x,y,z)ds &= \int_{t=a}^{b} f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}dt \\ &= \int_{t=a}^{b} f(\vec{r}(t))\left|\frac{d\vec{r}}{dt}\right| dt. \end{aligned} \tag{6.2.7}$$

See Example 20 in Section 6.2 of OSC3[3].

The last vector form is convenient as it covers both the 2D ad 3D versions.

The simple and important special case $f = 1$ gives a compact formula for the arc length of the curve,

$$\int_C ds = \int_{t=a}^{b} \left|\frac{d\vec{r}}{dt}\right| dt.$$

See Example 17 in Section 6.2 of OSC3[4].

6.2.1.3 Integrals Along Paths: Piecewise Smooth Curves

A path is another name for a piecewise smooth curve C: a collection of smooth curves $C_1, C_2, \ldots C_m$ that join end to end.

This is sometimes denoted $C = C_1 + C_2 + \cdots + C_m$.

The path integral along a path C is simply the sum of the line integrals along each smooth piece.

6.2.2 Vector Line Integrals: Integrating With Respect to Position Coordinates

6.2.2.1 Line Integrals in the Plane with Respect to the Coordinates, x and y

In some situations, the quantity to be summed is $f(x, y)\Delta x$ or $f(x, y)\Delta y$: for example if f is a force acting on an object in the x direction, $f(x, y)\Delta x$ is the work done by that force when the object moves distance Δx in that direction (no work is associated with movement perpendicular to the force).

Limits of sums give the work done by such a force in the x-direction as an object moves along curve C to be

$$\int_C f(x,y)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta x_i \tag{6.2.8}$$

called the line integral of f along C with respect to x. Likewise the line integral of f along C with respect to y is

$$\int_C f(x,y)\,dy = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta y_i \tag{6.2.9}$$

Much as with Equation (6.2.5), these line integrals can be computed with

$$\int_C f(x,y)\ dx = \int_{t=a}^{b} f(x(t),y(t))\frac{dx}{dt}dt \tag{6.2.10}$$

$$\int_C f(x,y)\ dy = \int_{t=a}^{b} f(x(t),y(t))\frac{dy}{dt}dt \tag{6.2.11}$$

However, the absence of the square root term can make these new integrals easier to work with.

[3]openstax.org/books/calculus-volume-3/pages/6-2-line-integrals
[4]openstax.org/books/calculus-volume-3/pages/6-2-line-integrals

See Example 18 in Section 6.2 of OSC3[5].

6.2.2.2 Line Integrals in Space Coordinates

Line integrals with respect to each of the space coordinate variables x, y and z can be defined similarly, so that for example

$$\int_C f(x,y,z)dz = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta z_i = \int_a^b f(x(t),y(t),z(t))\frac{dz}{dt}\,dt \tag{6.2.12}$$

6.2.2.3 Paired (and Tripled) Line Integrals and Vector Line Integrals

One often gets a sum of line integrals in the plane in both x and y, and these can be abbreviated

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy \tag{6.2.13}$$

This combination has a concise vector form, called a Vector Line Integral:

$$\int_C P(x,y)\,dx + Q(x,y)\;dy = \int_C \langle P(x,y), Q(x,y)\rangle \cdot \langle dx, dy\rangle \tag{6.2.14}$$

$$= \int_C \vec{F}\cdot d\vec{r} \tag{6.2.15}$$

where $\vec{F} = \langle P(x,y), Q(x,y)\rangle$ and for the position vector $\vec{r} = \langle x, y\rangle$, we use the short-hand

$$d\vec{r} = d\langle x,y\rangle = \langle dx, dy\rangle\,.$$

When P and Q are the components in the x and y directions of a force vector $\vec{F}$, this gives the total work done by the force as the object traverses curve C.

Likewise, integrals of the form

$$\int_C P(x,y,z)\;dx + \int_C Q(x,y,z)\;dy + \int_C R(x,y,z)\;dz = \int_C P\;dx + Q\;dy + R\;dz \tag{6.2.16}$$

arise—in particular for the example of the work done by a force—and they have the same vector form (6.2.15), now with $\vec{F} = \langle P, Q, R\rangle$.

See Example 20 in Section 6.2 of OSC3[6], and for the application of calculating work, see Examples 23 and 26.

For a more completely vector notation form, describe the path C as $\langle r\rangle(t),\ a \le t \le b$ and express each line integral above in terms of this: The integral in Equation (6.2.16) has the form

$$\int_C P\;dx + Q\;dy + R\;dz = \int_a^b \left[P(\vec{r}(t))\frac{dx}{dt} + Q(\vec{r}(t))\frac{dy}{dt} + R(\vec{r}(t))\frac{dz}{dt}\right]dt = \int_a^b \vec{F}(r(\vec{t}))\cdot\frac{d\vec{r}}{dt}\;dt \tag{6.2.17}$$

and in practice evaluation of $\int_C \vec{F}\cdot d\vec{r}$ is done via this form, with the formal substutution $d\vec{r} = \dfrac{d\vec{r}}{dt}\;dt$: this is sometimes called the line integral of $\vec{F}$ along C.

This integral $\displaystyle\int_C \vec{F}\cdot d\vec{r}$ can instead be defined rigorously by the familiar process of approximation by a sum of terms $\vec{F}(\vec{r}(t_i^*))\Delta\vec{r}_i$ and taking a suitable limit.

[5]openstax.org/books/calculus-volume-3/pages/6-2-line-integrals
[6]openstax.org/books/calculus-volume-3/pages/6-2-line-integrals

6.2.2.4 Vector Line Integrals in Terms of the Arc-length Differential ds: Circulation and Flux

Circulation Form. Recall from Equation (3.3.4) in Section 3.3, p. 31 that the unit tangent vector to C is $\vec{T} = d\vec{r}/ds$ and thus

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}) \cdot \frac{d\vec{r}}{ds}\ ds = \int_C \vec{F} \cdot \widehat{T}\ ds \tag{6.2.18}$$

getting back to a line integral with respect to arc length as defined in Equation (6.2.2).

For the example of $\vec{F}$ being the velocity in a fluid, the dot product in this integral is the rate of flow along the curve. Then for a closed curve, this integral measures the net flow around the curve: hence it is called the circulation around the curve.

Flux Form. In the plane there is another useful version: $\widehat{T} = \langle dx/ds, dy/ds\rangle$, so we can define a unit normal vector

$$\widehat{N} = \langle dy/ds, -dx/ds\rangle\,, \tag{6.2.19}$$

which is the rotation of $\widehat{T}$ a quarter turn clockwise. Thinking of $\widehat{T}$ as the "forward direction", this choice of unit normal points to the right of the curve.

(Note that this is not necessarily the same as the the principle unit normal vector defined in Equation (3.3.7) in Section 3.3, p. 31, which can be to the left or the right depending on which way the curve is turning.)

In the plane, this new choice has the advantage of always being defined, even at an "inflection point" where $d\widehat{T}/dt = \vec{0}$, and always continuous so long as $\widehat{T}$ is.

The integral of $\vec{F} \cdot \widehat{N}$ along a curve is the integral of the component of $\vec{F}$ normal to the curve; for example if $\vec{F}$ is the velocity in a fluid, this measures the rate of flow across the curve: it is thus known as the flux of $\vec{F}$ across C.

This equals the circulation of $\vec{G} = \langle -Q, P\rangle$, which is $\vec{F}$ rotated a quarter turn anti-clockwise:

$$\begin{aligned}\int_C \vec{F} \cdot \widehat{N}\ ds &= \int_C \langle P, Q\rangle \cdot \left\langle \frac{dy}{ds}, -\frac{dx}{ds}\right\rangle ds\\ &= \int_C \left(P\frac{dy}{ds} - Q\frac{dx}{ds}\right) ds\\ &= \int_C -Q\ dx + P\ dy\\ &= \int_C \langle -Q, P\rangle \cdot \widehat{T} ds\end{aligned} \tag{6.2.20}$$

This connection will be useful in Section 6.4, p. 104, where both the circulation and flux around a closed curve in the plane are related to a double integral over the region "inside" the curve, and also in later sections where these ideas are extended to integrals over surfaces and regions in space.

6.2.3 Reversing the Orientation of a Curve

A curve C described by a parameterization of a curve $x = f(t)$, $y = g(t)$, $a \leq t \leq b$ has an orientation, meaning a direction of motion from initial point $(x(a), y(a))$ to final point $(x(b), x(b))$.

If we reverse the orientation, for example with new parameterization $u = -t$ so that $x = f(-u)$, $y = g(-u)$, $-b \leq u \leq -a$, the new curve is denoted $-C$.

This reverses the order of limits on integration so that

$$\int_{-C} f(x,y)\ dx = -\int_C f(x,y)\ dx, \tag{6.2.21}$$

and so on, and for a vector line integral

$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_{C} \vec{F} \cdot d\vec{r} \tag{6.2.22}$$

which of course is equally valid in the 3D case.

See Example 19 in Section 6.2 of OSC3[7].

For the physical example where this integral gives the work done by force $\vec{F}$ on an object as it moves along the curve C, this says naturally that going in the opposite direction negates the amount of work done.

However, scalar line integrals are not changed by this reversal:

$$\int_{-C} f(x,y)\, ds = \int_{C} f(x,y)\, ds. \tag{6.2.23}$$

Compare this to Equation (6.2.20) for a line integral: there the integrand changes sign, due to $\widehat{T}$ negating when the orientation is reversed.

This is because Δs_i remains positive in Equation (6.2.2) whereas Δx_i and Δy_i in Equations (6.2.8) and (6.2.9) change sign.

6.2.4 Properties of Vector Line Integrals

Vector line integrals have many properties in common with definite integrals $\int_a^b f(x)\ dx$ in one dimension:

Theorem 6.2.2 For any continuous vector fields $\vec{F}$ and $\vec{G}$

1. $\displaystyle\int_C (\vec{F}+\vec{G}) \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} + \int_C \vec{G} \cdot d\vec{r}$
2. $\int_C k\vec{F} \cdot d\vec{r} = k \int_C \vec{F} \cdot d\vec{r}$, and as already seen
3. $\displaystyle\int_{-C} \vec{F} \cdot d\vec{r} = -\int_{C} \vec{F} \cdot d\vec{r}$

and these apply equally for path integrals, along piecewise smooth curves.

In comparison, scalar line integrals behave more like integrals over domains in two or three dimensions seen in Chapter 5, p. 65, and like the one dimensional versions $\int\limits_{[a,b]} f(x)\ dx$ introduced in Subsection 5.7.1, p. 85.

Study Guide. Study Section 6.2 of Calculus Volume 3[8]; in particular:

- The Definitions and Theorems.
- Examples 14–21, and according to your scientific interests, some of the application examples 22, 23 and 26 (and as usual, the Checkpoints following each).
- The true/false exercises 39–43.
- The following exercises (for ranges, do at least one in each): 49–53, 54, 55–58, 65–69.

6.3 Conservative Vector Fields

Revised on April 11.

References.

- OpenStax Calculus Volume 3, Section 6.3[1].

[7] openstax.org/books/calculus-volume-3/pages/6-2-line-integrals
[8] openstax.org/books/calculus-volume-3/pages/6-2-line-integrals
[1] openstax.org/books/calculus-volume-3/pages/6-3-conservative-vector-fields

6.3.1 The Fundamental Theorem for Path Integrals

For a function of a single variable $f(x)$, the Fundamental Theorem of Calculus says that its definite integral over interval $[a, b]$ can be evaluated by finding an anti-derivative F (i.e. a function whose derivative is f) and the simple calculation:

$$\int_a^b f(x)\,dx = F(b) - F(a) = [F(x)]_a^b$$

This can be extended first to line integrals of a vector field $\vec{F}$ along a smooth curve, with the role of the anti-derivative now played by a potential f:

Theorem 6.3.1 (The Fundamental Theorem for Path Integrals). For a conservative vector field $\vec{F}$, so that $\nabla f = \vec{F}$ for some scalar function f, then for the smooth curve C given by $\vec{r}(t)$, $a \le t \le b$,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = \left[f(\vec{r}(t))\right]_a^b \tag{6.3.1}$$

Proof. This comes from results in Section 6.2, p. 94 along with the Chain Rule.
For the 3D case, and a smooth curve C,

$$\begin{aligned}\int_C \nabla f \cdot d\vec{r} &= \int_a^b \nabla\, f(r(t)) \cdot \frac{d\vec{r}}{dt}\,dt \\ &= \int_a^b \left[\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right] dt \\ &= \int_a^b \frac{d}{dt} f(\vec{r}(t))dt = \left[f(\vec{r}(t))\right]_a^b\end{aligned}$$

For integrals along paths (piecewise smooth curves) we get the same result by adding up the integrals over each of the smooth pieces. For example, if smooth curve C_1 goes from point A_1 [position vector $\vec{r}_1$] to A_2 [$\vec{r}_2$] and smooth curve C_2 goes from point A_2 to A_3 [$\vec{r}_3$], they combine to form a path $C = C_1 + C_2$ from A_1 to A_3, and

$$\begin{aligned}\int_C \nabla f\,ds &= \int_{C_1} \nabla f\,ds + \int_{C_2} \nabla f\,ds \\ &= f(\vec{r}_2) - f(\vec{r}_1) + f(\vec{r}_3) - f(\vec{r}_2) \\ &= f(\vec{r}_3) - f(\vec{r}_1),\end{aligned}$$

which is still the change in the value of f between the initial and final points, as advertised. ■

See Examples 28 and 29 in Section 6.3 of OSC3[2].

6.3.2 Independence of Path for Integrals of Gradient Fields

The above results says that the value of the path integral of a gradient field depends only on the endpoints, not the curve used to connect those points. Thus for any other path P_2 with the same initial and final points, the path integral of ∇f has the same value.

Definition 6.3.2 We say that $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in domain D if this path integral has the same value for any two paths with the same initial and final points that stay within D. ◊

In this case we can write

$$\int_A^B \vec{F} \cdot d\vec{r}$$

for the common value of the integral along any path in D from point A to point B.

[2]openstax.org/books/calculus-volume-3/pages/6-3-conservative-vector-fields

See Example 30 in Section 6.3 of OSC3[3].

6.3.3 Closed Paths

One simple and important case is that of a closed path C: one whose initial and final points are the same, so that it is a "loop".

An integral $\int_C$ around a closed path is also denoted $\oint_C$ to emphasize this property of the path.

See Example 27 in Section 6.3 of OSC3[4].

Clearly the path integral of a gradient field over a closed path is zero. In fact,

Theorem 6.3.3 $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in domain D if and only $\oint_C \vec{F} \cdot d\vec{r} = 0$ for all closed paths in D.

