Notes for Math 220, *Calculus 2*

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Contents

Chapter 1

Integration (review from first semester calculus)

This chapter reproduces the introduction to integration in the final chapter of OpenStax Calculus Volume 1[1], as was covered at the end of MATH 120 *Introductory Calculus*; some class notes for that course are reproduced here for convenience.

Also check out the end of Chapter 1 Review in OpenStax Calculus Volume 2, including Key Terms[2], Key Equations[3] and Key Concepts[4].

References.

- OpenStax Calculus Volume 2, Chapter 1.[5]
- (a.k.a. OpenStax Calculus Volume 1, Chapter 5.[6])
- *Calculus, Early Transcendentals* by Stewart, Chapter 5.

1.1 Approximating Areas (and Distance Traveled)

References.

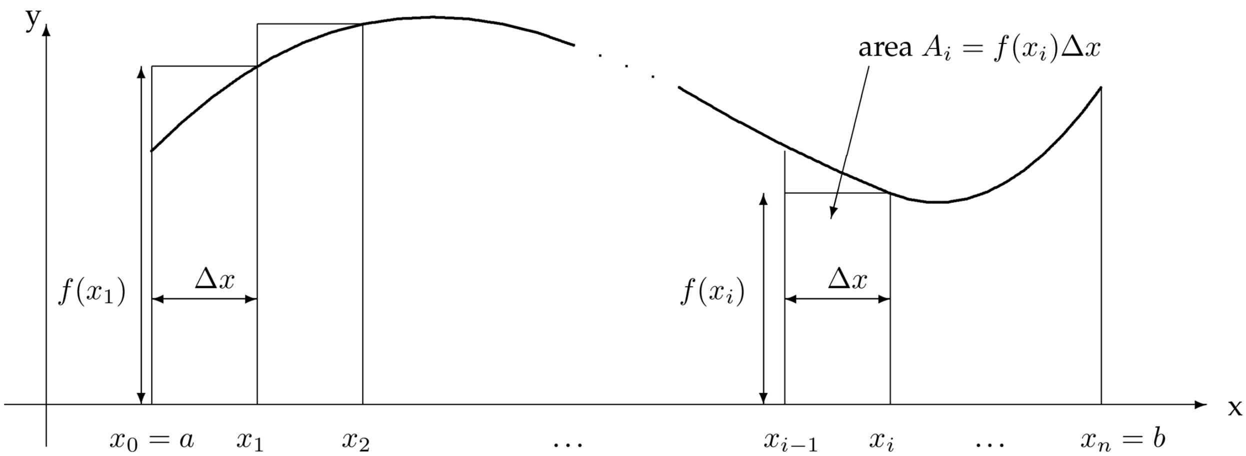
- OpenStax Calculus Volume 2, Section 1.1[1]
- *Calculus, Early Transcendentals* by Stewart, Section 5.1.

One of the beauties of mathematics is that often problems that seem to be very different turn out to have very similar mathematical representations and solutions. Two such problems are

- finding the area of a region with curved boundary, and
- finding the distance traveled when we know how the velocity varies with time.

[1] openstax.org/books/calculus-volume-1/
[2] openstax.org/books/calculus-volume-2/pages/1-key-terms
[3] openstax.org/books/calculus-volume-2/pages/1-key-equations
[4] openstax.org/books/calculus-volume-2/pages/1-key-concepts
[5] openstax.org/books/calculus-volume-2/pages/1-introduction
[6] openstax.org/books/calculus-volume-1/pages/5-introduction
[1] openstax.org/books/calculus-volume-2/pages/1-1-approximating-areas

Problem 1: The Area of a Region with Curved Upper Boundary. We can compute the area shown in the figure where the upper boundary is the curve $y = f(x)$, the lower boundary is the x axis, the left boundary is the line $x = a$, and the right boundary is the line $x = b$ as follows:



Divide the interval $[a, b]$ into n small subintervals each with length

$$\Delta x = \frac{b-a}{n}$$

and let $x_0, x_1, x_2, \dots, x_n$ be the end points of the subintervals, so that

$$x_0 = a, x_i = a + i\Delta x, x_n = b$$

The area A_i above the i-th subinterval $x_{i-1} < x < x_i$ will be approximately the area of a rectangle with width Δx and height $f(x_i)$:

$$A_i \simeq f(x_i)\Delta x$$

Adding these up we get the total area is given (approximately) by

$$A = \sum_{i=1}^{n} A_i \simeq R_n := \sum_{i=1}^{n} f(x_i)\Delta x. \tag{1.1.1}$$

This is the so-called **right-hand endpoint rule**, because we use the value of $f(x)$ at the right-hand end of each sub-interval $[x_{i-1}, x_i]$ as the height of the rectangle over that interval. An alternative is to use the height at the left end of each interval, giving the **left-hand endpoint rule**

$$A \simeq L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x. \tag{1.1.2}$$

Note: never use the value at both the left and right endpoints!

The Exact Area, Using Limits. If we used more subintervals (larger n and thus smaller Δx), we could get a better approximation, because the rectangles would fit the true area closer over the shorter intervals. If we can find the limit as $n \to \infty$ and $\Delta x \to 0$ of these approximating sums, then we can find the area exactly:

$$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x, = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x \tag{1.1.3}$$

It also turns out that the approximations L_n and R_n lead to the same limit, so long as $f(x)$ is continuous. We could also use rectangles with heights given by the function's value at intermediate points in each interval, such as the middle points $a + h/2$, $a + 3h/2$ ... $b - h/2$. In fact, we use this formula to *define* area in this situation; without calculus and limits, area has only really been defined for polygons.

