14.3 Partial Derivatives

For a differentiable function of one variable, $y = f(x)$, a single number $f'(a)$ describes the rate of change of the function value relative to change in the independent variable x away from $x = a$.

Change in the value of a function of two variables can be more complicated depending on the direction of the change. Consider the simple function $f(x, y) = 2x - y$ for (x, y) near $(0, 0)$. Moving a distance h in the positive x direction changes the value by $2h$; moving a distance h in the positive y direction changes the function value by $-h$; moving any distance in the direction $\langle 1, 2\rangle$ does not change the value at all. (Check this.)

The reason for this is that there are infinitely many directions in which the values of the independent variables can change. Thus we start by considering the effect of change along the coordinate axes: changing one independent variable while holding the other constant.

Partial Derivatives of $f(x, y)$ at a point

For $f(x, y)$ and a point (a, b) in its domain, consider the function given by allowing just x to vary while the y values is always b: $g(x) = f(x, b)$. Then $g'(a)$ is its rate of change at $x = a$, which is also the rate of change of $f(x, y)$ due to change in x alone at (a, b). This is called **the partial derivative of f with respect to x at** (a, b), denoted $f_x(a, b)$.

Using the definition of the derivative in terms of limits,

$$f_x(a, b) = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h} \tag{1}$$

Similarly, we define **the partial derivative of f with respect to y at** (a, b) by

$$f_y(a, b) = \lim_{h\to 0} \frac{f(a, b + h) - f(a, b)}{h}. \tag{2}$$

Partial Derivatives as Functions

For a given point (a, b), the partial derivatives at (a, b), $f_x(a, b)$ and $f_y(a, b)$, are numbers. As the point at which these are computed is allowed to vary, this gives functions:

Definition. *If f is a function of two variables, its* ***partial derivatives*** *are the functions f_x and f_y with values given by*

$$\begin{aligned} f_x(x, y) &= \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h} \\ f_y(x, y) &= \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h} \end{aligned}$$

To distinguish from partial derivatives, the derivative of a function of a single variable, f' or dy/dx, is sometimes called an **ordinary derivative**.

Notations

A great variety of notations are used in various places for partial derivatives. For $z = f(x, y)$, some are

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$
$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Note the symbol ∂ (a variant of δ, spoken "partial") used in place of the d of the Leibniz notation df/dx for derivatives of function of a single variable. For notation that does not depend on particular variable names, the prime notation f' is now ambiguous, so numbers 1 and 2 are used to indicate the position of the variable in forms like f_1 and $D_2 f$.

Rule for Computing Partial Derivatives of $f(x, y)$

To compute f_x, treat y as a constant, so that f is teated as a function of a single variable x and differentiate that. Likewise to compute f_y. See Example 1.

Geometrical Meaning

A geometrical interpretation of this is that when computing $f_x(a, b)$, one looks at the intersection of the graph of f with the plane $y = b$. This is the curve that is the graph of function $z = g(x) = f(x, b)$, but drawn on plane $y = b$ instead of on the $x - z$ plane. Then $f_x(a, b) = g'(b)$ which is the tangent slope of the curve at the point $P(a, b, c)$ with $c = f(a, b)$.

Likewise, $f_y(a, b)$ is the tangent slope at this point P to the curve given by the intersection of the graph of f with the plane $x = a$.

See Examples 2, 3, 4.

Functions of More than Two Variables

Extensions to functions of three or more variables are fairly intuitive. For example with $f(x, y, z)$, we have

$$f_z(x,y,z) = \lim_{h \to 0} \frac{f(x,y,z+h) - f(x,y,z)}{h}$$

When multiple variables are used with notation $u = f(x_1, x_2, \ldots, x_n)$,

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

As above, effectively all variables except one $[x_i]$ are treated as constants in the formula for f, and one then computes as one would the derivative with respect to that one remaining variable. Other notations include $\frac{\partial f}{\partial x_i} = \frac{\partial u}{\partial x_i} = f_{x_i} = f_i = D_i f$.

See Example 5.

Second Partial Derivatives

As with ordinary derivatives one can compute partial derivatives of partial derivatives. The most methodical notations are things like $(f_x)_y$ for the derivative with respect to y of the derivative with respect to x of f. A variety of more concise notations are used:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} = D_1^2 f = D_x^2 f$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} = D_2 D_1 f = D_y D_x f$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y} = D_1 D_2 f = D_x D_y f$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial x^y} = D_2^2 f = D_y^2 f$$

In the middle two, note the different order of x an y in the different notations, according to whether new derivatives are added to the right or the left. See Example 6.

Does the Order of Derivatives Matter?

In *mixed derivatives* like f_{xy} and f_{yx}, the order often does not matter, though this is far from obvious:

Theorem (Clairaut's Theorem)**.** *If f is defined on a disk D that contains the point (a, b), both partial derivatives f_{xy} and f_{yx} exist, and both are continuous on that disk, then*

$$f_{xy}(a, b) = f_{yx}(a, b)$$

In particular, if these two mixed derivatives exist and are continuous everywhere, they are equal everywhere.

See Example 7.

Higher Derivatives

Differentation can be done repeatedly (so long as the derivatives exist), and with any number of variables, with notations like f_{xyy}, f_{xyzx}, $\dfrac{\partial^3 f}{\partial x \partial y \partial z}$ and so on.

Repeated application of Clairaut's Theorem typically allows reordering of the derivatives, greatly reducing the number of different higher partial derivatives that need to be computed.

Partial Differential Equations

Many physical laws describe a function of several variables in terms of an equation relating its partial derivatives. Three of the most important basic ones for functions of two variables are

Laplace's Equation $u_{xx} + u_{yy} = 0$

The Wave Equation $u_{tt} = c^2 u_{xx}$

The Heat Equation $u_t = u_{xx}$

Exercise. *Verify that $u(t, x) = e^{-t} \sin x$ is a solution of the heat equation.*

See also Examples 8 and 9.

Homework

Exercises 1, 3, 4, 7, 8, 11-23, 33, 34*, 35, 36, 45, 46*, 47-50, 55-58, 65, 66*, 67.