Proof. If the integral of $\vec{F}$ around any closed path in D is zero and C_1, C_2 are two paths in D from point A to point B, then the path consisting of C_1 followed by $-C_2$ is a closed path in D, going from A to B and back again.
Thus the integral around this path is 0, and so

$$0 = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}.$$

So the integral is the same along either path: path independence.
Conversely, suppose the integral is path independent in D.
For any closed path C in D its initial and final point is the same.
The path integral on C is the same as on the path C_2 which just stays at that point.
Clearly the path integral along this "trivial" path C_2 is zero, so the integral along the closed path C is also zero. ■

6.3.4 Independence of Path Implies that a Field is Conservative

Not only are the path integrals of a conservative field path independent, but the converse is also true, at least in domains that are open and where any two points can be connected by a path.

Definition 6.3.4 A set in $\mathbb{R}^2$ or $\mathbb{R}^3$ is connected (or path connected) if any two points in the set are connected by a path that stays within that set. ♢

Theorem 6.3.5 The path integrals of a vector field $\vec{F}$ on an open connected domain D are path independent if and only if the field is conservative: that is, if it is of the form $\vec{F} = \nabla f$ for some function f.

Proof. We already know one half of this, so it only remains to show that path independence implies existence of such an f, which is is like finding an anti-derivative for $\vec{F}$.
For simplicity of notation this will be done only in 2D, so $\vec{F}(x, y) = \langle F_1(x, y), F_2(x, y)\rangle$. (As usual nothing much changes in 3D.)
Taking some point $A(a, b)$ as a "starting point", then for any suitable function f

$$f(x, y) - f(a, b) = \int_A^{(x,y)} \nabla f \cdot d\vec{r} = \int_A^{(x,y)} \vec{F} \cdot d\vec{r}$$

for any path C from A to $P(x, y)$.
Such paths exist due to connectedness, and any such path gives the same value due to path independence.
Choosing f to have value zero at A (this is a like choosing a constant of integration), the only possibility is

$$f(x, y) = \int_A^{(x,y)} \vec{F} \cdot d\vec{r}$$

[3]openstax.org/books/calculus-volume-3/pages/6-3-conservative-vector-fields
[4]openstax.org/books/calculus-volume-3/pages/6-3-conservative-vector-fields

But does this function have the correct partial derivatives?
Consider such a path that ends by coming in to point P from the left parallel to the x-axis, so the last part of the path is the straight line segment from (c, y) to (x, y) for some $c < x$.
Parameterizing the final straight part as $\vec{r}(t) = (t, y)$, $c \le t \le x$,

$$\begin{aligned} f(x,y) &= \int_A^{(c,y)} \vec{F}\cdot d\vec{r} + \int_{(c,y)}^{(x,y)} \vec{F}\cdot d\vec{r} \\ &= K(y) + \int_c^x \vec{F}(t,y)\cdot \frac{d\vec{r}(t)}{dt}\,dt, \text{ with } K(y) = \int_A^{(c,y)} \vec{F}\cdot d\vec{r} \\ &= K(y) + \int_c^x \langle F_1(t,y), F_2(t,y)\rangle \cdot \langle 1, 0\rangle \, dt \\ &= K(y) + \int_c^x F_1(t,y)\,dt \end{aligned}$$

Differentiating with respect to x, and using Part 1 of the Fundamental Theorem of Calculus gives

$$\frac{\partial f}{\partial x} = F_1(x,y).$$

Similarly, $\dfrac{\partial f}{\partial y} = F_2(x,y)$, so $\nabla f = \langle F_1, F_2\rangle = \vec{F}$, as needed. ■

See Examples 31 to 33 in Section 6.3 of OSC3[5].

6.3.5 Testing if a Vector Field is Conservative

Revised on April 11.It would be nice to be able to check if a vector field is conservative without computing an infinite number of path integrals! Fortunately this can be done: here only the 2D case will be described.

First, there is a simple condition that must be true for $\vec{F}$ to be conservative.

Theorem 6.3.6 If $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$ is conservative and both components are differentiable, then it satisfies the cross-partials condition

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{6.3.2}$$

By Theorem 6.3.5, p. 101 and Theorem 6.3.3, p. 101 above, the cross-partials condition is then also implied by the path-independence of $\int_C \vec{F}\cdot d\vec{r}$ on all paths in the domain and by having $\oint_C \vec{F}\cdot d\vec{r} = 0$ on any closed path in D.

Proof. If $\vec{F}$ is conservative, $P = \dfrac{\partial f}{\partial x}$, $Q = \dfrac{\partial f}{\partial x}$, so using Clairaut's Theorem,

$$\frac{\partial P}{\partial y} = \frac{\partial f}{\partial y \partial x} = \frac{\partial f}{\partial x \partial y} = \frac{\partial Q}{\partial x}.$$

■

See Examples 34 to 36 in Section 6.3 of OSC3[6].

We will see soon that the converse is also true under a restriction on the shape of the domain, but it is not true in every situation, as this exercise shows.

Exercise 6.3.7 Consider

$$\vec{F}(x,y) = P\hat{\imath} + Q\hat{\jmath} = \frac{-y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}$$

[5] openstax.org/books/calculus-volume-3/pages/6-3-conservative-vector-fields
[6] openstax.org/books/calculus-volume-3/pages/6-3-conservative-vector-fields

on its natural domain, all of $\mathbb{R}^2$ except the origin.
Show that this satisfies the cross-partials condition Equation (6.3.2) but its path integral along the circle of radius one going anti-clockwise around the origin is not zero, so $\vec{F}$ is not conservative.

The above cross-partials condition Equation (6.3.2) does guarantee that a vector field is conservative under the restriction that, loosely speaking, the domain has no holes in it.

This excludes the domain in Exercise 6.3.7, p. 102 which has a hole at the origin.

More precisely, we say a domain is simply connected if for any closed path, all the point inside the path are in the domain D.

See Checkpoint 25 in Section 6.3 of OSC3[7].

Then

Theorem 6.3.8 Cross-partials condition. If $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$ has simply connected domain D and P and Q are differentiable, then $\vec{F}$ is conservative if and only if it satisfies the cross-partials condition $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.
Again Theorem 6.3.5, p. 101 and Theorem 6.3.3, p. 101 thus show that on a simply connected domain, these two properties are also equivalent to the path-independence of $\int_C \vec{F} \cdot d\vec{r}$ on all paths in the domain, and to having $\oint_C \vec{F} \cdot d\vec{r} = 0$ on any closed path in the domain.

The proof can only be sketched in this course, and this will be done in Subsection 6.4.6, p. 107.

Example 6.3.9 For the vector field $\vec{F}$ in the previous exercise, but restricted to domain $x > 0$, note that this domain is simply connected and verify that $\vec{F}$ is the gradient of $\arctan(y/x)$ on that domain.
Thus, the integrals of this vector field around any closed path in this domain is zero. In fact the same is true so long as the path does not go around the hole at the origin. Roughly, the function f is the angle θ in polar coordinates, and the integral along a path is the change in this angle between the endpoints of the path. For a closed path that loops around the origin, this can be any multiple of 2π.
(Aside: the value of this integral along a closed path divided by 2π is called the winding number of the path around the origin.) □

6.3.6 Conservation of Energy

Perhaps the most important example of path integrals is the work done by a force $\vec{F}$ as an object moves along a path C given by $\vec{r}$, $a \le t \le b$,

$$W = \int_C \vec{F} \cdot d\vec{r}$$

This force also produces an acceleration $\vec{r}$", with $\vec{F}(\vec{r}) = m\vec{r}$", so the work done by the force as the object moves along the path is

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_a^b m\vec{r}'' \cdot \vec{r}' dt = \frac{m}{2}\int_a^b \frac{d}{dt}\left(\vec{r}' \cdot \vec{r}'\right) dt = \frac{m}{2}\left[\vec{r}' \cdot \vec{r}'\right]_a^b = \frac{m|v(b)|^2}{2} - \frac{m|v(a)|^2}{2}$$

Here $\vec{v} = \vec{r}'$ is the velocity so $|\vec{v}|$ is the speed and $\dfrac{m|v(t)|^2}{2}$ is half the mass times the speed squared: the kinetic energy. Thus, the work done by the force is the change in the kinetic energy of the object.

If the force is conservative, it is a gradient, so for a suitable function P, $\vec{F} = -\nabla P$.

Note the different sign for the potential energy of physics compared to the mathematical "potential" f discussed above!

[7] openstax.org/books/calculus-volume-3/pages/6-3-conservative-vector-fields

Thus work is also given by

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C -\nabla P \cdot d\vec{r} = -[P(\vec{r}(b) - P(\vec{r}(a)],$$

so the change in the potential energy is $-W$.

Defining the total energy as kinetic energy plus potential energy, its change is the sum of these changes, $W + (-W) = 0$: The total energy is conserved.

See Example 37 in Section 6.3 of OSC3[8].

Study Guide. Study Section 6.3 of OSC3[9]; in particular

- All the Definitions and Theorems.
- The Problem Solving Strategy for calculating a Potential for a vector field.
- All Examples (and the Checkpoints following each).
- The true/false exercises 99–102.
- The following exercises (for pairs and ranges, do at least one in each): 103, 106–111, 112&113, , 126&129.

6.4 Green's Theorem

Revised on April 11.

References.

- OpenStax Calculus Volume 3, Section 6.4[1].
- Calculus, Early Transcendentals by Stewart, Section 16.4.

Introduction. The Fundamental Theorem of Calculus in the form

$$\int_{[a,b]} \frac{df}{dx} dx = f(b) - f(a)$$

gives the integral of the derivative of a function over an interval in terms of the values of the function itself at the "edges" of that interval.

Green's Theorem is one of three results that extend this idea to multiple integrals, where the "edge" of a domain becomes a curve for double integrals and a surface for triple integrals, so the values at the edge must be integrated over such curves or surfaces.

6.4.1 Simple Closed Curves, Positive Orientation, and Green's Theorem

The simplest case is for a double integral over a region whose boundary is a simple closed curve, where simple means that the boundary curve does not intersect itself (except that its terminal point is the same as its initial point).

The value of a path integral can depend on the direction of movement along the path, which for a simple closed curve can be described as clockwise or anti-clockwise; to clarify, we specify a default direction of rotation:

[8] openstax.org/books/calculus-volume-3/pages/6-3-conservative-vector-fields
[9] openstax.org/books/calculus-volume-3/pages/6-3-conservative-vector-fields
[1] openstax.org/books/calculus-volume-3/pages/6-4-greens-theorem

Definition 6.4.1 Positive Orientation. A simple closed curve has positive orientation if it is traversed anti-clockwise (the ”trigonometric” direction).
Another way to think of this for a curve C that is the boundary of a region D is that when moving forward along the curve, the interior of the region is to the left. $\lozenge$

Alternative Notation. The use of positive orientation for the integral around a simple closed curve is sometimes indicated by $\oint_C$ and the boundary of D is sometime denoted ∂D, giving the alternative notations

$$\int_C Pdx + Qdy = \oint_C Pdx + Qdy = \oint_{\partial D} Pdx + Qdy$$

Theorem 6.4.2 Green’s Theorem (Circulation Form). Let D be a domain in the plane whose boundary can be described by a positively oriented simple closed curve $C = \partial D$.
For $f(x, y)$ a function that is continuous on D and on an open region surrounding D,

$$\oint_{\partial D} f(x, y)\ dx = -\iint\limits_D \frac{\partial f}{\partial y}\ dA \tag{6.4.1}$$

and

$$\oint_{\partial D} f(x, y)\ dy = \iint\limits_D \frac{\partial f}{\partial x}\ dA \tag{6.4.2}$$

Thus for $\vec{F} = \langle P, Q\rangle$ continuous on D and on an open region surrounding D, the circulation around the boundary of D is

$$\begin{aligned} \oint_{\partial D} \vec{F} \cdot d\vec{r} = \oint_{\partial D} \vec{F} \cdot \widehat{T}\ ds = \oint_{\partial D} P\ dx + Q\ dy \\ = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA \end{aligned} \tag{6.4.3}$$

6.4.2 Partial Proof of Green’s Theorem

Green’s Theorem can be verifed in several stages:

1. Verify Equation (6.4.1) on Type I regions. (This is where most of the work is.)
2. Verify it on a general region by dividing it into a collection of Type I regions, as done for double integrals in Section 5.2, p. 70.
3. Verify (6.4.2) similarly, except starting with Type II regions.
4. Combine these with $f = P$ in (6.4.1) and $f = Q$ in (6.4.2) to get Equation (6.4.3).

6.4.3 Verifying Equation (6.4.1) on Type I regions

Consider a Type I domain $D = \{(x, y) | a \le x \le b, B(x) \le y \le T(x)\}$.

Its boundary is the oriented path $\partial D = C_B + C_R + C_T + C_L$ where

1. $C_B = \{(x, B(x)), a \le x \le b\}$, the bottom of D going left to right,
2. $C_R = \{(b, y), B(x) \le y \le T(x)\}$, the vertical line at the right, if any,
3. $C_T = \{(x, T(x)), b \ge x \ge a\}$, the top of D, going right to left, and
4. $C_L = \{(a, y), T(x) \ge y \ge B(x)\}$, the vertical line at the left, if any.

The ”side” integrals $\int_{C_L} f\ dx$ and $\int_{C_R} f\ dx$ are zero because x is constant on these paths, so

$$\begin{aligned}\oint_{\partial D} f\ dx &= \int_{C_B} f\ dx + \int_{C_T} f\ dx \\ &= \int_{x=a}^{b} f(x, B(x))\ dx + \int_{x=b}^{a} f(x, T(x))\ dx \\ &= -\int_{x=a}^{b} [f(x, T(x)) - f(x, B(x))]\ dx\end{aligned}$$

The Fundamental Theorem of Calculus gives

$$\int_{y=B(x)}^{T(x)} \frac{\partial f(x,y)}{\partial y}\ dy = f(x, T(x)) - f(x, B(x))$$

leading to

$$\begin{aligned}\oint_{\partial D} f\ dx &= -\int_{x=a}^{b} \int_{y=B(x)}^{T(x)} \frac{\partial f}{\partial y}\ dy\ dx \\ &= -\iint_D \frac{\partial f}{\partial y}\ dA\end{aligned}$$

which is Equation (6.4.1).

6.4.4 Verifying Equation (6.4.1) on more general domains

As discusssed in Section 5.2, p. 70, integration over a more genereral domain in $\mathbb{R}^2$ can be done by cutting the domain into a collection of Type I domains. One method is to cut along each of the vertical lines that pass through points on the boundary where the tangent is vertical (which are the only points where there is a problem descibing the boundary with y as a function of x, as needed for ”Type I”)

The double integral of the whole domain is just the sum of these double integrals. On the other side of the equation, the boundaries of the new Type I pieces consist of:

- pieces of the boundary of the original domain, and
- the new edge pieces which arise in pairs on either side of each cut.

The first collection just add up to the original boundary curve, so give the desired path integral around the original boundary.