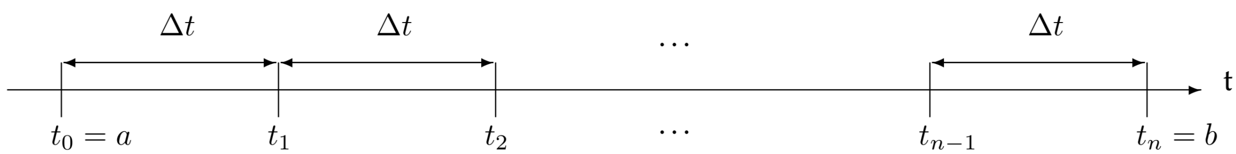
Later in this chapter we will learn how to evaluate such limits, at least for some functions f. However, the accurate approximations given by the sum formulas with small Δx are often also often useful in practice.

Problem 2: Displacement (net change in position) from Velocity. If we know a function that gives the velocity of an object at time t, that is we know $v = f(t)$ and we want to find the distance s that the object travels over a time interval $a \leq t \leq b$, we can proceed as follows: Divide the time interval $[a, b]$ into n small subintervals each with length

$$\Delta t = \frac{b-a}{n}$$

and let $t_0, t_1, t_2, \ldots, t_n$ be the end points of the subintervals, so that

$$t_0 = a, t_i = a + i\Delta t, t_n = b$$



Over the i-th subinterval $t_{i-1} < t < t_i$, the velocity can be approximated by its value at the start of that interval, $f(t_{i-1})$, so that we can use the familiar formula "distance = rate × time" to compute the distance s_i traveled (approximately) over that short time interval:

$$s_i \simeq f(t_{i-1})\Delta t. \tag{1.1.4}$$

Adding these up, the total distance traveled is given (approximately) by

$$s \simeq \sum_{i=0}^{n-1} f(t_i)\Delta t. \tag{1.1.5}$$

This is the *left-hand endpoint rule* (1.1.2) again, and again an alternative is to use the velocity at the end of each time interval; the *right-hand endpoint rule* (1.1.1).

From Approximations to the Exact Displacement. If we used more subintervals (larger n and thus smaller Δt), we could get a better approximation, because the velocity would be closer to being constant over the shorter intervals. If we can find the limit as $n \to \infty$ and $\Delta t \to 0$ of these approximating sums, then we can find the distance exactly.

$$s = \lim_{\Delta t \to 0} \sum_{i=1}^{n} f(t_i)\Delta t, = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i)\Delta t. \tag{1.1.6}$$

Comparing the Area and Distance Formulas. Note the similarity between Equation (1.1.3) for area under curve and Equation (1.1.6) for computing distance from a velocity function: this allows the same methods to be used for both the area and the distance problems, for both approximation and exact evaluation.

Indeed a great variety of the mathematical and scientific problems can be solved in terms of the same sort of "limit of a sum" formula, which makes evaluation of this quantity of great importance.

This is the topic for the rest of this course, and a major topic in Calculus II.

Study Guide. Study Calculus Volume 1, Section 5.1[2]; in particular, if you are unfamiliar with the Σ notation for sums, the first part of that section should help.

Study Exercises 15, 19, 23, 27, 29, and 43.

[2] openstax.org/books/calculus-volume-1/pages/5-1-approximating-areas

1.2 The Definite Integral

References.

- OpenStax Calculus Volume 2, Section 1.2[1]
- *Calculus, Early Transcendentals* by Stewart, Section 5.2.

The ``limit of sums'' formula seen in Section 1.1 for computing both distance traveled and area under a curve is also useful in many other cases, and the main goal of this chapter is to learn more about how to do this calculation in practice, without having to actually evaluate the sums or limits, but instead mostly using anti-derivatives.

In this section we make a careful statement of the quantity to be calculated, introduce some variants on the **Riemann sum** approximation of the area under a curve to make calculator approximations more accurate and efficient, and learn some properties akin to those for limits, derivatives and anti-derivatives: rules for sums, differences, constant multiples, etc.

A key calculus strategy: first approximate, then find a limit. There are many other problems that can be calculated by the above process of

- *approximating* a quantity by a sum of function values times a small interval width Δx, and then
- finding the *exact quantity* as the limit as the number of function values used goes to ∞ and Δx goes to 0.

Thus we need a name and notation for it:

Definition 1.2.1 Definite Integral, right-hand rule version. For $f(x)$ is a continuous function on the interval $a \leq x \leq b$, the **definite integral** of $f(x)$ over the interval $[a, b]$, denoted $\displaystyle\int_a^b f(x)\,dx$, is the numerical value given by the limit

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x, = \lim_{\Delta x\to 0} \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$, so $x_0 = a$ and $x_n = b$. ◊

Note that the notation

- modifies the ``Greek S'' $\Sigma$ to become the ``elongated S'' $\int$, and
- changes the ``Greek D'' $\Delta$ in $\Delta x$ to the ``small d'' in dx, to indicate that the limit was taken as $n \to \infty$ (i.e. $\Delta x \to 0$).

Other Choices for the Rectangle Heights and Widths. The sums of areas of rectangles used above to approximate the area under the curve is called a **Riemann Sum**, but the choice of using intervals of *equal width* with the height of each rectangle being the height of the curve at the *right endpoint* of each interval is not the only possibility: it was used partly because it makes the notation easiest. The intervals can instead vary in width, and the heights can instead be computed at other points x_i^* in each interval, like the left endpoints or the midpoints, or a different choice in each interval. Of these options, using the mid-point of each interval is intuitively the best choice, and this in fact can be proven to be the most accurate in some sense, to be seen in Calculus 2.

