16.5 Curl and Divergence

The curl of a vector field on $\mathbb{R}^3$ is an important quantity in the description of fluid flow and electromagnetic fields, and is also related to whether the vector field is conservative.

Definition. *For a differentiable vector field $\vec{F} = P\vec{\imath} + Q\vec{\jmath} + R\vec{k}$ on $\mathbb{R}^3$ then its **curl** is given by*

$$\operatorname{curl} \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} \tag{1}$$

Note that for a 2D vector field $\vec{F} = P(x,y)\vec{\imath} + Q(x,y)\vec{\jmath}$, this still makes sense with curl $\vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$. So in that case, curl $\vec{F} = \vec{0}$ is the mixed partials condition of Section 16.3, related to $\vec{F}$ being conservative: we will come back to this idea.

Curl and Rotation in a Fluid

The name "curl" refers to a measure of rotation. For example, the vector field $\vec{F} = -y\vec{\imath} + x\vec{\jmath}$ describes velocity of a fluid (say) going anti-clockwise around the z-axis: it has curl $\vec{F} = 2\vec{k}$ in which the direction $\vec{k}$ indicates the axis of rotation, the positive value indicates anti-clockwise direction as viewed from "above" down that z-axis, and the uniform magnitude indicating the uniform angular rate of rotation (which has period 2π).

∇ Short-hand for Gradient and Curl

A convenient short hand is based on thinking of the "nabla" or "del" symbol ∇ formally as a vector

$$\nabla = \vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

Formally multiplying this by f gives the formula for the gradient ∇f, and we can also compute the following formal cross product:

$$\begin{aligned}
\nabla \times \vec{F} &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&= \vec{\imath}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \vec{\jmath}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\
&= \operatorname{curl} \vec{F}.
\end{aligned}$$

Hence we sometimes use $\nabla \times \vec{F}$ as mnemonic notation for curl $\vec{F}$.

See Example 1.

The curl is involved in the 3D version of the mixed partials condition in Section 16.3:

Theorem. *The curl of a conservative vector field $\vec{F} = \nabla f$ vanishes: that is*

$$\operatorname{curl}(\nabla f) = \nabla \times (\nabla f) = \vec{0}.$$

As a partial converse, if a vector field $\vec{F}$ is defined on all of $\mathbb{R}^3$ and curl $\vec{F} = \vec{0}$*, then $\vec{F}$ is conservative.*

The verification of the first half is a straightforward calculation using Clairaut's Theorem: it is like three versions of the mixed partials condition for a conservative vector field in $\mathbb{R}^2$ seen in Section 16.3.

The proof of the converse part will be seen Section 16.8, as it requires Stokes' Theorem from that section. However, we can corroborate the theorem in many examples, by computing the potential f giving $\nabla f = \vec{F}$ once we have verified that curl$\vec{F} = \vec{0}$.

Aside: the converse is again true for a domain in $\mathbb{R}^3$ that is "simply connected" in the sense of "having no holes" but the details of that are omitted in this course.

See Examples 2 and 3.

Integrating the Tangential Component of a Vector Field around a Simple Closed Curve (with Respect to Arc Length)

Consider a region D in $\mathbb{R}^2$ with boundary the simple close curve C, parameterized with positive orientation as $\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath}$. This curve has unit tangent vector

$$\vec{T} = \frac{d\vec{r}}{ds} = \frac{dx}{ds}\vec{\imath} + \frac{dy}{ds}\vec{\jmath} = \left(\frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath}\right)\frac{dt}{ds}$$

so for a vector field $\vec{F} = P\vec{\imath} + Q\vec{\jmath}$ on $\mathbb{R}^2$, the component of the vector field tangent to the curve is

$$\vec{F}\cdot\vec{T} = (P\vec{\imath} + Q\vec{\jmath})\cdot\left(\frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath}\right)\frac{dt}{ds} = \left(P\frac{dx}{dt} + Q\frac{dy}{dt}\right)\frac{dt}{ds}$$

and the integral of this over the curve is

$$\oint_C \vec{F}\cdot\vec{T}ds = \oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \tag{2}$$

with the last step being Green's Theorem.

A Vector Form of Green's Theorem

Using the curl, the above result gives the following vector form of Green's Theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C \vec{F}\cdot\vec{T}ds = \iint_D (\text{curl}\,\vec{F})\cdot\vec{k}\,dA \tag{3}$$

This is sometimes also called the planar version of Stokes' Theorem: c.f. Section 16.8.

The Outward Normal Component of a Vector Field on a Simple Closed Curve

For a plane region D with positively oriented boundary curve $\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath}$, this boundary has unit normal vector

$$\vec{n} = \frac{dy}{ds}\vec{\imath} - \frac{dx}{ds}\vec{\jmath} \tag{4}$$

which points outwards from the region D. Thus vector field $\vec{F}$ has component $\vec{F}\cdot\vec{n}$ in the *outward normal direction*.