The motion along the new ”internal” edges is still anticlockwise, so with the respective regions being on the opposite side, keeping the ”inside” to the left requires that the motion be in the opposite direction on the two members of a pair, and so the integrals from each pair cancel out as seen in Subsection 6.2.3, p. 98).

6.4.5 Verifying Equation (6.4.2)

The second integral is similar; I sketch enough detail to show where the sign change arises.

This time one starts with a Type II domain $D = \{(x, y)|c \leq y \leq d, L(y) \leq x \leq R(y)\}$, whose boundary path is now $\partial D = C_R + C_T + C_L + C_B$ with

1. $C_B = \{(x, c), L(c) \leq x \leq R(c)\}$,
2. $C_R = \{(R(y), y), c \leq y \leq d\}$, the right boundary of D, going upward,
3. $C_T = \{(x, d), R(d) \geq x \geq L(d)\}$, and
4. $C_L = \{(L(y), y), d \geq y \geq c\}$, the left boundary of D, going downward.

The top and bottom integrals vanish as the side integrals did above, leaving

$$\begin{aligned}
\oint_{\partial D} Q(x,y)\ dy &= \int_{C_R} Q(x,y)\ dy + \int_{C_L} Q(x,y)\ dy \\
&= \int_{y=c}^{d} Q(R(y),y)\ dy + \int_{x=d}^{c} Q(L(y),y)\ dy \\
&= \int_{y=c}^{d} Q(R(y),y) - Q(L(y),y)\ dy \\
&= \int_{y=c}^{d} \int_{x=L(y)}^{R(y)} \frac{\partial Q(x,y)}{\partial x}\ dx\ dy \quad \text{using the FTC again} \\
&= \iint\limits_{D} \frac{\partial Q}{\partial x}\ dA.
\end{aligned}$$

6.4.6 An Application of Green's Theorem: When the Cross-Partials Condition Implies That A Vector Field is Conservative

The cross-partials condition for a vector field to be conservative on a simply connected domain, Theorem 6.3.8, p. 103, was left unproven in Section 6.3, p. 99.

The inference in one direction is already given by Theorem 6.3.6, p. 102 so we only need to verify that the cross-partials condition implies that the field is conservative.

We indicate why this is true by arguing that the cross-partials condition ensures that $\int_C \vec{F} \cdot d\vec{r} = 0$, which was seen in Theorem 6.3.5, p. 101 to be equivalent to being conservative.

If a closed path C is simple, it surrounds a simply connected domain D' within the simply connected domain D, because if anything inside C were not part of D, there would be a "hole" in D. (This is intuitive, but not a rigorous proof).

Green's theorem then gives

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy = \int_{D'} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int_{D'} 0\, dA = 0.$$

Next, for a non-simple path, divide it into pieces that are simple closed curves by cutting at each point where it crosses itself (again this is intuitive but not really proved here). The path integral around C is thus zero because it is the sum of the integrals around each of these simple closed curves, and each of those integrals is zero as argued above.

6.4.7 The Flux Form of Green's Theorem

Theorem 6.4.3 Green's Theorem, Flux Form. Let D be a domain in the plane whose boundary can be described by a positively oriented simple closed curve $C = \partial D$. with the "rightward unit normal" $\widehat{N} = \langle dy/ds, -dx/ds\rangle$ as defined in Equation (6.2.19), which for a positively oriented closed curve is the outward unit normal vector to the boundary curve: it points away from the domain.
For $\vec{F} = \langle P, Q\rangle$ continuous on D and on an open region surrounding D, the flux across the boundary of D is

$$\oint_{\partial D} \vec{F} \cdot \widehat{N}\ ds = \iint\limits_{D} \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\ dA \tag{6.4.4}$$

(This is the form that will be generalized to triple integrals in Section 6.8, p. 123.)
The quantity $\oint_C \vec{F} \cdot \widehat{N}\ ds$ is called the flux across the curve, and for $C = \partial D$, it is the flux out of domain D.

Proof.

$$\oint_{\partial D} \vec{F} \cdot \widehat{N}\ ds = \oint_{\partial D} \langle P, Q\rangle \cdot \langle dy/ds, -dx/ds\rangle\ ds = \oint_{\partial D} -Q\ dx + P\ dy$$

Swapping names $P \to -Q$, $Q \to P$ in (6.4.3) gives

$$\oint_{\partial D} \vec{F} \cdot \widehat{N}\ ds = \iint\limits_{D} \left(\frac{\partial P}{\partial x} - \frac{\partial(-Q)}{\partial y}\right)\ dA = \iint\limits_{D} \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\ dA$$

■

6.4.7.1 Source-free Vector Fields and Their Stream Functions

Added on April 9.

Definition 6.4.4 A vector field with no flux across any closed curve C is called source-free. ◊

Just as having no circulation about all closed curves in a conected domain leads to a description of the vector field in terms of a single function (a potential function), any source-free vector field on such a domain can be described by single function:

Definition 6.4.5 For a vector field $\vec{F} = \langle P, Q\rangle$, a function g satisfying

$$P = \frac{\partial g}{\partial y} \text{ and } Q = -\frac{\partial g}{\partial x} \tag{6.4.5}$$

is a stream function for $\vec{F}$. ◊

Theorem 6.4.6 A source-free vector field on a connected domain has a stream function.

Proof. This can be seen by noting that $\widehat{N}$ is got by rotating $\widehat{T}$ a quarter turn clockwise, so rotating $\vec{F} = \langle P, Q\rangle$ a quarter-turn anticlockwise to define $\vec{G} = \langle -Q, P\rangle$ gives

$$\begin{aligned}\vec{G} \cdot \widehat{T} ds &= \langle -Q, P\rangle \cdot \langle dx, dy\rangle = P\ dy - Q\ dx = \langle P, Q\rangle \cdot \langle dy, -dx\rangle \\ &= \vec{F} \cdot \widehat{N} ds\end{aligned}$$

Thus the source-free condition on $\vec{F}$ gives $\oint_C \vec{G} \cdot \widehat{T}\ ds = \oint_C \vec{F} \cdot \widehat{N}\ ds = 0$. That is, by Theorem 6.3.5, p. 101, $\vec{G}$ is conservative: there is a function g with $\vec{\nabla} g = \langle g_x, g_y\rangle = \vec{G} = \langle -Q, P\rangle$, so

$$P = \frac{\partial g}{\partial y} \text{ and } Q = -\frac{\partial g}{\partial x}$$

■

6.4.8 Green's Theorem for Non-simply Connected Domains

If a domain D has "holes", it can intuitively be considered as coming from a larger domain E by removing one or more simply connected pieces H_1, H_2, etc., so that $E = D \cup H_1 \ldots$.

Then intuitively

$$\iint\limits_{E} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA = \iint\limits_{D} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA + \iint\limits_{H_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA + \cdots$$

or

$$\iint\limits_{D} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA = \iint\limits_{E} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA - \iint\limits_{H_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA\ \cdots$$

All the integrals at right are over simply connected regions, so Green's Theorem in the form of Equation (6.4.3) allows each to be written as a circulation.

This motivates

Theorem 6.4.7 For a region D consisting of an enclosing simply connected region E minus simply connected "holes" H_1 and so on, so that it has "outer boundary" ∂E and inner boundaries around each hole,

$$\iint\limits_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA = \oint\limits_{\partial E} P\ dx + Q\ dy - \oint\limits_{\partial H_1} P\ dx + Q\ dy\ \cdots$$

That is, the integral of $Q_x - P_y$ over region D is the circulation around the outer boundary curve minus the circulation around each of the inner boundary curves.

Proof. The more careful proof of this is done by cutting with curves C_1 and so on from the outer boundary of D to each of the holes, giving a simply connection region whose boundary consists of

- The outer boundary ∂E, cut into several pieces.
- The boundary of each hole, but traversed clockwise, so that the pieces are $-\partial H_1$ and so on.
- The two sides of each of the cut curves: C_1, $-C_1$ and so on.

The cuts do not change what is in the region, so the double integral is unchanged. On the other hand, the integrals along the two sides of each cut cancel out, leaving the boundary circulation integral as

$$\begin{aligned}&\oint_{\partial D} P\ dx + Q\ dy + \oint_{-\partial H_1} P\ dx + Q\ dy\ ...\\ &= \oint_{\partial D} P\ dx + Q\ dy - \oint_{\partial H_1} P\ dx + Q\ dy\ ...\end{aligned}$$

■

Because of this, the boundary of such a non-simply connected region is sometimes denoted

$$\partial D = \partial E - \partial H_1 \ldots$$

and then Equation Theorem 6.4.2, p. 105 still holds.

Study Guide. Study Section 6.4 of OSC3[2]; in particular

- The concept of a positively oriented curve in the plane (anti-clockwise rotation)
- Both versions of Green's Theorem.
- All Examples and Checkpoints.
- One or several exercises from each of the following ranges: 146–151, 152–160, 161–169, 174–179.

6.5 Divergence and Curl

Revised on April 11.

References.

- OpenStax Calculus Volume 3, Section 6.5[1].
- Calculus, Early Transcendentals by Stewart, Section 16.5.

[2]openstax.org/books/calculus-volume-3/pages/6-4-greens-theorem
[1]openstax.org/books/calculus-volume-3/pages/6-5-divergence-and-curl

6.5.1 Divergence

6.5.1.1 The Divergence of a Two Dimensional Vector Field

For two dimensional vector fields, the object ∇ or $\vec{\nabla}$ introduced in Equation (6.1.4) of Section 6.1, p. 92 is formally the vector $\vec{\nabla} = \hat{\imath}\dfrac{\partial}{\partial x} + \hat{\jmath}\dfrac{\partial}{\partial y}$, suggesting the dot product

$$\vec{\nabla} \cdot \vec{F} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y}\right) \cdot (\hat{\imath}P + \hat{\jmath}Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

This is the two dimensional divergence, also denoted Div $\vec{F}$, so the flux form of Green's Theorem (6.4.4) can be written as the Two Dimensional Divergence Theorem

$$\oint_{\partial D} \vec{F} \cdot \widehat{N}\ ds = \iint\limits_{D} \vec{\nabla} \cdot \vec{F}\, dA \tag{6.5.1}$$

For the example of $\vec{F}$ describing fluid velocity, $\vec{\nabla} \cdot \vec{F}$ is related to "net outflow", underlying the name "divergence".

Divergence-free fields.
Theorem 6.5.1 A source-free vector field in the plane is also divergence-free. That is, "$\oint_C \langle P, Q\rangle \cdot \widehat{N} ds = 0$ implies $P_x + Q_y = 0$".

Proof. From Theorem 6.4.6, p. 108 a source-free field has a stream function g: $P = g_y$ and $Q = -g_x$. Thus Clairault's Theorem gives

$$P_x + Q_y = (g_y)_x + (-g_x)_y = 0.$$

■

As we have seen many times before, the converse is also true when the domain is also simply connected:

Theorem 6.5.2 Divergence Test for Source-Free Vector Fields. A continuous and differentiable vector field $\vec{F} = \langle P, Q\rangle$ on a simply connected domain is source-free if and only if it is divergence-free.

Proof. The main idea of the proof is conecting to the analogous Theorem 6.3.8, p. 103 by the strategy of "rotating by a quarter turn" seen in the proof of The Flux form of Green's Theorem 6.4.3, p. 107. ■

See Example 50 in Section 6.5 of OSC3[2].

One important example is that magnetic fields are always descibed by divergence-free vector fields.

See Example 49 in Section 6.5 of OSC3[3].

6.5.1.2 The Divergence of a Three Dimensional Vector Field

For a vector field in three dimensions, $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$, the divergence of F is

$$\text{Div } \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

See Example 48 in Section 6.5 of OSC3[4].

The extension of Equation (6.5.1) to three dimensions is the Divergence Theorem 6.8.1, p. 123 to be seen in Section 6.8, p. 123.

[2] openstax.org/books/calculus-volume-3/pages/6-5-divergence-and-curl
[3] openstax.org/books/calculus-volume-3/pages/6-5-divergence-and-curl
[4] openstax.org/books/calculus-volume-3/pages/6-5-divergence-and-curl

6.5.2 Curl

The first, "circulation" form of Green's Theorem Theorem 6.4.2, p. 105 suggests that the quantity $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ integrated measures (anti-clockwise) rotation in the vector field, perpendicur to the plane, and so around the z-azis.

It is useful to think of this as a vector quantity

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

and for vector fields in space, extending this to corresponding measures of rotation with respect to the x and y axes. This lead to the quantity sometimes called the rotation, Rot $\vec{F}$ when the vector field describes the velocity in a fluid, but now more often call the curl of a vector field on $\mathbb{R}^3$:

Definition 6.5.3 For a differentiable vector field $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$ on $\mathbb{R}^3$ its curl is given by

$$\text{Curl } \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k} \tag{6.5.2}$$

◊

See Examples 52 and 53 in Section 6.5 of OSC3[5].

This is an important quantity in the description not only of fluid flow but also electro-magnetic fields, and it is related to whether the vector field is conservative.

For example, with a 2D vector field $\vec{F} = P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath}$, this is just Curl $\vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$, and so the condition Curl $\vec{F} = \vec{0}$ is the cross-partials condition of Section 6.3, p. 99 related to $\vec{F}$ being conservative.

Using this 2D case of the curl, the original form of Green's Theorem in Equation (6.4.3) can be put in the form

$$\oint_{\partial D} \vec{F} \cdot d\vec{r} = \oint_{\partial D} \vec{F} \cdot \vec{T} ds = \iint_D (\text{Curl } \vec{F}) \cdot \hat{k}\, dA \tag{6.5.3}$$

This is sometimes also called the planar version of Stokes' Theorem 6.7.2, p. 120 to be seen in Section 6.7, p. 119.

Curl and Rotation in a Fluid. The name "curl" refers to a measure of rotation. For example, the vector field $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$ describes velocity of a fluid (say) going anti-clockwise around the z-axis: it has Curl $\vec{F} = 2\hat{k}$ in which the direction $\hat{k}$ indicates the axis of rotation, the positive value indicates anti-clockwise direction as viewed from "above" down that z-axis, and the uniform magnitude indicating the uniform angular rate of rotation (which has period 2π).

The $\vec{\nabla}$ Short-hand for the Curl. The formal $\vec{\nabla}$ shorthands for the gradient Grad$f = \vec{\nabla} f$ and the divergence Div$f = \vec{\nabla} \cdot \vec{F}$ also have a counterpart for the curl, via the formal cross product:

$$\begin{aligned}
&\vec{\nabla} \times \vec{F} \\
&\quad = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&\quad = \hat{\imath}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \hat{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\
&\quad = \text{Curl } \vec{F}.
\end{aligned}$$

Hence we sometimes use $\vec{\nabla} \times \vec{F}$ as mnemonic notation for Curl $\vec{F}$.