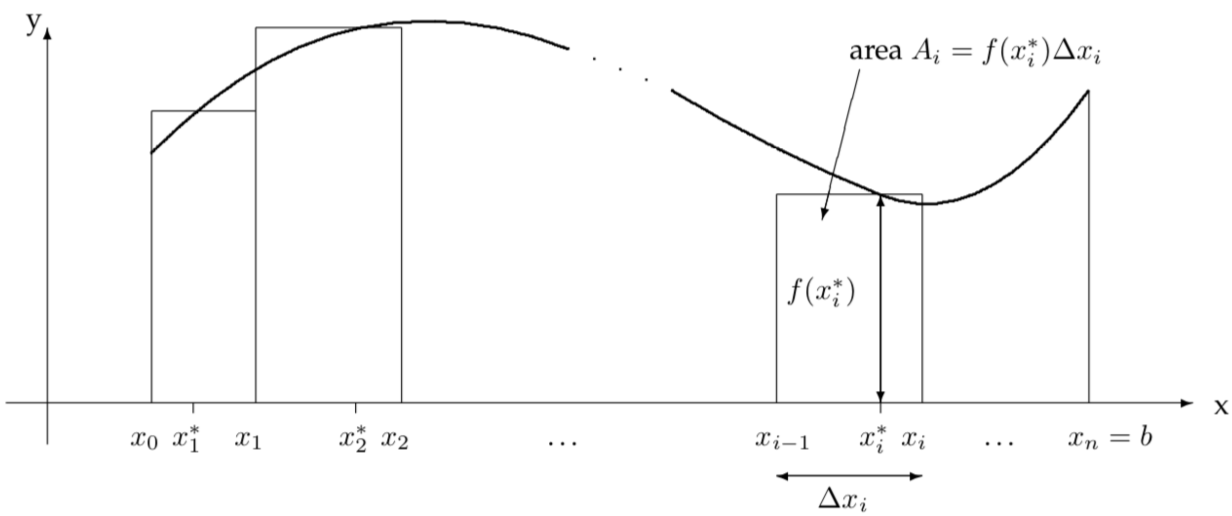
The most general form of the approximation for area under the curve allows for the interval $[a, b]$ to be divided into possibly unequal intervals by x values $a = x_0 < x_1 \cdots < x_n = b$, with widths

$$\Delta x_1 = x_1 - x_0, \ldots, \Delta x_i = x_i - x_{i-1}, \ldots, \Delta x_n = x_n - x_{n-1},$$

[1] openstax.org/books/calculus-volume-1/pages/5-2-the-definite-integral

taking any point x_i^* within each sub-interval $[x_{i-1}, x_i]$ to get the height of a rectangle on that sub-interval. Then the approximate area under the curve is the general *Riemann sum Approximation*

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 \cdots f(x_n^*)\Delta x_n.$$



It can be shown that even with varying intervals widths and choices of where in each interval to compute the rectangle height, the approximations all get close to the same value when the widths of all the subintervals are very small (no rectangle width Δx_i bigger than some maximum width Δx), so long as $f(x)$ is continuous on $[a, b]$. The proof is omitted here; it is seen is advanced calculus courses.

This gives the most general definition:

Definition 1.2.2 Definite Integral, with all Riemann Sum Approximations. If $f(x)$ is a continuous function on the interval $a \leq x \leq b$, with the x_i, Δx_i and x_i^* as above and $\Delta x_i \leq \Delta x$, then

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i \tag{1.2.1}$$

◊

The Mid-point Rule. Of these approximations, using the mid-point of each interval is intuitively the best choice. % and this in fact can be proven to be the most accurate in some sense. %, to be seen in Calculus 2. It is still simplest to use n intervals of equal width $h = \Delta x = (b-a)/n$, which gives the n**-point midpoint rule approximation**

$$\int_a^b f(x)\,dx \approx M_n = h\sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right) = h\sum_{i=1}^{n} f(a+(i-1/2)h),$$

where $x_i = a + ih$ ($x_0 = a, x_1 = a + h$, etc.)

This sum can be evaluated on calculators with something like

```
sum(seq(f(a+(i-0.5)*h),i,1,n))*h
```

or the slightly quirky but easier to type version

```
sum(seq(f(x),x,a+h/2,b,h))*h
```

This uses x values $a + h/2$, $a + 3h/2$ and so on, continuing so long as the value is less than b. Actually the last value used is $b - h/2$, but using the upper limit of b is safer; if you use $b - h/2$, a slight rounding error can cause that last x value to be omitted!

Properties of the Definite Integral. Thinking of definite integrals as areas under curves or displacements given by velocities, the following facts are intuitive. We will soon see a simple way to verify them, using anti-derivatives.

1. $\displaystyle\int_a^b c\, dx = c(b-a)$ where c is any constant.
2. $\displaystyle\int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx$ where c is any constant.
3. $\displaystyle\int_a^b f(x) + g(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$
4. $\displaystyle\int_a^b f(x) - g(x)\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$
5. $\displaystyle\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$
6. $\displaystyle\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$
7. $\displaystyle\int_a^a f(x)\, dx = 0$

Comparison Properties of the Definite Integral.

1. If $f(x) \geq 0$ for $a \leq x \leq b$, then
$$\int_a^b f(x)\, dx \geq 0.$$
2. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then
$$\int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx.$$
3. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then
$$m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a).$$

The last is a cousin of the Mean Value Theorem.

Study Guide. Study Calculus Volume 1, Section 5.2[2]; in particular

- Examples 8 to 13
- Checkpoints 8 to 12
- Exercises 61, 65, 73, 75, 79, 81, 89, 91, 93, 99, 101 and 107.

[2] openstax.org/books/calculus-volume-1/pages/5-2-the-definite-integral

1.3 The Fundamental Theorem of Calculus

References.

