12.2 Vectors

The earliest meaning of a vector relates to movement from one place to another (like a mosquito as a vector for malaria), and this leads to the geometrical idea of a vector describing the displacement from one location to another.

For example, points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in $\mathbb{R}^3$ determine the **three dimensional vector** or *directed line segment* going from P_1 to P_2, denoted $\overrightarrow{P_1P_2}$.

In algebraic terms, this is described by the changes in the values of each of the three coordinates, so we denote this $\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$. Note that only the change in position or *displacement* counts, so many different pairs of points represent the same vector. Thus we often name a vector without reference to a particular pair of endpoints, with an over-arrow like $\vec{a}$, or with boldface like $\mathbf{a}$.

Using vectors as "bundled coordinates"

Any vector $\vec{a} = \langle x, y, z \rangle$ can be considered with the origin $O(0, 0, 0)$ as its starting point so that it ends at $P(x, y, z)$: $\vec{a} = \overrightarrow{OP}$. Thus vectors give a convenient way to bundle the values of all the coordinates of a point P into a single object, and we can sometimes think of vectors as equivalent to points.

But as we shall see, vectors have added features. Thus we call the set of all three dimensional vectors V_3, to distinguish from the set $\mathbb{R}^3$ of points.

All this works in the plane too of course, giving **two dimensional vectors**, notation $\langle x, y \rangle$.

The Length or Magnitude of a Vector

Vectors share some but not all of the properties of real numbers (which we now often call **scalars** to distinguish them from vectors) like length which resembles the absolute value of a scaler; addition; and and a form of multiplication.

The **length** or **magnitude** of a vector $\vec{a} = \langle x, y \rangle$ or $\vec{a} = \langle x, y, z \rangle$ is the distance between its endpoints, denoted $|\vec{a}|$ or $\|\vec{a}\|$ to mimic the absolute value or magnitude of a scalar. As seen in the previous section, this is

$$|\vec{a}| = |\langle x, y \rangle| = \sqrt{x^2 + y^2} \quad \text{for a 2D vector,} \tag{1}$$

$$|\vec{a}| = |\langle x, y, z \rangle| = \sqrt{x^2 + y^2 + z^2} \quad \text{for a 3D vector.} \tag{2}$$

Adding Vectors

Since vectors describe displacements or movement from one point to another, one can combine several vectors, by making one move and then the other. If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, combining these two displacements changes the first $[x]$ coordinate by $a_1 + b_1$ and so on, so the combined displacement is described by the new vector $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$. This is called the sum of the vectors $\vec{a}$ and $\vec{b}$, and we write

$$\vec{a} + \vec{b} = \langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle. \tag{3}$$

Clearly this is **commutative** like addition of real numbers:

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \tag{4}$$

Also, there is a natural zero vector $\vec{0} = \langle 0, 0, 0 \rangle$ representing no change in position, with

$$\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}. \tag{5}$$

Scalar Multiples of Vectors

Repeated addition of copies of the same vector give natural number multiples of a vector, like

$$2\langle a_1, a_2, a_3 \rangle = \langle a_1, a_2, a_3 \rangle + \langle a_1, a_2, a_3 \rangle = \langle 2a_1, 2a_2, 2a_3 \rangle.$$

This suggests the natural generalization to defining the scalar-by-vector product for any scalar (real number) c:

$$c\vec{a} = c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle \tag{6}$$

Geometrically, $c\vec{a}$ describes a displacement parallel to that described by $\vec{a}$ but of magnitude different by a factor $|c|$,and in the opposite direction if c is negative.

It is routine to check that this multiplication is **distributive** in both factors and **associative** where it makes sense:

$$(c+d)\vec{a} = c\vec{a} + d\vec{a}, \quad c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}, \quad (cd)\vec{a} = c(d\vec{a}). \tag{7}$$

The Magnitude of Products

From the formula above for the length of a vector, it can be checked that the magnitude of a scalar-vector product is the product of the magnitudes:

$$|c\vec{a}| = \sqrt{(ca_1)^2 + (ca_2)^2 + (ca_3)^2} = \sqrt{c^2(a_1^2 + a_2^2 + a_3^2)}$$
$$= \sqrt{c^2}\sqrt{a_1^2 + a_2^2 + a_3^2} = |c||\vec{a}|. \tag{8}$$

Vector Subtraction and Vector-Scalar Division

Subtraction as always is defined in terms of addition: $\vec{a} - \vec{b}$ must be the vector that satisfies $(\vec{a} - \vec{b}) + \vec{b} = \vec{a}$, and this has to be

$$\vec{a} - \vec{b} = \langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle. \tag{9}$$

Likewise division by a scalar can be defined in terms of multiplication:

$$\vec{a}/c = \frac{1}{c}\vec{a} \tag{10}$$

Basic Vectors

Two and three dimensional coordinates were described in terms of getting to a point P with a succession of three moves, parallel to each axis in turn.

The displacement described by a vector can be broken up into three such displacements, which can in turn be written as multiples of displacements by distance one:

$$\vec{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle = a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle.$$

Thus any vector can be written in terms of the three special vectors appearing in the last line: $\vec{a} = a_1 \vec{\imath} + a_2 \vec{\jmath} + a_3 \vec{k}$ where

$$\vec{\imath} = \langle 1, 0, 0 \rangle \qquad \vec{\jmath} = \langle 0, 1, 0 \rangle \qquad \vec{k} = \langle 0, 0, 1 \rangle \tag{11}$$

These are known as the **standard basis vectors** for the set V_3 of vectors.

The space V_2 of two dimensional vectors has standard basic vectors

$$\vec{\imath} = \langle 1, 0 \rangle, \ \vec{\jmath} = \langle 0, 1 \rangle$$

the duplicate names goes with the common intuition that V_2 is the subset of V_3 containing the vectors of form $\langle a_1, a_2, 0 \rangle = a_1 \vec{\imath} + a_2 \vec{\jmath}$.

Unit Vectors

Unit vectors are vectors of length one, like the standard basis vectors above. They are often used to indicate a direction of motion. For any non-zero vector $\vec{a}$, there is a unique unit vector $\vec{u}$ with the same direction,

$$\vec{u} = \frac{1}{|\vec{a}|} \vec{a} = \frac{\vec{a}}{|\vec{a}|}$$

Example (4). *Find the unit vector in the direction of the vector* $\langle 2, -1, -2 \rangle = 2\vec{\imath} - \vec{\jmath} + 2\vec{k}$.

What is Missing?

Although we have seen how do do with vectors much of what can be done with real numbers, a few things are missing: we have not defined a product of two vectors, nor the quotient of two vectors or the inverse of a vector.

In the next sections we will see two versions of the product of vectors, but neither makes division or inverses possible.

Homework: Exercises 4-6, 13, 14*, 15-20, 21-23, 24*, 25-27.

Also work Examples 1-4 in the text, and Exercise 5 if you are interested in applications to physics or engineering.