[5] openstax.org/books/calculus-volume-3/pages/6-5-divergence-and-curl

6.5.3 Some Connections Between Div, Curl and Grad

Added on April 11.

Gradients are "Curl Free", and vice versa on Simply Conected Domains. The curl is involved in the 3D version of the mixed partials condition in Section 6.3, p. 99:

Theorem 6.5.4

(1) The curl of a conservative vector field $\vec{F} = \text{Grad} f$ vanishes: that is

$$\text{Curl}(\text{Grad} f) = \vec{\nabla} \times (\vec{\nabla} f) = (\vec{\nabla} \times \vec{\nabla}) f = \vec{0}.$$

(The final form is given as a mnemonic: if you think of $\vec{\nabla}$ as being a true vector, its cross product with itself should be zero.)

(2) On a simply conected domain, the converse is also true: if Curl $\vec{F} = \vec{0}$ then $\vec{F}$ is a gradient.

See Example 56 in Section 6.5 of OSC3[6].

Proof of part (1). The verification of the first half is a straightforward calculation using Clairaut's Theorem: it is like three versions of the mixed partials condition for a conservative vector field in $\mathbb{R}^2$ seen in Section 6.3, p. 99. ■

Proof of part (2); postponed. The proof of the converse (2) is difficult; the main ideas will be sketched in Section 6.7, p. 119, using Stokes' Theorem, p. 120 from that section.
As a hint, in the plane this is the conclusion of Theorem 6.3.8, p. 103. ■

Curls are "Divergence Free".
Theorem 6.5.5 If $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$ is a vector field on $\mathbb{R}^3$ and all components have continuous second derivatives, then

$$\text{Div Curl } \vec{F} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0.$$

Proof. This can again be verified by straightforward calculation and using Clairaut's Theorem. ■

See Example 55 in Section 6.5 of OSC3[7].

The Laplacian of a Function and the Laplace Operator. There is one other very important combination of divergence, curl and gradient:

$$\text{Div Grad} f = \vec{\nabla} \cdot \vec{\nabla} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

This is the Laplacian of f often abbreviated as $\vec{\nabla}^2 f$ or Δf. For functions of two variables this is

$$\Delta f = \vec{\nabla}^2 f = \vec{\nabla} \cdot \vec{\nabla} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

This operation Δ is called the Laplace operator, with the symbolic form

$$\Delta = \vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The Laplacian of a vector field is the vector field got by applying the Laplace operator to each component in turn; that is:

$$\Delta \vec{F} = \Delta P\, \hat{\imath} + \Delta Q\, \hat{\jmath} + \Delta R\, \hat{k}$$

[6]openstax.org/books/calculus-volume-3/pages/6-5-divergence-and-curl
[7]openstax.org/books/calculus-volume-3/pages/6-5-divergence-and-curl

Summary: All the Second Derivative Combinations. Here are fomulas for all the combinations of Div, Curl and Grad that make sense:

- Div Grad$f = \vec{\nabla} \cdot \vec{\nabla} f = \vec{\nabla}^2 f = \Delta f$
- Curl Grad$f = \vec{\nabla} \times \vec{\nabla} f = \vec{0}$
- Div Curl $\vec{F} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$
- and the final ones combine in one formula:

$$\text{Curl Curl } \vec{F} = \text{Grad Div } \vec{F} - \text{DivGrad}\vec{F};$$

  that is,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - \vec{\nabla}^2 \vec{F}.$$

6.5.4 Some Fundamental Differential Equations of Physics

The Laplacian arises in several of the fundamental equations for physics:

$$\begin{array}{rr} \text{Laplace's Equation} & \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0 \\ \text{The Poisson Equation} & \Delta u = f, \\ \text{The Heat Equation} & \dfrac{\partial u}{\partial t} = \Delta u, \\ \text{The Wave Equation} & \dfrac{\partial^2 u}{\partial t^2} = \Delta u, \text{and} \\ \text{The Schrödinger Equation} & i\hbar \dfrac{\partial \psi}{\partial t} = -\dfrac{\hbar^2}{2m}\Delta\psi + V(\vec{x})\psi \end{array}$$

used in the description of fluid motion, electric fields, heat conduction, waves in water and in electro-magnetic fields, and in quantum mechanics.

Study Guide. Study Section 6.5 of OSC3[8]; in particular

- The definitions of divergence and of curl.
- The concept of a source-free vector field: a vector field $\vec{F} = \langle P(x,y), Q(x,y)\rangle$ for which there is a stream function $g(x,y)$ for which $P = g_y, Q = -g_x$ from Subsubsection 6.4.7.1, p. 108, and its connection to a field being divergence-free: $\vec{\nabla} \cdot \vec{F} = 0$; see Theorems 14 and 15.
- Theorem 16: Div Curl $\vec{F} = 0$, or $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$ (as if $\vec{\nabla}$ were a real vector.)
- Theorem 17, which can be restated as $(\vec{\nabla} \times \vec{\nabla})f = 0$ (again as if $\vec{\nabla}$ were a real vector.)
- Theorem 18, a restatement of the cross-partial condition of Theorem 6.3.8, p. 103 in Section 6.3, p. 99.
- Examples 48–50, 52–56, and the Checkpoints following each.
- The T/F Exercises 207–211 and one or several exercises from each of the following ranges: 212–221, 222–231, 256, 257, 258.

6.6 Surface Integrals

Revised on April 18.

References.

- OpenStax Calculus Volume 3, Section 6.6[1].

[8] openstax.org/books/calculus-volume-3/pages/6-5-divergence-and-curl
[1] openstax.org/books/calculus-volume-3/pages/6-6-surface-integrals

- Calculus, Early Transcendentals by Stewart, Section 16.7.

6.6.1 Parametric Surfaces

Just as not all plane curves can be conveniently described as the graph of a function $y = f(x)$ and are instead best described parametrically as $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$, so surfaces are often decribed parametrically, but now with two parameters:

$$\vec{r}(u, v) = x(u, v)\hat{\imath} + y(u, v)\hat{\jmath} + z(u, v)\hat{k} \tag{6.6.1}$$

for (u, v) in some domain D in $\mathbb{R}^2$.

The set of all points (x, y, z) in $\mathbb{R}^3$ given by this form is the parametric surface S, with the parametric equations

$$x = x(u, v),\ y = y(u, v),\ z = z(u, v) \tag{6.6.2}$$

See Examples 58 to 60 in Section 6.6 of OSC3[2].

Grid Curves: Curves on a Surface with Constant u or v. In visualizing and drawing surfaces it can be useful to consider

1. lines with constant v value, $v = v_0$, which are the parameterized curves $\vec{r}(u) = \vec{r}(u, v_0)$, and
2. lines with constant u value, $u = u_0$: $\vec{r}(v) = \vec{r}(u_0, v)$.

These are grid curves.

Surfaces of Revolution. A familiar example is surfaces of revolution, given by rotating a curve around some axis. The surface given by rotating the curve $y = f(x)$, $a \le x \le b$ about the x-axis can be described using the two parameters x and the angle θ measuring rotation around that axis, starting at the y axis and going towards the z-axis:

$$x = x, y = f(x)\cos\theta, z = f(x)\sin\theta,\ \text{domain } D = [a, b] \times [0, 2\pi].$$

Tangent Planes. To find the tangent plane to S at the point $P(x_0, y_0, z_0)$ given by parameter values (u_0, v_0), one can first find two tangent directions,so that their cross product is a normal to the surface and to the tangent plane.

Natural choices are the tangent lines given by using the linearizations of $x(u, v)$, $y(u, v)$ and $z(u, v)$at this point and varying one parameter while holding the other constant. Note that these are tangent lines to the grid curves.

Varying u while fixing $v = v_0$ gives linearizations

$$\begin{aligned}
x &= x(u_0, v_0) + \left[\frac{\partial x}{\partial u}(u_0, v_0)\right](u - u_0),\\
y &= y(u_0, v_0) + \left[\frac{\partial y}{\partial u}(u_0, v_0)\right](u - u_0),\\
z &= z(u_0, v_0) + \left[\frac{\partial z}{\partial u}(u_0, v_0)\right](u - u_0).
\end{aligned}$$

This gives the tangent vector in the u direction

$$\vec{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\hat{\imath} + \frac{\partial y}{\partial u}(u_0, v_0)\hat{\jmath} + \frac{\partial z}{\partial u}(u_0, v_0)\hat{k}$$

[2] openstax.org/books/calculus-volume-3/pages/6-6-surface-integrals

Similarly, varying v with fixed $u = u_0$ gives tangent vector

$$\vec{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\hat{\imath} + \frac{\partial y}{\partial v}(u_0, v_0)\hat{\jmath} + \frac{\partial z}{\partial v}(u_0, v_0)\hat{k}$$

If the cross product $\vec{r}_u(u_0, v_0) \times \vec{r}_v(u_0, v_0)$ is non-zero then it is a normal vector to S at point P and the tangent plane there is

$$(\langle x, y, z\rangle - \langle x_0, y_0, z_0\rangle) \cdot (\vec{r}_u(u_0, v_0) \times \vec{r}_v(u_0, v_0)) = \vec{0}$$

If the cross product $\vec{r}_u \times \vec{r}_v$ is non-zero for all (u, v) in D, this is caled a regular parameterization and the surface S is called smooth: loosely, it has no corners or creases, and there is a well-defined tangent plane everywhere on the surface.

See Example 61 in Section 6.6 of OSC3[3].

This is analagous to the condition $\vec{r}'(t) \neq \vec{0}$ in Section 3.2 for a curve to be smooth and with well-defined tangent vector.

6.6.2 Surface Area of a Parametric Surface

The area of the surface S can now be defined, and the intuitive approach is to look at the area of the infinitesimal part of the surface near a point with parameters (u, v) given by varying u by an infinitesimal amount du and v by dv. This surface is an infinitesimal parallelogram with edges given by the above linearization as $\vec{r}_u du$ and $\vec{r}_v dv$, and so of infinitesimal area dS,

$$dS = \|\vec{r}_u du \times \vec{r}_v dv\| = \|\vec{r}_u \times \vec{r}_v\| du\ dv \tag{6.6.3}$$

Note that if the surface is a flat region in the x-y plane, so $z(u, v) = 0$, then

$$\|\vec{r}_u \times \vec{r}_v\| = \left\| \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \hat{k} \right\| = \left| \frac{\partial(x, y)}{\partial(u, v)} \right|$$

and so this becomes the infinitesimal planar area dA of Equation (5.7.11) in Section 5.7, p. 85.

Combining these infinitesimal areas by integration motivates the following definition.

Definition 6.6.1 For a smooth parametric surface S given by

$$x = x(u, v),\ y = y(u, v),\ z = z(u, v)$$

with (u, v) in D, the surface area of S is

$$A(S) = \iint_D dS = \iint_D \|\vec{r}_u \times \vec{r}_v\|\ du\ dv = \iint_D \|\vec{r}_u \times \vec{r}_v\|\ dA$$

where dA refers to area in the (u, v) plane. ◊

Exercise 6.6.2 Compute the surface area of a sphere, parameterizing it with spherical coordinates.

See Examples 62, 63 and 64 in Section 6.6 of OSC3[4].

Alternatively, one can start from the beginning: approximate the areas of small parts of the surface, sum them, take a limit, and recognizing the limit as the above integral.

[3]openstax.org/books/calculus-volume-3/pages/6-6-surface-integrals
[4]openstax.org/books/calculus-volume-3/pages/6-6-surface-integrals

The Surface Area of the Graph of a Function. In the case where the surface is the graph of a function $z = g(x,y)$, the parameters can be simply x and y, and straightforward calculations give

$$\|\vec{r}_x \times \vec{r}_y\| = \|\langle -\partial g/\partial x, -\partial g/\partial y, 1\rangle\| = \sqrt{1 + (\partial g/\partial x)^2 + (\partial g/\partial y)^2}$$

so

$$dS = \sqrt{1 + (\partial g/\partial x)^2 + (\partial g/\partial y)^2}\ dA \tag{6.6.4}$$

Thus the surface area is

$$A(S) = \iint\limits_D dS = \iint\limits_D \sqrt{1 + (\partial g/\partial x)^2 + (\partial g/\partial y)^2}\ dx\ dy \tag{6.6.5}$$

(Compare this to the formula (1.2.4) for the arc length of the graph of a function in Section 1.2, p. 4.)

Exercise 6.6.3 Compute the surface area of a sphere again, this time by considering a hemisphere as the graph of a function.

6.6.3 Surface Integral of a Scalar-Valued Function

The formula for surface area seen in the previous section was based on the formula

$$dS = \|\vec{r}_u \times \vec{r}_v\|\ dA$$

for the area dS of an infinitesimal fragment of a parametric surface $\vec{r}(u,v)$ in terms of the corresponding infinitesimal area $dA = du\ dv$ in the (u,v) plane of the parameters. To integrate a function over a surface, such as summing a density $f = f(x,y,z)$ over the surface to compute a total mass, start with the idea that the mass on a small fragment of the surface is density times area:

$$f\ dS = f\|\vec{r}_u \times \vec{r}_v\|\ dA$$

Thus the natural form for the integral of function f over surface S is

$$\iint\limits_S f\ dS = \iint\limits_D f(\vec{r}(u,v))\|\vec{r}_u \times \vec{r}_v\|\ dA \tag{6.6.6}$$

This can be taken as the definition of the integral over a surface, or one can define the integral in terms of limits of sums and derive this result, as already seen in various cases in these notes.

The formula for surface area in the previous section is just the case of integrating the constant function $f = 1$.

See Examples 66 and 67 in Section 6.6 of OSC3[5].

Surface Integrals for Graphs of Functions. When the surface is the graph of $z = f(x,y)$ over domain D, using the form for the infinitesimal area dS in Equation (6.6.4) and $f(x,y,z) = f(x,y,g(x,y))$ gives

$$\iint\limits_S f\,dS = \iint\limits_D f(x,y,g(x,y))\sqrt{1 + (\partial g/\partial x)^2 + (\partial g/\partial y)^2}\ dA$$

See Example 68 in Section 6.6 of OSC3[6].

[5] openstax.org/books/calculus-volume-3/pages/6-6-surface-integrals
[6] openstax.org/books/calculus-volume-3/pages/6-6-surface-integrals

6.6.4 Oriented Surfaces

To integrate the normal component of a vector field over a surface, we need something like the outward unit normal vector $\widehat{N}$ constructed for a positively oriented curve in Subsection 6.4.7, p. 107.

There are two choices of unit normal at each point of a smooth surface, and we need to make a consistent choice over the whole surface which varies continuously, so that the normal can be considered as pointing "outward", or "upward" for example.

Definition 6.6.4 A surface is oriented if there is a continuous function $\widehat{N}$ defined everywhere on the surface whose value at each point is a unit vector normal to the tangent plane at that point. ◊

Note that for any oriented surface, there are two choices of orientation for the surface, since negating one suitable $\widehat{N}$ gives another. The two-way choice here is akin to the choice of positive (anti-clockwise) orientation for a curve.