- OpenStax Calculus Volume 1, Section 5.3[1]
- *Calculus, Early Transcendentals* by Stewart, Section 5.3.

The Fundamental Theorem of Calculus relates derivatives to definite integrals, giving an easy way to evaluate many definite integrals using antiderivatives. One half is the formula

$$\int_a^b f(x)\,dx = F(b) - F(a), \tag{1.3.1}$$

true when F is *any* antiderivative of f on the interval $[a,b]$: $F' = f$.

This equation can for example be corroborated for some simple definite integrals whose values are clear from geometry:

Example 1.3.1 For $f(x) = c$, any constant, all antiderivatives have the form $cx + C$ (C another constant), so Equation (1.3.1) says

$$\int_a^b c\,dx = F(b) - F(a) = (cb + C) - (ca + C) = c(b - a),$$

the expected area of the rectangle of width $b - a$, height c under this curve. □

Example 1.3.2 For $f(x) = x$, antiderivatives have the form $x^2/2 + C$ (C a constant),

$$\int_a^b x\,dx = F(b) - F(a) = \left(\frac{b^2}{2} + C\right) - \left(\frac{a^2}{2} + C\right) = \frac{b^2}{2} - \frac{a^2}{2} = \frac{a+b}{2}(b-a)$$

which is the area of the trapezoid under this line $y = x$: the difference of the areas of two right triangular regions, or the width of the trapezoid times its average height. (Case $a = 0$ is the area $1/2 \cdot b \cdot b$ of a right triangle of width b, height b.) □

Next, one that requires a far less obvious anti-derivative:

Example 1.3.3

1. Verify that $f(x) = \sqrt{1 - x^2}$ has antiderivative $F(x) = \dfrac{x\sqrt{1-x^2} + \arcsin(x)}{2}$.
2. Sketch a graph of $y = \sqrt{1 - x^2}$ on interval $[-1, 1]$.
3. Use this graph to explain why the value of $2\int_{-1}^{1} \sqrt{1-x^2}dx$ should be π.
4. Verify this by evaluating this integral, using FTC, Equation (1.3.1).

□

The moral here is that we would like to know how to find many more anti-derivatives, like the one seen above.

Getting Antiderivatives from Definite Integrals. To understand why the above result is true, we do something a bit more ambitious: using the definite integral over intervals of variable width $[a, x]$ for x between a and b, so that the value of the definite integral depends on the choice of x. This gives a function of x, which turns out to be an antiderivative of f.

[1] openstax.org/books/calculus-volume-1/pages/5-3-the-fundamental-theorem-of-calculus

That is, we define a function $g(x)$, $a \leq x \leq b$ by

$$g(x) = \int_a^x f(t)\,dt$$

Let us compute the derivative of g, using the definition

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}.$$

The numerator in the difference quotient is

so

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)\,dt.$$

Intuitively, the value of $f(t)$ over the small interval is close to $f(x)$, so the area given by the interval is close to that of a rectangle of height $f(x)$, width h. That is, the integral here is approximately $f(x)h$, so that the difference quotient is approximately $f(x)$, and then the limit gives

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt = \lim_{h \to 0} \frac{1}{h} f(x)h = f(x).$$

This can be shown more carefully using the Extreme Value Theorem and the comparison properties of definite integrals.

So as claimed, $g(x) = \displaystyle\int_a^x f(t)\,dt$ is an antiderivative of f:

$$\frac{d}{dx} \int_a^x f(t)\,dt = f(x).$$

Getting Definite Integrals from Antiderivatives. If F is any antiderivative of f on interval $[a, b]$, it differs from the above antiderivative g only by an added constant, so $F(x) = \displaystyle\int_a^x f(t)\,dt + C$. Thus

The last is true because the variable name used, t or x, has no effect on the value of a definite integral, which is a number, not a function of x or of t.For F any antiderivative of f on interval $[a, b]$,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The difference here arises so often that it is useful to have a short-hand for it:

$$[F(x)]_a^b = F(b) - F(a), \quad = \begin{array}{r} F(b) \\ -F(a) \end{array}$$

I sometimes use the "vertical" form at right above to keep straight which term is added and which subtracted.

Integration and differentiation as inverse processes. The two parts of the Fundamental Theorem of Calculus can be summarized by the idea that integration and differentiation are like inverses:

- Computing the integral of a function f [to upper limit x] and then differentiating the result gets you back to where you started: function f.

- Differentiating a function F (getting $f = F'$) and then integrating over an interval $[a, x]$ the result gets you back to where you started: function F (up to adding a constant.)

Study Guide. Study Calculus Volume 1, Section 5.3[2]; in particular

- Theorems 4 and 5
- Examples 17, 18, 20 and 21
- Checkpoints 16, 17 and 19
- Exercises 149, 153, 155, 157, 161, 171, 177, 179, 183, 190, 191 and 195.

For further practice, look at several exercises from each of the following ranges: 148–159, 160–163, 170–189, 190–193, and 194–197.

1.4 Integration Formulas and the Net Change Theorem

References.

- OpenStax Calculus Volume 1, Section 5.4[1]
- *Calculus, Early Transcendentals* by Stewart, Section 5.4.

Now that we have seen the connection between antiderivatives and definite integrals, it is convenient to recast antiderivatives in terms of integrals, and use the notation of integrals when calculating with antiderivatives. Thus, just as $\int_a^x f(t)\,dt$ gives one antiderivative of f (a different one for each different choice of a) we denote the general antiderivative by dropping the specific choice a, and simplifying a bit:

Definition 1.4.1 The **Indefinite Integral of f with respect to x** is the most general function $F(x)$ having $F'(x) = f(x)$, including an arbitrary added constant. This is denoted

$$\int f(x)\,dx$$

The function inside this expression is called the **integrand**.

