15.6 Surface Area

Our goal is to compute the area of a surface S described as the graph of a function, $z = f(x, y)$, for points (x, y) in some domain D (initially a rectangle.)

We follow a familiar procedure of approximation and limits to get an answer in terms of integrals:

1. Divide the domain into small rectangles R_{ij} of dimensions Δx by Δy.
2. Approximate the part of the surface over each rectangle by some small surface T_{ij} whose area we can compute.
3. Sum the areas of all these pieces T_{ij} to approximate the total surface area.
4. Take the limit of this sum to get the exact surface area.
5. Recognize this limit of a sum as a double integral.

The main challenge is finding a good choice for the surface pieces T_{ij}: we use suitable parallelograms tangent to the actual surface.

A domain R that is rectangle can be divided into small rectangles R_{ij} each with one corner at point (x_i, y_j) and side of lengths Δx, Δy. Consider the point (x_i, y_j, z_{ij}), $z_{ij} = f(x_i, y_j)$ on the surface above (x_i, y_j), and the tangent plane to the surface at this point. The part of this tangent plane that lies over R_{ij} will be the surface T_{ij} that we use to approximate this part of surface S. It is a parallelogram, described once we know the vectors that run along the two edges above the edges of R_{ij} from (x_i, y_j) to $(x_i + \Delta x, y_j)$ and from (x_i, y_j) to $(x_i, y_j + \Delta y)$.

Using the linearization as in Eq. (2) of page 1 of Section 14.4,

- the point over (x_i, y_j) is (x_i, y_j, z_{ij}),
- the point over $(x_i + \Delta x, y_j)$ is $(x_i + \Delta x, y_j, z_{ij} + f_x(x_i, y_j)\Delta x)$, and
- the point over $(x_i, y_j + \Delta y)$ is $(x_i, y_j + \Delta y, z_{ij} + f_y(x_i, y_j)\Delta y)$.

Thus the edges of the parallelogram T_{ij} give vectors

$$\vec{a} = \Delta x \vec{\imath} + f_x(x_i, y_j)\Delta x \vec{k}, \quad \vec{b} = \Delta y \vec{\jmath} + f_y(x_i, y_j)\Delta y \vec{k}.$$

The area of this parallelogram $A(T_{ij}) = |\vec{a} \times \vec{b}|$ and

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \Delta x & 0 & f_x(x_i, y_j)\Delta x \\ 0 & \Delta y & f_x(x_i, y_j)\Delta y \end{vmatrix} \\ &= -f_x(x_i, y_j)\Delta x \Delta y \vec{\imath} - f_y(x_i, y_j)\Delta x \Delta y \vec{\jmath} + \Delta x \Delta y \vec{k} \\ &= [-f_x(x_i, y_j)\vec{\imath} - f_y(x_i, y_j)\vec{\jmath} + \vec{k}]\Delta A \end{aligned}$$

so

$$A(T_{ij}) = \sqrt{1 + [f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2}\,\Delta A = \sqrt{1 + |\nabla f(x_i, y_j)|^2}\,\Delta A$$

When we sum these and take the limit, we get the area of S as the double integral

$$A(S) = \iint_D \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA = \iint_D \sqrt{1 + |\nabla f|^2}\, dA \qquad (1)$$

Homework Study both Examples in the text, and do Exercises 1-5, 6*, 7-12, 13(a), 14(a)*.