See Example 69 in Section 6.6 of OSC3[7].

Non-orientable Surfaces: The Möbius Strip etc.. Not all surfaces are oriented: the Möbius strip is a famous example of a surfaces where one can move from one side to the other by traveling along the surface, so starting on the side indicated by a continuous unit normal, as you move around without jumping sides, the normal should continue to point to your side, but you can move around returning to the same point but on the opposite side, so the normal at the same point must now point in the opposite direction.

This makes it impossible to choose between the two possible unit normals at each point in a way that is continuous over the whole surface.

Fortunately all the most familiar common types of surfaces are oriented: graphs of (differentiable) functions, level sets of (differentiable) functions, and ones with a single overall parameterization.

Oriented Surfaces: Globally Parameterized Surfaces. Smooth surfaces that can be described by a single "global" parametrization $\vec{r}(u,v)$ with $\vec{r}_u \times \vec{r}_v \neq \vec{0}$ describing the whole surface are oriented. This is because that cross product is a continuous normal, giving unit normal

$$\widehat{N} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} \tag{6.6.7}$$

Note how the non-zero conditionfor smoothness is necessary and sufficient for this to make sense everywhere on the surface.

Exercise 6.6.5 Compute this unit normal for the sphere or radius R parametrized with the spherical angular coordinates ϕ, and θ, $\vec{r}(\psi,\theta) = R\sin\phi\cos\theta\hat{\imath} + R\sin\phi\sin\theta\hat{\jmath} + R\cos\phi\hat{k}$.
What happens if we call this $\vec{r}(\theta,\psi)$ instead?

Oriented Surfaces: Graphs of Functions. The surface given by the graph of a function $z = g(x,y)$ is globally parametrized by x and y, and so is oriented. Formula (6.6.7) applied to $\vec{r}(x,y) = \langle x, y, g(x,y)\rangle$ gives

$$\widehat{N} = \frac{\vec{r}_x \times \vec{r}_y}{\|\vec{r}_x \times \vec{r}_y\|} = \frac{\langle -\partial g/\partial x, -\partial g/\partial y, 1\rangle}{\sqrt{1 + (\partial g/\partial x)^2 + (\partial g/\partial y)^2}} \tag{6.6.8}$$

Since the third component is always positive, this is the "upward" unit normal.

Exercise 6.6.6 Compute this unit normal for the hemisphere given as the graph of $z = \sqrt{R^2 - x^2 - y^2}$.

[7]openstax.org/books/calculus-volume-3/pages/6-6-surface-integrals

Oriented Surfaces: Level Sets of Functions. More generally surfaces defined as a level set by an equation $f(x, y, z) = C$, for C a constant, and with $\vec{\nabla} f \neq \vec{0}$ everywhere on the surface are oriented. For then $\vec{\nabla} f$ is a normal to the surface, and

$$\widehat{N} = \frac{\vec{\nabla} f}{\|\vec{\nabla} f\|} \tag{6.6.9}$$

is a suitable unit normal, pointing in the direction of increasing values of f.

It can be shown, using The Implicit Function Theorem in 3D, p. 56, that such a surface is the union of a collection of parametric surfaces; indeed, as mentioned in that theorem, the parameters for each piece can be two of the cartesian coordinates: each piece can be the graph of a function $z = g(x, y)$, $x = g(y, z)$ or $y = g(x, z)$.

Exercise 6.6.7 Compute this unit normal for the sphere given as the level set $x^2 + y^2 + z^2 = R^2$.

Choice of Orientation. Note that there are two choices of unit normal vector function for any oriented surface, as we can always negate $\widehat{N}$. Thus a particular choice of orientation has to be specified, such as outwards from or into a domain D when S is the boundary of D.

It will be then convenient to combine the infinitesimal area dS at a point on the surface with the chosen orientation $\widehat{N}$ of the surface there into the vector differential

$$d\vec{S} = \widehat{N} dS \tag{6.6.10}$$

For a parametrized surface we then have $\vec{r}(u, v)$ $dS = \|\vec{r}_u \times \vec{r}_v\| dA$ and choosing $\widehat{N} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$ as above,

$$d\vec{S} = \widehat{N} dS = (\vec{r}_u \times \vec{r}_v)\ dA$$

For the case of the graph of a function $z = g(x, y)$, combining Equation (6.6.8) for $\widehat{N}$ with Equation (6.6.4) for dS gives

$$d\vec{S} = \widehat{N} dS = \langle -\partial g/\partial x, -\partial g/\partial y, 1\rangle\ dA \tag{6.6.11}$$

Closed Surfaces and Their Outward Unit Normal. A surface S that is the boundary of solid region E is called a closed surface, and then a natural choice of normal is outwards from the surface, which is called the positive orientation.

This is easily specified when the region is given by an inequality $f(x, y, z) \leq k$ [or several such inequalities] because then the boundary is the surface is given by $f(x, y, z) = k$ [or several such equations], and the outward normal is $\widehat{N} = \frac{\vec{\nabla} f}{\|\vec{\nabla} f\|}$ as seen above.

For example, the unit sphere centered at the origin considered as the boundary of the ball $f(x, y, z) = x^2 + y^2 + z^2 \leq 1$ has outward unit normal $\widehat{N} = \langle x, y, z\rangle$.

Exercise 6.6.8 Evaluate $d\vec{S}$ for the surface of the ball $x^2 + y^2 + z^2 \leq R^2$.

6.6.5 Surface Integral of a Vector-Valued Function

The path integral $\oint_C \vec{F} \cdot \widehat{N}\ ds$ around an oriented closed curve has a three dimensional counterpart for oriented surfaces:

Definition 6.6.9 For $\vec{F}$ a continuous vector field defined on an oriented surface S and $\widehat{N}$ a continuous unit normal vector for S, the surface integral of $\vec{F}$ over S is

$$\iint_S \vec{F} \cdot \widehat{N}\ dS, = \iint_S \vec{F} \cdot d\vec{S}. \tag{6.6.12}$$

This is also called the flux of $\vec{F}$ over S. $\Diamond$

See Examples 70 and 71 in Section 6.6 of OSC3[8].

The Integral of a Vector Field over a Parametrized Surface. When surface S is parametrized as $\vec{r}(u,v)$ for (u,v) in domain D, we get a nice simplification:

$$d\vec{S} = \widehat{N}\ dS = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}\|\vec{r}_u \times \vec{r}_v\|\ dA = (\vec{r}_u \times \vec{r}_v)\ dA$$

and so the flux is

$$\iint\limits_S \vec{F}\cdot\widehat{N}\ dS = \iint\limits_S \vec{F}\cdot d\vec{S} = \iint\limits_D \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\ dA \tag{6.6.13}$$

Avoiding the division and the square root in the formula for $\|\vec{r}_u \times \vec{r}_v\|$ can help when it comes to evaluating the integral.

When the surface is the graph of a function $z = g(x,y)$, Equation (6.6.11) gives the form

$$\begin{aligned}\iint\limits_S \vec{F}\cdot d\vec{S} &= \iint\limits_D \vec{F}\cdot\langle -\partial g/\partial x, -\partial g/\partial y, 1\rangle\ \ dA \\ &= \iint\limits_D -P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\ dA \end{aligned} \tag{6.6.14}$$

Study Guide. Study Section 6.6 of OSC3[9]; in particular

- All the Definitions.
- Examples 58–64 and 66–71, and the Checkpoints following each.
- The T/F Exercises 269–271 and one or several exercises from each of the following ranges: 273–278, 281–283, and 284–286.

6.7 Stokes' Theorem

Revised on April 18.

References.

- OpenStax Calculus Volume 3, Section 6.7[1].
- Calculus, Early Transcendentals by Stewart, Section 16.8.

Green's Theorem (Circulation Form), p. 105 says that for D a region in the plane and C its boundary, a positively oriented simply connected curve,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA \tag{6.7.1}$$

Considering D as a surface in $\mathbb{R}^3$ that lies in the plane $z = 0$, it has unit normal vector $\widehat{N} = \vec{k}$, and in the notation of the previous section, $\vec{k}\ dS = \vec{N}\ dS = d\vec{S}$. Also, the integrand at right is the third component of

[8] openstax.org/books/calculus-volume-3/pages/6-6-surface-integrals
[9] openstax.org/books/calculus-volume-3/pages/6-6-surface-integrals
[1] openstax.org/books/calculus-volume-3/pages/6-7-stokes-theorem

the curl of $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$. Thus, Equation (6.7.1) can be expressed as

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \text{Curl } \vec{F} \cdot d\vec{S}.$$

Stokes' Theorem extends this to the case where D is replaced by an oriented surface S in $\mathbb{R}^3$ with continuous unit normal $\widehat{N}$ and bounded by the piecewise smooth curve $C = \partial D$, with the integral around C done with an orientation that is compatible with the choice of the unit normal for S.

To do this, we need to extend the idea of "anti-clockwise" or "positive" orientation of a planar closed curve.

Orientation of a Surface and its Boundary Curve.
Definition 6.7.1 For an oriented surface with chosen continuous unit normal $\widehat{N}$ and whose boundary $C = \partial S$ is a smooth curve with unit tangent $\widehat{T}$, the boundary is positively oriented with respect to the surface if $\widehat{N} \times \widehat{T}$ points into the surface. ♢

For example, for a planar region with upward normal $\widehat{N} = \hat{k}$ and boundary curve parameterized with $\widehat{T} = \langle dx/ds, dy/ds\rangle$, the cross-product $\widehat{N} \times \widehat{T} = \langle -dy/ds, dx/ds\rangle$, which is the vector got by rotating $\widehat{T}$ a quarter turn anti-clockwise, or to the left relative to the direction $\widehat{T}$: this is indeed "into the surface" if the curve is traversed anti-clockwise.

Intuitively, if one imagines walking around the edge C of region S in the direction of the parameterization and with one's "up" direction as indicated by $\widehat{N}$, then the surface S is to one's left.

Alternatively, if one views the surface from the side indicated by $\widehat{N}$, the boundary is traversed anti-clockwise, matching our description of positive orientation for plane curves.

In the case where normal $\widehat{N}$ has positive z component, as when the surface is the graph of a function, positive orientation is moving anti-clockwise around C as seen from "above", meaning from higher z values.

Theorem 6.7.2 Stokes' Theorem. For a piecewise-smooth oriented surface S with positively oriented piecewise-smooth boundary curve ∂S and outward unit normal $\widehat{N}$,

$$\oint_{\partial S} \vec{F} \cdot \vec{T}\ ds = \iint_S \text{Curl } \vec{F} \cdot \widehat{N}\ dS;$$

that is,

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} \tag{6.7.2}$$

See Example 73 in Section 6.7 of OSC3[2].

Proof for the surface as the graph of a function. We start by proving this in the case that the surface is a graph $z = g(x, y)$, $x = g(y, z)$ or $y = g(x, z)$), and then deal with the general case by dividing the surface into a union of "easy pieces" of these three types. This mimics the strategy used to prove Theorem 6.4.2, p. 105, where the easy pieces were regions of Type I or Type II.
First, using Equation (6.6.14) for $d\vec{S}$ on the right-hand side,

$$\begin{aligned}
&\iint_S \text{Curl } \vec{F} \cdot \widehat{N}\ dS \\
&= \iint_S \text{Curl } \vec{F} \cdot \langle -\partial g/\partial x, -\partial g/\partial y, 1\rangle\ dA \\
&= \iint_S \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle \cdot \langle -\partial g/\partial x, -\partial g/\partial y, 1\rangle\ dA
\end{aligned}$$

$$= \iint_S -\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\frac{\partial g}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\frac{\partial g}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dA$$

$$= \iint_S \frac{\partial Q}{\partial z}\frac{\partial g}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial g}{\partial y} + \frac{\partial Q}{\partial x} - \frac{\partial R}{\partial y}\frac{\partial g}{\partial x} - \frac{\partial P}{\partial z}\frac{\partial g}{\partial y} - \frac{\partial P}{\partial y}\, dA \tag{6.7.3}$$

Next the contour integral along a parameterization $x = x(t), y = y(t), z = g(x(t), y(t))$, $a \leq t \leq b$ of the boundary curve, with $(x(t), y(t))$ going anti-clockwise in the x–y plane, is

$$\begin{aligned}
&\int_{\partial D} \vec{F} \cdot d\vec{r} \\
&= \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right) dt \\
&= \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt}\right)\right) dt \\
&= \int_a^b \left(\left(P + R\frac{\partial g}{\partial x}\right)\frac{dx}{dt} + \left(Q + R\frac{\partial g}{\partial y}\right)\frac{dy}{dt}\right) dt \\
&= \int_a^b \left(P + R\frac{\partial g}{\partial x}\right) dx + \left(Q + R\frac{\partial g}{\partial y}\right) dy
\end{aligned}$$

and by Green's theorem this gives

$$\begin{aligned}
&\int_{\partial D} \vec{F} \cdot d\vec{r} \\
&= \iint_D \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial g}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial g}{\partial x}\right)\right] dA \\
&= \iint_D \Bigg[\left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial g}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial g}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial g}{\partial x}\frac{\partial g}{\partial y} + R\frac{\partial^2 g}{\partial x \partial y}\right) \\
&- \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial g}{\partial y} + \frac{\partial R}{\partial y}\frac{\partial g}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial g}{\partial y}\frac{\partial g}{\partial x} + R\frac{\partial^2 g}{\partial y \partial x}\right)\Bigg] dA
\end{aligned}$$

The two pairs of terms involving $\frac{\partial R}{\partial z}\frac{\partial g}{\partial x}\frac{\partial g}{\partial y}$ and $R\frac{\partial^2 g}{\partial x \partial y}$ cancel, leaving

$$\begin{aligned}
&\int_{\partial D} \vec{F} \cdot d\vec{r} \\
&= \iint_D \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial g}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial P}{\partial y} - \frac{\partial P}{\partial z}\frac{\partial g}{\partial y} - \frac{\partial R}{\partial y}\frac{\partial g}{\partial x} dA
\end{aligned} \tag{6.7.4}$$

The right-hand side here is is the same as in (6.7.3), completing the verification for surfaces $z = g(x, y)$: the same verification strategy clearly works for the two other "graph directions" $x = g(y, z)$ and $y = g(x, z)$. ∎

Proof sketch for the general case. For other surfaces, the basic idea is to cut the surface up into pieces, each of which is a graph in one of the three directions. This will be described here for a parametric surface, or a collection of parametric surface pieces.

As noted in Subsection 6.6.4, p. 117, Oriented Surfaces, p. 117, a smooth level surface $G(x, y, z) = 0$ with unit normal $\widehat{N} = \vec{\nabla} G / \|\vec{\nabla} G\|$ defined everywhere can be described as a collection of parametric surface pieces, (indeed, graphs of functions), so the result works there too.