The differential "dx" is essential! For example, we can verify that

$$\int x\,t\,d\mathbf{x} = x^2 t/2 + C \qquad \text{while} \qquad \int x\,t\,d\mathbf{t} = x t^2/2 + C.$$

◊

Example 1.4.2

$$\int x^2\,dx = \frac{x^3}{3} + C$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \sec^2 x\,dx = \tan x + C$$
$$\int \tan x\,dx = \log|\sec x| + C$$
$$\int \ln x\,dx = x\ln x - x + C$$

[2] openstax.org/books/calculus-volume-1/pages/5-3-the-fundamental-theorem-of-calculus
[1] openstax.org/books/calculus-volume-1/pages/5-4-integration-formulas-and-the-net-change-theorem

$$\int 2xe^{x^2}\,dx = e^{x^2} + C$$

□

How do we know that these are correct?

Differentiate the formula at right and verify that this gives the integrand, the function inside the integral expression at left.

Activity 1.4.1 Make a list of as many indefinite integrals as you can. This will be a useful reference when working exercises keep adding to this list as you learn more.

Rather than list numerous sums and constant multiples, the list can start with the two general "combining" rules that we have for *sums*, *differences* and *constant multiples* of antiderivatives, rephrased as facts about integrals:

$$\int cf(x)\,dx = c\int f(x)\,dx \quad \text{for any constant c.}$$

$$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

But beware: we have ***no rules*** for *products* or *quotients* or *compositions* of functions.

Simplify first! As usual, it often helps to simplify the function as much as possible before looking for antiderivatives. both by using the above rules to break up sums and differences and extract constant factors, and by using other algebraic rules and trigonometric facts.

Connection to Definite Integrals. The Fundamental Theorem of Calculus gives

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

and we denote the difference here with the shorthand forms

$$\big[F(x)\big]_a^b = F(x)\big]_a^b = F(b) - F(a)$$

so we can now use the indefinite integral notation for the antiderivative:

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b = \int f(x)\,dx\Bigg]_a^b$$

(I prefer always using matching left and right brackets to avoid any possible ambiguity, but some texts use the right bracket only.)

Integrals of Derivatives and the Net Change Theorem. The indefinite integral a function is the *general* antiderivative, so the indefinite integral of the derivative f' of function f is the general antiderivative of the derivative. The original function f itself is one such antiderivative, so all that remans is to add an arbitrary constant:

$$\int_a^b f'(x)\,dx = f(b) - f(a), = \big[f(x)\big]_a^b$$

This says that the definite integral of the rate of change of a quantity gives the net change in the quantity.

For $f(t)$ a Velocity, Displacement is Net Change. For example, if function f gives position and the independent variable is time, the rate of change is velocity, $v(t) = f'(t)$, so the definite integral of velocity from a to b is the *net* change in position between times a and b, the *displacement*, not the *total* distance traveled:

The *displacement* between times a and b is $\displaystyle\int_a^b v(t)\,dt = \int_a^b f'(t)\,dt = f(b) - f(a)$.

This is what we saw in Problem 2 of Section 1.1, motivating the idea of the definite integral. Geometrically, this is the difference between the area under the positive part of the graph of $v = f'$ and the area below the negative part.

Total Distance Traveled is Total Change. On the other hand, the total distance traveled is the "total change" of position, given by integrating the rate of change of position without regard to direction: this is *speed*, which is the magnitude of the velocity, $|v(t)|$.

The *total distance traveled* between times a and b is $\displaystyle\int_a^b |v(t)|\,dt = \int_a^b |f'(t)|\,dt$.

Geometrically, this is the total area between the graph of v and the t-axis, adding area above and below the axis. It is not given by the simple formula $f(b) - f(a)$, so how can it be computed?

The answer is to break the integral up into several integrals over several intervals such that on each interval, $v = f'$ is either positive throughout or negative throughout. Then each integral is of the form either $\int_c^d f'(t)\,dt$ or $\int_c^d -f'(t)\,dt$, and so each can be evaluated easily by the Net Change Theorem, as either $f(d) - f(c)$ or $f(c) - f(d)$. Adding these positive pieces gives the total distance traveled.

Study Guide. Study Calculus Volume 1, Section 5.4[2]; in particular

- Theorem 6
- Examples 23–26, 28 and 29
- Checkpoints 21, 22 and 24
- Exercises 207, 209, 211 and 223.

For further practice, look at several exercises from each of the following ranges: 207-212 and 223-228.

1.5 Substitution

References.

- OpenStax Calculus Volume 1, Section 5.5[1]
- *Calculus, Early Transcendentals* by Stewart, Section 5.5.

So far we are rather limited in our ability to calculate antiderivatives and integrals because, unlike with derivatives, knowing indefinite integrals for two functions does not in general allow us to calculate the indefinite integral of their product, quotient, or composition. However, we can find a rule that will help up with ***some*** products and compositions, using the same strategy that lead us to our first few antiderivatives: take a fact about derivatives and "invert" it.

Surprisingly, it is the Chain Rule that is most useful, because the derivative of a composition is a certain product, and thus running it backwards gives an antiderivative for that product: ***almost*** a product rule for indefinite integrals.