Two common physical examples related to fluids (in 2D!) are

- For a fluid with velocity field $\vec{v}$, $\vec{v}\cdot\vec{n}$ is the speed of flow out of region D perpendicular across the boundary.
- For a fluid in region D with pressure $\vec{p}$, $\vec{p}\cdot\vec{n}$ is the magnitude of the pressure acting perpendicularly on the boundary.

Integrating the Outward Normal Component of a Vector Field around a Simple Closed Curve

The integral of this outward normal component around the curve describes for example the total outflow from region D or the total force on the boundary, and is given similarly to above by

$$\oint_C \vec{F}\cdot\vec{n}\,ds \;=\; \oint_C (P\vec{\imath}+Q\vec{\jmath})\cdot\left(\frac{dy}{dt}\vec{\imath}-\frac{dx}{dt}\vec{\jmath}\right)dt \;=\; \oint_C -Q\,dx+P\,dy \;=\; \iint\limits_D \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)dA \tag{5}$$

The last step comes by changing P to $-Q$ and Q to P in Eq. (2) for the tangential component.

The Divergence of a Two Dimensional Vector Field

For two dimensional vector fields, the quantity ∇ above is just $\nabla=\vec{\imath}\frac{\partial}{\partial x}+\vec{\jmath}\frac{\partial}{\partial y}$, so formally one can compute the dot product

$$\nabla\cdot\vec{F}=\left(\vec{\imath}\frac{\partial}{\partial x}+\vec{\jmath}\frac{\partial}{\partial y}\right)\cdot(\vec{\imath}P+\vec{\jmath}Q)=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}$$

This is the two dimensional **divergence**, also denoted div $\vec{F}$, so Eq. (5) can be written as the *Two Dimensional Divergence Theorem*

$$\oint_C \vec{F}\cdot\vec{n}\,ds=\iint\limits_D \text{div}\,\vec{F}\,dA,\ \text{or}\ \iint\limits_D \nabla\cdot\vec{F}\,dA \tag{6}$$

This shows that for $\vec{F}$ describing fluid velocity, div $\vec{F}$ is related to “net outflow”, underlying the name “divergence”.

The Divergence of a Three Dimensional Vector Field

For a vector field in three dimensions, $\vec{F}=P\vec{\imath}+Q\vec{\jmath}+R\vec{k}$, the **divergence** of F is

$$\text{div}\,\vec{F}=\nabla\cdot\vec{F}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$$

The extension of Eq. (6) to three dimensions is the Divergence Theorem to be seen in Section 16.9.

See Example 4.

Along with the previous result $\text{curl}(\nabla f)=\nabla\times\nabla f=0$, we have

Theorem. *If* $\vec{F}=P\vec{\imath}+Q\vec{\jmath}+R\vec{k}$ *is a vector field on* $\mathbb{R}^3$ *and all components have continuous second derivatives, then* div curl $\vec{F}=\nabla\cdot(\nabla\times\vec{F})=0$.

This can again be verified by straightforward calculation and using Clairaut’s Theorem.

See Example 5.

The Laplacian of a Function and the Laplace Operator

There is one other very important combination of divergence, curl and gradient:

$$\text{div}\,\nabla f = \nabla\cdot\nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

This is the **Laplacian of f** seen in Section 14.3, often abbreviated as $\nabla^2 f$ or Δf. For functions of two variables this is

$$\nabla^2 f = \nabla\cdot\nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

This *operation* $\nabla^2 f$ is called the **Laplace operator.**

The Laplacian of a vector field is the vector field got by applying the Laplace operator to each component in turn; that is:

$$\nabla^2 \vec{F} = \nabla^2 P\,\vec{\imath} + \nabla^2 Q\,\vec{\jmath} + \nabla^2 R\,\vec{k}$$

Aside: Some Fundamental Differential Equations of Physics

The Laplacian arises in several of the fundamental equations for physics:

$$\begin{aligned}
\text{Laplace's Equation} \quad & \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \\
\text{The Poisson Equation} \quad & \nabla^2 u = f, \\
\text{The Heat Equation} \quad & \frac{\partial u}{\partial t} = \nabla^2 u, \\
\text{The Wave Equation} \quad & \frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \text{ and} \\
\text{The Schrödinger Equation} \quad & i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\vec{x})\psi
\end{aligned}$$

used in the description of fluid motion, electric fields, heat conduction, waves in water and in electro-magnetic fields, and in quantum mechanics.

Homework Exercises 1-7, 8*, 12*, 13,14,15*,16-20