Writing the normal as $\widehat{N} = \langle n_x, n_y, n_z \rangle$, The Implicit Function Theorem in 3D, p. 56 can be used to show that if $n_z \neq 0$ at a point of the surface $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, one can solve nearby for the parameters u and v in terms of x and y, and thus get z in terms of x and y: this part of the surface is a graph $z = g(x, y)$, as handled in the first part of this proof.

Similarly, having either $n_x \neq 0$ or $n_y \neq 0$ at a point allows the surface to be descrbed as a graph in one of

the other two directions.
Since $\widehat{N} \neq 0$ anywhere, at least one of these three conditions holds at every point on the surface, so every part of the surface can be covered by at least one of these types of graph.
This huge collection of "graph pieces" covers the surface many times over; then one can cut the collection and the domains down to cover each point once, except for overlap along the edges of their domains.
Then the sum of the surface integrals over each piece is the total surface integral, while the sum of the boundary integrals over the pieces is

- the integral around the boundary of the whole surface, plus
- various path integrals along the internal edges produced by cutting.

These internal edge path integrals come in pairs going in opposite directions (along the "opposite sides" of each cut), and so cancel out, leaving just that outside boundary integral, as needed. ■

See Example 74 in Section 6.7 of OSC3[3], converting a surface integral into an easier line integral, and Example 75 converting a line integral into an easier surface integral.

Velocity Fields, Circulation, and Curl. Stokes' Theorem helps with the interpretion of the curl in the context of the velocity field of a fluid, $\vec{v}$.

The circulation of a velocity field $\vec{v}$ around a curve C is the net flow around this curve in the direction of the curve's orientation, given by integrating the tangential part $\vec{v} \cdot \vec{T}$ of the velocity with respect to arc length: $\oint_C \vec{v} \cdot \vec{T}\, ds$.

Stokes' Theorem says that for any oriented surface S with this curve as boundary, positively oriented, this is

$$\oint_C \vec{v} \cdot \vec{T}\, ds = \oint_C \vec{v} \cdot d\vec{r} = \iint_S \text{Curl } \vec{v} \cdot \vec{N} dS,$$

so the circulation is given by integrating the part of the curl normal to the surface. (This shows a sense in which the curl measures anti-clockwise rotation in a velocity vector field.)

If you are interested in applications to electric and magnetic fields, see the subsection Interpretation of Curl at the end of Section 6.7 of OSC3[4] and Example 76 there.

Curl$\vec{F} = \vec{0}$ implies that $\vec{F}$ is conservative. Stokes' Theorem is the main ingredient in proving the second, converse half of Theorem 6.5.4, p. 112 in Section 6.5, p. 109; the part that requires a simply connected domain.

The other ingredient is showing that for any closed curve C in the simply connected domain D, one can find a surface S with C as its boundary.

Proving this takes some work: the main idea is the shrink the curve down to a point, sweeping out a surface as you go. Being simply connected is the condition which ensures that this shrinking can be done without getting stuck on a hole in the domain.

With these two ingredients, Curl$\vec{F} = \vec{0}$ ensures that

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl } \vec{F} \cdot d\vec{S} = 0$$

for any closed curve C in the domain, and as seen in Theorem 6.3.5, p. 101 this zero integral around any closed path is equivalent to $\vec{F}$ being conservative: there is a function $f(x, y, z)$ with Grad$f = \vec{F}$.

Study Guide. Study Section 6.7 of OSC3[5]; in particular

- Stokes' Theorem (of course).

[2] openstax.org/books/calculus-volume-3/pages/6-7-stokes-theorem
[3] openstax.org/books/calculus-volume-3/pages/6-7-stokes-theorem
[4] openstax.org/books/calculus-volume-3/pages/6-7-stokes-theorem
[5] openstax.org/books/calculus-volume-3/pages/6-7-stokes-theorem

- Examples 73–75, and the Checkpoints following each.
- One or several exercises from each of the ranges 326–331 and 332–335.

6.8 The Divergence Theorem

Revised on April 21.

References.

- OpenStax Calculus Volume 3, Section 6.8[1].
- Calculus, Early Transcendentals by Stewart, Section 16.9.

Theorem 6.8.1 The Divergence Theorem. Let E a region in $\mathbb{R}^3$ (of appropriate shape as discussed below), ∂E with its boundary S being smooth and with outward unit normal $\widehat{N}$, and $\vec{F}$ a vector field whose components all have continuous partial derivatives on E and an open region containing it.
Then the outward flux of $\vec{F}$ through S is

$$\iint_{\partial E} \vec{F} \cdot \widehat{N}\ dS = \iiint_{E} \text{Div } \vec{F}\ dV$$

or in alternate notation,

$$\iint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_{E} \vec{\nabla} \cdot \vec{F}\ dV \tag{6.8.1}$$

See Example 77 in Section 6.8 of OSC3[2].

Verifying the Divergence Theorem for a Simple Solid Region. For what sort of domains is this true?

It is easiest to verify for domains that are of all three types 1, 2, and 3 discussed in Section 5.4 on triple integrals. That is, any line parallel to any of the three axes that passes through the region does so in a single line segment, entering and leaving exactly once. In other words, the boundary can be divided into a ”top” and a ”bottom”, and into a ”left” and a ”right”, and into a ”front” and a ”back”.

Such regions are called simple solid regions.

For $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$ and with the unit normal written as $\widehat{N} = n_x\hat{\imath} + n_y\hat{\jmath} + n_z\hat{k}$, the Divergence Theorem expanded in components is

$$\iint_{S} Pn_x + Qn_y + Rn_z\ dS = \iiint_{E} \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\ dV$$

This can be broken into a sum of three parts, and is verified by verifying each of

$$\iiint_{E} \frac{\partial P}{\partial x}\, dV = \iint_{S} P\hat{\imath} \cdot \widehat{N} dS, \quad \iiint_{E} \frac{\partial Q}{\partial y}\, dV = \iint_{S} Q\hat{\jmath} \cdot \widehat{N} dS, \quad \iiint_{E} \frac{\partial R}{\partial z}\, dV = \iint_{S} R\hat{k} \cdot \widehat{N} dS.$$

The verification for each is done in the same way, so let us look at just the last,

$$\iiint_{E} \frac{\partial R}{\partial z}\, dV = \iint_{S} R\hat{k} \cdot \widehat{N} dS = \iint_{S} R\hat{k} \cdot d\vec{S} \tag{6.8.2}$$

[1] openstax.org/books/calculus-volume-3/pages/6-8-the-divergence-theorem
[2] openstax.org/books/calculus-volume-3/pages/6-8-the-divergence-theorem

Since the domain E is of type 1, the boundary surface S consists of a top T given by $z = t(x, y)$ and a bottom B given by $z = b(x, y)$, each over the domain $D \in \mathbb{R}^2$ of all possible (x, y) pairs occurring in D, and the solid can be described as

$$E = \{(x, y, z) | (x, y) \in D, b(x, y) \leq z \leq t(x, y)\}.$$

The triple integral in Equation (6.8.2) becomes

$$\iiint\limits_E \frac{\partial R}{\partial z}\, dV = \iint\limits_D \left[\int_{z=b(x,y)}^{z=t(x,y)} \frac{\partial R}{\partial z}\, dz \right] dA$$

The integral in brackets can be evaluated with the Fundamental Theorem of Calculus (which is the one dimensional little brother of the Divergence Theorem) so

$$\iiint\limits_E \frac{\partial R}{\partial z}\, dV = \iint\limits_D R(x, y, t(x, y)) - R(x, y, b(x, y))\, dA \tag{6.8.3}$$

Starting at the other end, dividing the surface integral in Equation (6.8.3) into integrals over the top and bottom parts gives

$$\iint\limits_S R\hat{k} \cdot d\vec{S} = \iint\limits_T R\hat{k} \cdot d\vec{S} + \iint\limits_B R\hat{k} \cdot d\vec{S} \tag{6.8.4}$$

For the top surface T, the outward unit normal is the upward unit normal for the graph $z = t(x, y)$ and the integral can be evaluated using the formula in Section 6.7, p. 119

$$\iint\limits_S \vec{F} \cdot d\vec{S} = \iint\limits_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA$$

for the current simple case with $P = Q = 0$,

$$\iint\limits_T R\hat{k} \cdot d\vec{S} = \iint\limits_T \langle 0, 0, R(x, y, t(x, y)\rangle \cdot d\vec{S} = \iint\limits_D R(x, y, t(x, y))\ dA \tag{6.8.5}$$

On the bottom, the outward normal to E is the downward normal to the graph $z = b(x, y)$, so the sign is reversed:

$$\iint\limits_B R\hat{k} \cdot d\vec{S} = \iint\limits_D -R(x, y, b(x, y))\, dA \tag{6.8.6}$$

Combining Equations (6.8.3), (6.8.4), (6.8.5) and (6.8.6) gives Equation (6.8.2), the ”z” part of the divergence theorem for simple solids. As said above, the other two parts are verified similarly.

The Basic Idea for Other Domain Shapes. The Divergence Theorem formula extends to other regions, with the basic idea being to ”dice” a solid region into a collection of simple solid regions; a strategy like that used for the proofs of Green’s Theorem and Stokes’ Theorem.

The total volume integral is the sum of the volume integrals over each simple piece, and so is the sum of the surface integrals over all the pieces.

This sum of surface integrals is the desired integral over the surface of the whole region plus extra integrals over the new surfaces produced by the cutting into simple pieces.

These extra bits of surface come in pairs on either side of each cut, with the outward normal being in the opposite direction on the two sides of the cut, so that the values of those extra surface integrals are negatives of each other, and thus those pairs of extra surface integrals cancel out.

That leaves just the desired surface integrals over the boundary of the domain.

See Example 78 in Section 6.8 of OSC3[3], and Example 79 for an application to fluid flow.

The Flux of an Electric Field Through a Surface. The electric field around a point charge of Q at the origin of the coordinates is

$$\vec{E}(\vec{x}) = \frac{Q\vec{x}}{4\pi\epsilon_0|\vec{x}|^3}.$$

Using the Divergence Theorem, it can be shown (Exercise!) that

$$Q = \iint_{\partial D} \varepsilon_0 \vec{E} \cdot d\vec{S}. \tag{6.8.7}$$

for any surface ∂D that is the boundary of a region D containing the origin (and thus containing this point charge):

1. First compute the integral directly for the case of ∂D a sphere of radius a centered at the origin.
2. Then use the Divergence Theorem to get the general version.
3. Extend to a collection of point charges $Q_1 \dots Q_n$ at locations $\vec{x}_1 \dots \vec{x}_n$ inside region D, with the integral then giving the total charge in the region.

See Example 80 Section 6.8 of OSC3[4].

Gauss's Law. Equation (6.8.7) is a case of Gauss's Law of Electrostatics. In general, Gauss's Law says that the above flux integral always gives the total charge inside a region with boundary the surface S: this allows the charge within a region to be determined by measurements only at the boundary of the region.

Indeed, Gauss discovered the Divergence Theorem through his work on electrostatics.

See Theorem 21 and Example 81 Section 6.8 of OSC3[5].

Study Guide. Study Section 6.8 of OSC3[6]; in particular

- The Overview of Theorems, covering all the main integration theorems going back to the Fundamental Theorem of Calculus.
- The Divergence Theorem (of course).
- Examples 77 and 78, and the Checkpoints following each.
- If interested in physical applications, see Theorem 21 and Examples 79 and 80.
- One or several exercises from each of the ranges 385–388 and 390–394.

Chapter Review. When reviewing this chapter, also look at the end of chapter review material in OpenStax Calculus Volume 3, including Key Terms[7], Key Equations[8] and Key Concepts[9].

[3] openstax.org/books/calculus-volume-3/pages/6-8-the-divergence-theorem
[4] openstax.org/books/calculus-volume-3/pages/6-8-the-divergence-theorem
[5] openstax.org/books/calculus-volume-3/pages/6-8-the-divergence-theorem
[6] openstax.org/books/calculus-volume-3/pages/6-8-the-divergence-theorem
[7] openstax.org/books/calculus-volume-3/pages/6-key-terms
[8] openstax.org/books/calculus-volume-3/pages/6-key-equations
[9] openstax.org/books/calculus-volume-3/pages/6-key-concepts

Appendix A

Rules for Derivatives and Integrals

A.1 Rules for Derivatives

Sums, differences, constant factors.

$$\frac{d}{dx}(kf(x)) = k\frac{df}{dx}$$
$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{df}{dx} \pm \frac{dg}{dx}$$

Products, Quotients and Compositions.

$$\frac{d}{dx}(f(x)g(x)) = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx}$$
$$\frac{d}{dx}(f(x)/g(x)) = \frac{\frac{df}{dx}g(x) - f(x)\frac{dg}{dx}}{g^2(x)}$$
$$\frac{d}{dx}(f(g(x)) = f'(g(x))\frac{dg}{dx}$$

That is, with $u = g(x), y = f(u)$,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

A.2 Rules for Integrals

Sums, differences, constant factors.

$$\int kf(x)\,dx = k\int f(x)\,dx$$
$$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Substitution.

$$\text{If } \int f(x)\,dx = F(x) + C$$

then $\displaystyle\int f(g(x))g'(x)\,dx = F(g(x)) + C$

That is, with $u = g(x)$,

$$\int f(u(x))\frac{du}{dx}dx = \int f(u)\,du$$

Integration by Parts.

$$\int u(x)\frac{dv}{dx}dx = u(x)v(x) - \int v(x)\frac{du}{dx}dx$$

That is, $\displaystyle\int u\,dv = uv - \int v\,du$

Appendix B

Reduction Formulas For Integrals

This is a collection of reduction formulas; for a more comprehensive list of integrals, see OpenStax Calculus Volume 2, Appendix A: Table of Integrals.[1]

B.1 Integrals Involving Exponential or Trigonometric Functions

$$\int u^n e^{au}\ du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\ du \tag{B.1.1}$$

$$\int \sin^n u\ du = -\frac{1}{n}\sin^{n-1} u\cos u + \frac{n-1}{n}\int \sin^{n-2} u\ du \tag{B.1.2}$$

$$\int \cos^n u\ du = \frac{1}{n}\cos^{n-1} u\sin u + \frac{n-1}{n}\int \cos^{n-2} u\ du \tag{B.1.3}$$

$$\int \tan^n u\ du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\ du \tag{B.1.4}$$

$$\int \cot^n u\ du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\ du \tag{B.1.5}$$

$$\int \sec^n u\ du = \frac{1}{n-1}\tan u\sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\ du \tag{B.1.6}$$

$$\int \csc^n u\ du = \frac{-1}{n-1}\cot u\csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u\ du \tag{B.1.7}$$

$$\int u^n \sin u\ du = -u^n\cos u + n\int u^{n-1}\cos u\ du \tag{B.1.8}$$

$$\int u^n \cos u\ du = u^n\sin u - n\int u^{n-1}\sin u\ du \tag{B.1.9}$$

$$\int \sin^n u\cos^m u\ du = \int (1-\cos^2 u)^k \cos^n u\sin u\ du = -\int (1-v^2)^k v^n\ dv,\ m = 2k+1,\ v = \cos u \tag{B.1.10}$$

$$\int \sin^n u\cos^m u\ du = \int (1-\sin^2 u)^k \sin^m u\cos u\ du = \int (1-v^2)^k v^m\ dv,\ n = 2k+1,\ v = \sin u \tag{B.1.11}$$