[2] openstax.org/books/calculus-volume-1/pages/5-4-integration-formulas-and-the-net-change-theorem
[1] openstax.org/books/calculus-volume-1/pages/5-5-substitution

Getting some integrals involving products, quotients and compositions. To get an idea of how the Substitution Rule will work, let us first get a few examples of integrals of products by working backwards from some derivatives.

$$\frac{d}{dx}(\sin x)^3 = 3(\sin x)^2 \frac{d}{dx}(\sin x) = 3\sin^2 x \cos x \text{ , so dividing by 3,}$$

$$\int \sin^2 x \cos x \, dx = \frac{1}{3}(\sin x)^3 + C \tag{1.5.1}$$

$$\frac{d}{dx}\ln(\cos x) = \frac{1}{\cos x}\frac{d}{dx}(\cos x) = -\frac{\sin x}{\cos x} \text{ , so}$$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\ln(\cos x) + C \tag{1.5.2}$$

Reversing a Chain Rule Calculation. Let us look at the first calculation above in reverse order. Factor $\cos x$ has antiderivative $\sin x$, which also appears in the other factor, so

$$\sin^2 x \cos x = (\sin x)^2 \frac{d}{dx}(\sin x)$$

Using the name u for this repeated term $\sin x$, this is $u^2 \dfrac{du}{dx}$.

Since u^2 has antiderivative $u^3/3$,

$$u^2 \frac{du}{dx} = \frac{d}{du}\left(\frac{u^3}{3}\right)\frac{du}{dx}.$$

This is what the Chain Rule gives for

$$\frac{d}{dx}\left(\frac{u^3}{3}\right), = \frac{d}{dx}\frac{(\sin x)^3}{3}.$$

Thus $\sin^2 x \cos x$ has antiderivative $\dfrac{(\sin x)^3}{3}$.

Reversing a Chain Rule Calculation. In this calculation, the antiderivatives for $\cos x$ and u^2 have been combined to get this new antiderivative. In terms of indefinite integrals, we have used the two simple indefinite integrals

$$\int \cos x \, dx = \sin x + C$$

and

$$\int u^2 du = \frac{u^3}{3} + C$$

plus the Chain Rule $\dfrac{d}{dx}f(u) = f'(u)\dfrac{du}{dx}$ to get

$$\int \sin^2 x \cos x \, dx = \frac{\sin^3 x}{3} + C.$$

The Substitution Rule. Suppose that we seek the (indefinite) integral of a function of the special "composition-product-derivative" form $f(g(x))g'(x)$, and we know an antiderivative F for f. Then the Chain Rule gives

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x) = f(g(x))g'(x)$$

so $F(g(x))$ is an antiderivative of this function, and

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

On the other hand, if we define $u = g(x)$ then $g'(x) = \dfrac{du}{dx}$, $F(g(x)) = F(u)$, and $\int f(u)\,du = F(u) + C$. Combining these results

$$\int f(g(x))g'(x)\,dx = \int f(u)\frac{du}{dx}\,dx = \int f(u)\,du = F(u) + C. \tag{1.5.3}$$

In practice, the emphasis is on choosing the new quantity u and changing to it as the variable, which involves getting a differential du in the integral formula in place of dx. To be precise,

Theorem 1.5.1 Substitution Rule. *If $u = g(x)$ is differentiable with range covering some interval I, and function f is continuous on that interval I, then*

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

That is, one can effectively make a substitution with the differential formula $du = \dfrac{du}{dx}dx$ inside integral formulas, and this helps so long as the rest of the formula can also be expressed entirely in terms of the new variable u. ***Note well:*** for this substitution method to be useful, one must completely convert to u from x, not have a mix of both variables in the transformed integral.

Choosing u. The key in practice is finding a suitable choice of u, and there may be more than one worth trying. One strategy is to use the quantity inside a composition as u, since u is the "inside" part of the Chain Rule. *This composition should then multiply some other factor containing just the derivative of u.*

Another strategy is that to seek some quantity such that both its and its derivative appear in the integrand, *with the derivative simply multiplying the rest of the integrand.*

Example 1.5.2 Find $\int x^3 \cos(x^4 + 2)\,dx$.

Start by thinking of options for u. □

Example 1.5.3 Find $\int \cos x \sin x\,dx$.

Start by seeking several possible options for u. □

Example 1.5.4 Find $\int \sqrt{2x+1}\,dx$.

Here there is no product, just a composition, so what do we do about the need for factor du/dx?

It still helps to try the inside of a composition as u. □

Example 1.5.5 Find $\int \dfrac{x}{\sqrt{1-4x^2}}dx$. □

Example 1.5.6 Find $\int e^{3x}\,dx$. □

Example 1.5.7 Find $\int \sqrt{1+x^2}\,x^5\,dx$. □

Example 1.5.8 Find $\int \tan x\,dx$. □

Substitution in Definite Integrals. Often the easiest way to deal with definite integrals is to first seek an indefinite integral, and then use the FTC. However with substitution, this involves the step of converting back from a function of new variable u to the original variable x, and it may be easier to avoid that by converting everything to the new variable, including the limits of integration.

Theorem 1.5.9 The Substitution Rule for Definite Integrals. *If $u = g(x)$ is differentiable with range covering some interval I, g' is continuous, and function f is continuous on that interval I, then*

$$\int_{x=a}^{x=b} f(g(x))g'(x)\,dx = \int_{u=c}^{x=d} f(u)\,du, \quad \textit{with } c = g(a), d = g(b). \tag{1.5.4}$$

That is, for F an antiderivative of f,

$$\int_{x=a}^{x=b} f(g(x))g'(x)\,dx = [F(u)]_{u=g(a)}^{u=g(b)}. \tag{1.5.5}$$

I write the integral limits as "$x = a$", "$u = b$" and so on to emphasize that u must *completely* displace x, in *three* places:

- in the formula for the integrand, $f(u)$ replaces $f(g(x))$;
- in the differential, du replaces $\dfrac{du}{dx}dx = g'(x)dx$; and
- in the limits of integration, $c = g(a)$ and $d = g(b)$ replace a and b.

Example 1.5.10 Evaluate $\displaystyle\int_0^4 \sqrt{2x+1}\,dx$. □

Example 1.5.11 Evaluate $\displaystyle\int_1^2 \frac{1}{(3-5x)^2}\,dx, \; = \int_1^2 \frac{dx}{(3-5x)^2}$. □

Example 1.5.12 Evaluate $\displaystyle\int_1^e \frac{\ln x}{x} dx$.

Try it both ways: using the above formula , and by first finding the indefinite integral as a function of x and then using the FTC. □

Short-cuts From Symmetry. For even and odd functions integrated over a symmetric interval $[-a, a]$, the intervals simplify:

- (a) If $f(x)$ is odd $[f(-x) = -f(x)]$ then $\displaystyle\int_{-a}^a f(x)\,dx = 0$.
- (b) If $f(x)$ is even $[f(-x) = f(x)]$ then $\displaystyle\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$.

Example 1.5.13 Evaluate $\displaystyle\int_{-2}^2 x^6 + 1\,dx$. □

Example 1.5.14 Evaluate $\displaystyle\int_{-1}^1 \frac{\tan x}{1+\sec^2 x} dx$. □

Study Guide. Study Calculus Volume 1, Section 5.5[2]; in particular

- Theorem 7
- the *Problem Solving Strategy* that follows it

[2] openstax.org/books/calculus-volume-1/pages/5-5-substitution

- Examples 30–33 (and maybe 34 and 35)
- Checkpoints 25–28, (and maybe 29 and 30)
- and one or several exercises from each of the following ranges: 256–260, 261–270, 271–287 and 292–297; Some suggested selections are Exercises 257, 261, 265, 271, 275, 281, 293, 297.

As noted above, for definite integrals one can either do it as described there (Theorem 8, Examples 34 and 35, Checkpoints 29 and 30) or (a) first get the indefinite integral $\int f(x)dx = F(x) + C$ using substitution and then (b) use FTC: $\int_a^b f(x)dx = F(b) - F(a)$.

1.6 Integrals Involving Exponential and Logarithmic Functions — Summary

References.

- OpenStax Calculus Volume 1, Section 5.6[1]

This section of the OpenStax text just introduces a couple useful new indefinite integrals, and then gives some example and practicee of using them in combination with substitutions; these notes just provide a brief study guide to that.

The main new integrals here are:

$$\int e^x \, dx = e^x + C$$
$$\int a^x \, dx = \frac{1}{\ln a} a^x + C, \text{ and}$$
$$\int \ln x \, dx = x \ln(x) - x + C = (x-1)\ln x$$

along with

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

already seen.

Study Guide. Study Calculus Volume 1, Section 5.6[2]; in particular

- Examples 37, 38, 39, 41, 44, 45, 47, 48
- the Checkpoints that immediately follow each of those Examples
- and a few Exercises from each of the ranges 320–325, 328–339, and 355–357.

1.7 Integrals Resulting in Inverse Trigonometric Functions — Summary

References.

- OpenStax Calculus Volume 1, Section 5.7[1]

As with Section 1.6 , this section of the OpenStax text just introduces a few useful indefinite integrals, and then gives some example and practice with using them in combination with substitutions; often simple ones of the form $u = ax$; these notes just provide a brief guide to that.

[1] openstax.org/books/calculus-volume-1/pages/5-6-integrals-involving-exponential-and-logarithmic-fu
[2] openstax.org/books/calculus-volume-1/pages/5-6-integrals-involving-exponential-and-logarithmic-fu
[1] openstax.org/books/calculus-volume-1/pages/5-7-integrals-resulting-in-inverse-trigonometric-funct

The two very useful integrals here are

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin(x/a) + C, \quad a > 0, \text{and}$$
$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\arctan(x/a) + C$$

The third one

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}(|x/a|) + C, \quad a > 0$$

is also occasionally useful, but less often.

Study Guide. Study Calculus Volume 1, Section 5.7[2]. All Examples and Checkpoint items are worth looking at; then do a few Exercises from each of the ranges 391–394, 397–400, and 411–414.

[2] openstax.org/books/calculus-volume-1/pages/5-7-integrals-resulting-in-inverse-trigonometric-funct

Chapter 2

Applications of Integration

References.

- OpenStax Calculus Volume 2, Chapter 2. Applications of Integration[1]
- *Calculus, Early Transcendentals* by Stewart, Chapter 6.

2.1 Areas between Curves

References.

- OpenStax Calculus Volume 2, Section 2.1[1].
- *Calculus, Early Transcendentals* by Stewart, Section 6.1.

Introduction. In Section 1.1 the area below a curve and above the x-axis between $x = a$ and $x = b$ was described in terms of a limit of Riemann sums, and then the Fundamental Theorem of Calculus Equation (1.3.1) in Section 1.3 allowed us to avoid the sums and limits, and get the result simply as a definite integral, evaluated by finding an anti-derivative $F(x)$ and evaluating $F(b) - F(a)$. This strategy will be used to solve a wide variety of problems:

1. Approximate the solution of some problem by a sum of general form $\sum f(x_i)\Delta x$, with Δx the spacing between the successive values $x_1, x_2, \ldots$.
2. Get the exact solution as a limit of these approximating sums, of the form $\lim\limits_{\Delta x \to 0} \sum f(x_i)\Delta x$
3. Recognise this result as some definite integral $\int_a^b f(x)\,dx$.
4. Evaluate if possible by finding the indefinite integral $F(x) + C$ of $f(x)$, so that the result is $[F(x)]_a^b = F(b) - F(a)$.

We start in this section with some variations on the area problem, and later see other applications like computing volumes, curve lengths, and surface areas.