[1] openstax.org/books/calculus-volume-2/pages/a-table-of-integrals

$$\int \sin^n u \cos^m u \; du = \int \left(\frac{1-\cos 2u}{2}\right)^{k_1} \left(\frac{1+\cos 2u}{2}\right)^{k_2}, \; n = 2k_1, \; m = 2k_2 \tag{B.1.12}$$

$$\int \sin^n u \cos^m u \; du = -\frac{\sin^{n-1} u \cos^{m+1} u}{n+m} - \frac{n-1}{n+m} \int \sin^{n-2} u \cos^m u \; du, \; n \geq 2 \tag{B.1.13}$$

$$\int \sin^n u \cos^m u \; du = \frac{\sin^{n+1} u \cos^{m-1} u}{n+m} + \frac{m-1}{n+m} \int \sin^n u \cos^{m-2} u \; du, , \; m \geq 2 \tag{B.1.14}$$

B.2 Integrals Involving Inverse Trigonometric Functions

$$\int u^n \sin^{-1} u \; du = \frac{1}{n+1}\left[u^{n+1} \sin^{-1} u - \int \frac{u^{n+1} du}{\sqrt{1-u^2}}\right], \; n \neq -1 \tag{B.2.1}$$

$$\int u^n \cos^{-1} u \; du = \frac{1}{n+1}\left[u^{n+1} \cos^{-1} u + \int \frac{u^{n+1} du}{\sqrt{1-u^2}}\right], \; n \neq -1 \tag{B.2.2}$$

$$\int u^n \tan^{-1} u \; du = \frac{1}{n+1}\left[u^{n+1} \tan^{-1} u + \int \frac{u^{n+1} du}{1+u^2}\right], \; n \neq -1 \tag{B.2.3}$$

B.3 Integrals Involving $\sqrt{a+bu}$

$$\int u^n \sqrt{a+bu}\, du = \frac{2}{b(2n+3)}\left[u^n (a+bu)^{3/2} - na \int u^{n-1} \sqrt{a+bu}\, du\right] \tag{B.3.1}$$

$$\int \frac{u^n}{\sqrt{a+bu}}\, du = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)} \int \frac{u^{n-1}}{\sqrt{a+bu}}\, du \tag{B.3.2}$$

$$\int \frac{1}{u^n\sqrt{a+bu}}\, du = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)} \int \frac{1}{u^{n-1}\sqrt{a+bu}}\, du \tag{B.3.3}$$

Appendix C

Strategy for Evaluating Integrals

C.1 A few general tactics for integration

In trying to evaluate an integral with the collection of methods we have seen in Chapters 1 and 3, some general ideas are worth keeping in mind:

Find antiderivatives first It is often best to start by seeking any one antiderivative $F(x)$ for the function involved and deal later with the constant of integration in an indefinite integral and the limits of a definite integral.
However when using substitutions on a definite integral, you can avoid substituting back to the original variable by instead changing the limits of integration to the corresponding values for the new variable.

Start with the easiest possibilities Several techniques might apply to one integral, so start try the easiest first (recognizing a previously known integral, from memory or tables) and work through to the more sophisticated options (like inverse trigonometric substitutions and integration by parts.)

Repeat as necessary All methods except recognizing a known integral give one or several new simpler integrals, so the process needs to be applied repeatedly until every part of the answer is found by reducing to a previously known integral.

C.2 A detailed strategy for integration

Here is one detailed approach to evaluating integrals. It is based on the discussion in Section 3.5 and covers all the methods we have seen in class. For practice and test review, I strongly recommend selecting exercises from Section 3.5, so that you must decide what methods to use as well as then applying the methods correctly: that is the way integration is in real life.

C.2.1 Use tables of integrals and known integrals

The easiest approach is to recognize an integral as one you have already worked out how to evaluate. The corollary of this is that you should memorize the most common integrals, and collect the ones that you encounter often but have not memorized (yet) on a formula sheet. The formulas should be as flexible as possible with adjustable constants to avoid routine substitutions: for example, not $\int \cos x \; dx = \sin x + C$ but instead

$$\int \cos ax \; dx = \frac{1}{a} \sin ax + C$$

C.2.2 Do basic simplifications

Simplify first is a good strategy in many mathematical situations: try to simplify the function involved before starting on the calculus itself.

The most general basic simplifications are breaking up sums and differences into separate integrals, taking constant factors out in front of each integral (including division by constant factors), and rewriting roots as fractional powers.

It is also often useful to eliminate divisions by rewriting powers in the denominator as negative powers in the numerator, and using trig identities like converting a factor $\cos x$ in the denominator into a factor $\sec x$ in the numerator.

For example, $\displaystyle\int \frac{x^2}{7} - \frac{1}{\sin 2x} + \frac{5x}{\sqrt[3]{1+x^2}}\, dx$ could be rewritten as

$$\frac{1}{7}\int x^2\, dx - \int \csc 2x\, dx + 5\int x(1+x^2)^{-1/3}\, dx$$

One important special type of simplification is used with integrals of products of powers of trigonometric functions, which will be discussed below.

C.2.3 Substitution

If the above steps do not give the solution, the easiest of the two most powerful general tools is substitution, especially with some compositions and products. That is, finding a function $u(x)$ so that the integrand is in the form $f(u)u'(x)$, or $f(u)\dfrac{du}{dx}$. Then you can use the ``cancellation of differentials'' idea

$$\int f(u)\frac{du}{dx}\, dx = \int f(u)\, du$$

to get a new, simpler integration problem.

That puts you back at the beginning with a new hopefully simpler integral. If the new integral is not easier to evaluate, the substitution was not useful, so try something else: another substitution or a different method.

C.2.4 Choosing a substitution function $u(x)$

As u will often appear inside a composition, one common choice is the function inside a power or other composition. In the example above of $5x(1+x^2)^{-1/3}$, you could try $u = 1 + x^2$.

You must then check if the rest of the term is the derivative of u, up to a constant factor. In the current example $du/dx = 2x$ and the remainder of the term apart from the power of u is $5x$, which does match the derivative up to a harmless constant factor of $5/2$, so this substitution will work. (Exercise: complete this integral.)

In general, you must look to see if the term to be integrated consists of just the derivative of u times some expression that can be put in terms of u only (with no stray x terms.) For $\int \sin(x)\cos(x)\, dx$, the substitution $u = \sin(x)$ gives

$$\int u\frac{du}{dx}\, dx = \int u\, du = \frac{u^2}{2} + C = \frac{1}{2}\sin^2(x) + C$$

and also substitution $u = \cos(x)$ gives

$$\int -u\frac{du}{dx}\, dx = -\int u\, du = -\frac{u^2}{2} + C = -\frac{1}{2}\cos^2(x) + C$$

which despite initial appearances ("$u^2/2$" vs "$-u^2/2$") agree, due to $\cos^2(x) + \sin^2(x) = 1$.

This shows that there can be more than one useful substitution.

C.2.5 Integration by Parts

Our second and last general tool is Integration by Parts, as summarized by

$$\int u\,dv = uv - \int v\,du.$$

Here u and v are both functions of the actual integration variable x, so that in more detail the rule is

$$\int u(x)\frac{dv}{dx}\,dx = u(x)v(x) - \int v(x)\frac{du}{dx}\,dx$$

Note that if one does a definite integral directly, the formula becomes

$$\int_{x=a}^{x=b} u\,dv = \Big[uv\Big]_{x=a}^{x=b} - \int_{x=a}^{x=b} v\,du$$

The word "parts" refers to the fact that only one part of the integrand gets integrated, at least initially: dv/dx is integrated to find v, while u gets differentiated. The main hint that this method might be useful is that the function is a product of several functions and we know how to integrate at least one of them: particularly, powers of x, trigonometric functions, exponentials, logarithms, and inverses of familiar functions. There are always many different ways to choose which factor is to be the factor u to be differentiated (with all the rest of the integrand going into the factor dv/dx to be integrated; or in other words into the differential dv). Some guidelines are

- Try to integrate the most complicated part you can.
- It it necessary that you can integrate the function that goes into the differential dv.
- It is desirable for the function u to have a simple derivative, and three very common choices are positive integer powers of x, logarithms, and any inverses of familiar functions.

C.2.6 Inverse Substitution, especially with trigonometric functions

All substitutions can be done in inverse form, where one specifies $x = g(u)$, $dx = g'(u)du$ instead of $u = f(x)$, $du = f'(x)dx$: g is the inverse of f. This has the great advantage that the formulas for x and dx automatically put everything in terms of the new variable u: there are never any stray x terms. Of course, the inverse g might be a messier function to work with, so this method is at its best when f is itself the inverse of a familiar function, and the most common examples are when the forward substitution function f is an inverse trigonometric function, so that g is a basic trigonometric function: therefore, the new variable is traditionally called θ instead of u. The three main cases are

- For an integral containing integer powers of $\sqrt{a^2 - x^2}$ and of x, use $x = a\sin\theta$, so $dx = a\cos\theta\,d\theta$ and $\sqrt{a^2 - x^2} = a\cos\theta$.
- For an integral containing integer powers of $\sqrt{a^2 + x^2}$ and of x, use\\ $x = a\tan\theta$, so $dx = a\sec^2\theta\,d\theta$ and $\sqrt{a^2 + x^2} = a\sec\theta$.
- For an integral containing integer powers of $\sqrt{x^2 - a^2}$ and of x, use\\ $x = a\sec\theta$, so $dx = a\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2 - a^2} = a\tan\theta$.

In each case, an appropriate right-triangle summarizes all the formulas needed. The resulting integrals involve products of powers of trigonometric functions, and often the methods of section~F below are needed to evaluate them. In that case you can get solutions involving sine and coinse on multiples of a new variable θ. These can be put in erms of just $\sin\theta$ and $\cos\theta$ using the double angle formulas

$$\sin 2\theta = 2\sin\theta\cos\theta, \qquad \cos 2\theta = 2\cos^2\theta - 1, \quad = 1 - 2\sin^2\theta$$

C.2.7 Special simplifications and substitutions for products of trigonometric functions

It often helps to convert into an expression with just sines and cosines using $\tan x = \sin x/\cos x$ and so on. The example above of $\int \sin x \cos x dx$ illustrates some important special trigonometric substitutions: With products of integer powers of sine and cosine

- if there is an odd power of cosine one can gather a term like $\cos x dx$ at the end and use the substitution $\sin x = u$, $\cos x\, dx = du$, leaving over an even power of $\cos x$ which can be written as an integer power of $\cos^2 x$ which in turn gets converted with

  $$\cos^2 x = 1 - \sin^2 x = 1 - u^2$$

  Then all the remaining x terms are in terms of $\sin x$, which becomes u, so you get an integral of a polynomial (easy), or rational function if there were some negative powers (see below).

- if there is an odd power of sine, one can gather a term like $\sin x dx$ and use the substitution {\boldmath $\cos x = u$, $\sin x\, dx = -du$}, and deal with the remaining even power of $\sin x$ using

  $$\sin^2 x = 1 - \cos^2 x = 1 - u^2.$$

- if there are products of even powers of both sine and cosine (including the case where only one of these functions is present, like $\sin^4 x$), one can reduce the powers using the half-angle formulas

  $$\cos^2 x = \frac{1}{2}(1 + \cos 2x), \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

  Then expand, simplify, and if necessary apply further trigonometric simplifications to some parts. Eventually you will get an odd power of sine or cosine in each term, plus a constant. For example

  $$\begin{aligned} \sin^2 x \cos^2 x &= \frac{1}{2}(1 - \cos 2x)\frac{1}{2}(1 + \cos 2x) \\ &= \frac{1}{4}(1 - \cos^2 2x) \\ &= \frac{1}{4}\sin^2 2x \\ &= \frac{1}{4} \cdot \frac{1}{2}(1 - \cos 4x) \\ &= \frac{1}{8} - \frac{1}{8}\cos 4x \end{aligned}$$

C.2.8 Integration of rational functions (ratios of polynomials)

It is possible to integrate any rational function $f(x) = \dfrac{P(x)}{Q(x)}$ if you can factorize the polynomial $Q(x)$ in the denominator into linear factors $(x - r)$ from roots plus irreducible quadratic factors $x^2 + bx + c$. Irreducible means that the quadratic has no real roots, which by the discriminant test means that $b^2 < 4c$.

It is easy and convenient to divide $P(x)$ and $Q(x)$ by a constant so that the lead coefficient in $Q(x)$ is one; this form is assumed from now on.

Also, if the numerator has the same or higher degree as the denominator, the function should first be simplified to the sum of a polynomial plus a proper rational function, one with numerator of lower degree than the denominator. This is done by synthetic division of polynomials.

The integration is done by rewriting the rational function as the sum of a polynomial plus terms that can be integrated with a few basic rules. The main integration rules needed are

$$\int \frac{du}{u - r} = \ln|u - r| + C$$

$$\int \frac{du}{(u-r)^n} = \frac{-1}{n-1}\frac{1}{(u-r)^{n-1}} + C, \quad n > 1$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$$

$$\int \frac{u}{a^2+u^2}du = \frac{1}{2}\ln(a^2+u^2) + C$$

plus occasionally completing the square to get into the latter two forms.

1. The first step is to convert to the case of a proper rational function, one with the polynomial $P(x)$ in the numerator of degree less than $Q(x)$, if this is not already true.

   Example C.2.1
   $$\frac{x^3+3x+1}{x^2-4} = \frac{x(x^2-4)+7x+1}{x^2-4} = x + \frac{7x+1}{x^2-4}.$$
   □

   The polynomial part is easy to integrate, so I will deal from now on only with proper rational functions.

2. Next, factorize the denominator into linear and irreducible quadratic factors.

   Example C.2.2 The denominator in Example 1 has roots 2 and -2, and so
   $$\frac{7x+1}{x^2-4} = \frac{7x+1}{(x-2)(x+2)}.$$
   □

   Sometimes the factorization will have to involve quadratic factors that have no real roots and so cannot be written in terms of two linear factors. This is true if the discriminant $b^2 - 4c < 0$.

   For an irreducible quadratic factor x^2+bx+c, complete the square to get the form $x^2+bx+c = (x+e)^2+a^2$, where $e = b/2$, $a^2 = c - b^2/4$.

   Example C.2.3 $\dfrac{13}{x^3-4x^2+13x}$. The denominator here has only one real root 0:
   $$\frac{13}{x^3-4x^2+13x} = \frac{13}{x(x^2-4x+13)} = \frac{13}{x[(x-2)^2+3^2]}.$$
   □

3. Next the rational function can be written as a sum of constant multiples of simple functions, for which we know the integrals. The simple functions needed are as follows

   - For a factor $(x-r)$ in the denominator, a term like
     $$\frac{A}{x-r}$$
   - For a factor $(x-r)^n$ in the denominator, n terms
     $$\frac{A_1}{x-r}, \quad \frac{A_2}{(x-r)^2}, \text{ and up to } \frac{A_n}{(x-r)^n} \text{ (the power in the original denominator)}$$
   - For an irreducible quadratic factor $(x+e)^2+a^2$ in the denominator, two terms, which can be put together as
     $$\frac{A+Bx}{a^2+(x+b)^2}$$

- For the most complicated case of a repeated irreducible quadratic factor like $[a^2+(x+b)^2]^n$, put together the previous two ideas: a succession of terms each with a linear factor like $A+Bx$ on top, and powers of $a^2+(x+b)^2$ on the bottom, ranging up to the same n-th power as in the original denominator.

Example C.2.4 For Examples 2 and 3,

$$\frac{7x+1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}.$$

and

$$\frac{13}{x[(x-2)^2+3^2]} = \frac{A}{x} + \frac{B+Cx}{(x-2)^2+3^2}$$

□

4. The constants in the expansion can be determined, and the first step is to clear the denominator: multiply through by the denominator $Q(x)$ of the original rational function to get an equation between two polynomials.

Example C.2.5 For the two examples above we get

$$7x+1 = A(x+2) + B(x-2)$$

and

$$13 = A[(x-2)^2+3^2] + (B+Cx)x.$$

□

5. To find the numerical values of the constants in this equation, one method (not my favorite) is to expand out the right hand side into multiples of powers of x, including a constant, set the coefficients of each power of x equal to those at left, and solve the resulting simultaneous equations for the constants A, B, etc.

However this equation solving can be made easier or avoided entirely by the strategy of strategic substitution: first looking at the easier equations you get by substituting each root of the denominator $Q(x)$ in for x.

When there are repeated factors or irreducible quadratic factors, this substitution method will only give some of the constants. To get the others, one can seek other easy equations by substituting in a few other "nice" integer values such as $x = 0, 1, -1, \ldots$.

Example C.2.6 For the first example function above, $x=-2$ and $x=2$ give

$$-13 = B(-4), \quad 15 = A(4)$$

which immediately give $B = 13/4$, $A = 15/4$:

$$\frac{7x+1}{(x^2-4)} = \frac{7x+1}{(x-2)(x+2)} = \frac{15/4}{x-2} + \frac{13/4}{x+2}.$$

□

Example C.2.7 For the second example, $x=0$ is the only root to substitute in, giving $13 = A[4+9]$ so $A=1$. Using this A value and substituting also $x=1$ and $x=-1$ gives

$$13 = [1+9] + (B+C), 13 = [9+9] - (B-C)$$

or $B+C=3$, $B-C=5$, with the solution $B=4$, $C=-1$:

$$\frac{13}{x^3-4x^2+13x} = \frac{13}{x[(x-2)^2+3^2]} = \frac{1}{x} + \frac{4-x}{(x-2)^2+3^2}.$$

The integral of this can be found using the four basic forms above plus a substitution $u = x - 2$. (You get $\ln|x| + \arctan((x-2)^2 + 3^2) - \frac{1}{2}\ln((x-2)^2 + 3^2) + C$.) $\square$

6. Finally, you can evaluate the resulting integrals, using the four basic integrals above:

$$\int \frac{7x+1}{(x^2-4)}\,dx = \int \frac{15/4}{x-2} + \frac{13/4}{x+2}\,dx = \frac{15}{4}\ln|x-2| + \frac{13}{4}\ln|x+2| + C$$

and

$$\int \frac{13}{x^3 - 4x^2 + 13x}\,dx = \int \frac{1}{dx} + \frac{4-x}{(x-2)^2+3^2}\,dx = \ln|x| + \arctan((x-2)^2 + 3^2) - \frac{1}{2}\ln((x-2)^2+3^2) + C$$

where the substitution $u = x - 2$ is needed for the last term.

C.2.9 Final steps: make sure that you answer the original question

Remember a few things that must be done before you are finished:

- Substitute if necessary to get the solution in terms of the original variable.
- For an indefinite integral, add the constant of integration: if $F(x)$ is an antiderivative of $f(x)$,

$$\int f(x)\,dx = F(x) + C.$$

- For a definite integral, evaluate at the limits of integration:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

- For a definite integral using substitution $u = g(x)$, you might instead do the evaluation in the new u variable to avoid converting back to the original variable:

$$\int_{x=a}^{x=b} f(g(x))g'(x)\,dx = \int_{u=c}^{u=d} f(u)\,du,\ c = g(a),\ d = g(b).$$

- Similarly, for a definite integral using inverse substitution $x = h(t)$, you might do the evaluation in the new variable using.

$$\int_{x=a}^{x=b} f(x)\,dx = \int_{t=c}^{t=d} f(h(t))h'(t)\,dt,$$

But you have to solve equations to get the new limits, c from a and d from b: $h(c) = a$, $h(d) = b$.

Appendix D

Some Formulas Worth Knowing

It is worth knowing all the Differentiation Rules from the Reference Pages at the back of the text, and all the Basic Forms for integrals, items 1 to 20 in the Table of Integrals in those Reference Pages. Appendix A is a guide to checking your knowledge of these.

Beyond that, here are some important formulas, mostly integrals.

There are some useful formulas not given here but better reviewed in the notes where their usage is explained. In particular see Section 7.4 on integrating rational functions and 7.8 on approximate (numerical) integration.

$$\begin{aligned}
\cosh x &= \frac{e^x + e^{-x}}{2}, \ \sinh x = \frac{e^x - e^{-x}}{2}. \\
\tanh x &= \frac{\sinh x}{\cosh x}, \ \text{etc.} \\
\cos^2(ax) &= \frac{1}{2}[1 + \cos(2ax)] \\
\sin^2(ax) &= \frac{1}{2}[1 - \cos(2ax)] \\
\sin(ax)\cos(ax) &= \frac{1}{2}\sin(2ax) \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 \cdots, |x| <; 1
\end{aligned}$$

The area between curves $y = g(x) \leq f(x)$, $a \leq x \leq b$. This has infinitesimal area differential

$$dA = (f(x) - g(x))dx$$

giving area integral

$$A = \int dA = \int_{x=a}^{b} (f(x) - g(x))dx$$

The solid produced by rotating curve $y = f(x)$ about the horizontal axis. This solid can be sliced perpendicular to the x-axis into disks with volume differential

$$dV = (\text{disk area})dx = \pi r^2 dx = \pi y^2 dx$$

giving volume integral

$$V = \int dV = \int_{x=a}^{b} \pi r^2 dx = \int_{x=a}^{b} \pi y^2 dx = \int_{x=a}^{b} \pi [f(x)]^2 dx$$

As always I recommend using the versions in terms of variables like x and y or geometric quantities like the radius r, without using function names like $f(x)$: you will know the formula for y, or be able to work out the relevant formula for the radius r.

The solid produced by rotating the region $g(x) \leq y \leq f(x)$, $a \leq x \leq b$ about the horizontal axis. This can be sliced perpendicular to the x-axis into annuli with volume differential

$$dV = (\text{annulus area})dx = \pi R^2 - \pi r^2 dx,\ R = f(x) > r = g(x)$$

giving volume integral

$$V = \int dV = \pi \int_{x=a}^{b} R^2 - r^2 dx$$

The solid produced by rotating the region $g(x) \leq y \leq f(x)$, $0 \leq a \leq x \leq b$ about the vertical axis. This can be sliced into cylindrical shells of volume differential

$$dV = (\text{shell circumference})(\text{shell height})dx = 2\pi r[f(x) - g(x)]dx = 2\pi x[f(x) - g(x)]dx$$

(since the radius of each shell is $r = x \geq 0$) giving volume integral

$$V = \int dV = \int_{x=a}^{b} 2\pi x[f(x) - g(x)]dx$$

The average value of the function $f(x)$ on interval $a \leq x \leq b$.

$$\bar{f} = f_{ave} = \frac{1}{b-a} \int_{x=a}^{b} f(x)\, dx$$

Substitution in a Definite Integral with $u = g(x)$.

$$\int_{x=a}^{b} f(u(x)) \frac{du}{dx} dx = \int_{u=c}^{d} f(u) du,\ c = g(a),\ d = g(b)$$

Note that the limits of integration change, from x values to u values.

Integration by Parts.

$$\int u\, dv = uv - \int v\, du \text{ or } \int u(x) \frac{dv}{dx} = u(x)v(x) - \int v(x) \frac{du}{dx}\, dx$$

With definite integrals, this becomes

$$\int_{x=a}^{b} u\, dv = [uv]_{x=a}^{b} - \int_{x=a}^{b} v\, du$$

Note that the limits of integration are still x values, as du and dv are shorthands $du = \dfrac{du}{dx}dx$ and $dv = \dfrac{dv}{dx}dx$: the dummy variable in the integration is still x.

For Integrals Involving the Square Root of a Quadratic.

- With $\sqrt{a^2 - x^2}$, try $x = a \sin\theta$ so that $\sqrt{a^2 - x^2} = a\cos\theta$, $dx = a\cos\theta\, d\theta$.
- With $\sqrt{a^2 + x^2}$, try $x = a\tan\theta$ so that $\sqrt{a^2 + x^2} = a\sec\theta$, $dx = a\sec^2\theta\, d\theta$.
- With $\sqrt{x^2 - a^2}$, try $x = a\sec\theta$ so that $\sqrt{x^2 - a^2} = a\tan\theta$, $dx = a\tan\theta\sec\theta\, d\theta$.

For more general quadratics, first extract the coefficient of x^2 as a common factor and then complete the square

$$x^2 + bx + c = [x + b/2]^2 + c - (b/2)^2$$

Then the substitution $u = x + b/2$ is useful, and $a^2 = c - (b/2)^2$ or $(b/2)^2 - c$, whichever is positive.

For a Separable Differential Equation $\dfrac{dy}{dx} = f(x)g(y)$. Write in differential form

$$\frac{1}{g(y)}dy = f(x)dx$$

and integrate, but beware of division by zero: check the case $g(y) = 0$.

The Arc Length of the Curve $y = f(x)$, $a \le x \le b$. The arc length differential is

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

so the total arc length is

$$L = \int ds = \int_{x=a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

The parametric curve $x = f(t)$, $y = f(t)$. This has slope at point $P(x(t), y(t))$ given by

$$m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

but beware of division by zero, whihc happens at points where $\dfrac{dx}{dt} = 0$. There, the above formula needs to be simplified before using it.

The arc length of the above parametric curve.. For $\alpha \le t \le \beta$ the arc length differential is

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

so the total arc length is

$$L = \int ds = \int_{t=\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Polar coordinates. To convert between polar coordinates r, θ and cartesian (rectangular) coordinates x, y, use

$$x = r\cos\theta,\ y = r\sin\theta,$$
$$r^2 = x^2 + y^2,\ \tan\theta = \frac{y}{x}$$

with special handling of points on the y-axis: the ones with $x = 0$.

The polar curve $r = f(\theta)$. This is a case of parametric curve, with angle θ as the parameter and

$$x = r\cos\theta = f(\theta)\cos\theta,\ y = r\sin\theta = g(\theta)\sin\theta$$

The slope of polar curve $r = f(\theta)$. This is given by combining formulas above, but simplifies to

$$m = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

The area of the region inside polar curve $r = f(\theta)$, $a \le \theta \le b$. This has area differential

$$dA = \frac{1}{2}r^2 d\theta = \frac{1}{2}(f(\theta))^2 d\theta$$

so the area is

$$A = \int dA = \int_{\theta=a}^{b} \frac{1}{2}r^2 d\theta = \frac{1}{2}\int_{\theta=a}^{b} (f(\theta))^2 \, d\theta.$$

The arc length of the above polar curve. This can be derived using formulas above for parametric curves but simplifies nicely: the arc length differential is

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

so the total arc length is

$$L = \int ds = \int_{\theta=a}^{b} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Integral error bound for the series $S = \sum a_n$ with $a_n = f(n)$, f positive and decreasing. The error R_N for the partial sum $S_N = \sum_{n=1}^{N} a_n$ as an approximation of $S = \sum a_n$ has a size limit

$$0 \le R_N = S - S_N \le \int_N^\infty f(n)\, dn$$

The Taylor Series.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 \cdots$$

and the N-the degree Taylor polynomial

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \cdots \frac{f^{(N)}(a)}{N!}(x-a)^N,$$

the remainder

$$R_N(x) = f(x) - T_N(x)$$

has an upper size limit

$$|R_N(x)| \le \frac{M}{(N+1)!}|x-a|^{N+1}$$

if you have a number M such that $|f^{(n)}(x)| \le M$. If the later is only true for some x, $|x-a| \le d$, then the limit on $R_N(x)$ is also only for these x values.

Appendix E

Some Trigonometry

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
1 + \tan^2 x &= \sec^2 x \\
\cos^2(ax) &= \frac{1}{2}[1 + \cos(2ax)] \\
\sin^2(ax) &= \frac{1}{2}[1 - \cos(2ax)] \\
\sin(ax)\cos(ax) &= \frac{1}{2}\sin(2ax) \\
\sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y \\
\cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \\
\cos x \cos y &= \frac{1}{2}\left[\cos(x - y) + \cos(x + y)\right] \\
\sin x \sin y &= \frac{1}{2}\left[\cos(x - y) - \cos(x + y)\right] \\
\sin x \cos y &= \frac{1}{2}\left[\sin(x - y) + \sin(x + y)\right]
\end{aligned}$$

Values at key angles. In the first two quadrants, the main values with simple forms are

θ	$0(\rightarrow)$	$\frac{\pi}{6}$	$\frac{\pi}{4}\ (\nearrow)$	$\frac{\pi}{3}$	$\frac{\pi}{2}\ (\uparrow)$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}(\nwarrow)$	$\frac{5\pi}{6}$	$\pi(\leftarrow)$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	DNE	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

When sketching curves, it can help to know some numerical values:

$$\sqrt{2} \approx 1.4142 \quad \text{and} \quad \sqrt{3} \approx 1.7321$$

leading to

$$\frac{1}{\sqrt{2}} \approx 0.7071, \quad \frac{1}{\sqrt{3}} \approx 0.5774, \quad \text{and} \quad \frac{\sqrt{3}}{2} \approx 0.8660.$$