Approximating the Area Between Two Curves Using Thin Rectangular Strips. Consider the first goal of approximating the area of the region bounded above by curve $y = f(x)$, below by curve $y = g(x)$, at left by vertical line $x = a$ and at right by $x = b$. We can use the following strategy to get approximations of this area.

[1] openstax.org/books/calculus-volume-2/pages/2-introduction

[1] openstax.org/books/calculus-volume-2/pages/2-1-areas-between-curves

(It is very similar to the approach used in Section 1.1 to approximate the area under a curve and we use a similar strategy several more times in this course, so I strongly recommend that you make sketches illustrating the strategy as we go along.)

1. First, we can slice the region into n thin vertical strips all of equal width $\Delta x = (b-a)/n$, with the cuts being the vertical lines $x = x_i$, $x_i = a + ih$, $1 \le i \le n-1$. (The extremities are at $x_0 = a$ and $x_n = b$.) Thus strip 1 is the interval $[x_0, x_1]$, ending at x_1, and so on: strip number i is $[x_{i-1}, x_i]$.
2. The area of each strip can be approximated by that of a rectangle roughly fitting the strip: width Δx and height the vertical distance between the curves at some point within the strip.
3. Where do we measure the height of the i-th rectangle? At any x value x_i^* within the strip, so $x_{i-1} \le x_i^* \le x_i$. (One natural choice is the midpoints, $x_i^* = (x_{i-1} + x_i)/2$.)
4. The i-th rectangular strip extends vertically from $y = g(x_i^*)$ to $y = f(x_i^*)$ and so has height $f(x_i^*) - g(x_i^*)$ and area
$$(f(x_i^*) - g(x_i^*))\Delta x.$$
5. The total area of the region between the curves is approximated by the sum of these strip areas:
$$A_n = \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*))\Delta x.$$

The Exact Area Between Two Curves. We can improve the above approximation A_n by increasing the number of strips n and so decreasing their width Δx, and it is reasonable that the exact area is the limit as $n \to \infty$:

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*))\Delta x. \tag{2.1.1}$$

Now the good news, critical to a lot of what we do in this course: *we do not need to do anything with this limit and sums except compare this formula*

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*))\Delta x$$

to the definition of the definite integral

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

(see Equation (1.2.1) in Section 1.2) with the same meaning for the x_i, x_i^* and so on.

This comparison gives the area of this region between the curves and vertical lines as

$$A = \int_a^b f(x) - g(x)\,dx \tag{2.1.2}$$

Example 2.1.1 The area between $y = e^x$, $y = x$, $x = 0$ and $x = 1$.

- Sketch the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded at the sides by $x = 0$ and $x = 1$.
- Find its area.

□

Regions Bounded by Intersections of Curves. Sometimes a region is naturally enclosed by two curves $y = f(x)$, $y = g(x)$ that intersect, and the left and right extremities come from those intersections, rather than from explicitly given x values. Then the limits of integration come from solving $f(x) = g(x)$ to find the "corners".

Example 2.1.2 The area between $y = x^2$ and $y = 2x - x^2$.

- Sketch the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$
- Find its area.

□

Which is the Top, Which is the Bottom? Sometimes it is is not initially obvious which function describes the top and which the bottom, and instead you have to work this out. Also, the curves can swap places if they intersect repeatedly. The safe way is to note that the height of the region at any x value is always $|f(x) - g(x)|$, so the area is

$$A = \int_a^b |f(x) - g(x)|\, dx$$

However, to find anti-derivatives it helps to have a formula that does not involve absolute values. The way to do that is work out the intervals where $f(x) > g(x)$ giving height $f(x) - g(x)$ and those where the opposite is true so the height is $g(x) - f(x)$; then integrate these different height functions over each such interval, and add the results. Only one anti-derivative need by found: just flip its sign! In turn, the way to find the x values that divide these intervals is to note that they are the intersections of the curves, found by solving $f(x) = g(x)$: a sketch can then help to work out the top and bottom in each sub-interval. (Inequalities are often best handled like this, by first solving for equality.)

Example 2.1.3 The area surrounded by $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \pi/2$.

- Sketch the region enclosed by the curves $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \pi/2$
- Find its area.

□

Rotating and Chopping to Simplify, and Geometry. Some regions are not best described by top and bottom curves as functions of x. Instead, the best approach might be to look at things sideways, with curves at left and right and using y as the variable of integration. Another strategy is to divide the region into several simpler pieces and add their areas.

(And sometimes, basic geometry is the better way to compute an area!)

Example 2.1.4 The area between the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

- Sketch the region enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.
- Then try several different strategies for computing its area.

□

2.2 Determining Volumes by Slicing

References.

- OpenStax Calculus Volume 2, Section 2.2[1].
- *Calculus, Early Transcendentals* by Stewart, Section 6.2.

Approximating volume. To compute the volume of a solid, we can largely mimic the strategy used for areas, starting by slicing the solid perpendicular to the rotation axis (x-axis) into thin pieces whose volume can be easily approximated. It is often natural to get the volume of a piece being "area times thickness", with the thickness being Δx. If the position of each slice depends on its position x along the axis, and the area

[1] `openstax.org/books/calculus-volume-2/pages/2-2-determining-volumes-by-slicing`

depends on the position through a function $A(x)$, then the slice volume is roughly $A(x)\Delta x$. More carefully, the approximate volume of the i-th slice is $A(x_i^*)\Delta x$, where x_i^* is some x-value lying within the slice.

From approximate slice volume to total volume. Once we have worked out this slice volume approximation, the rest is all much as before, with slice area $A(x)$ here doing what slice height $f(x) - g(x)$ did for area between curves. Leaving aside the details (seen in the text and in class), the key idea is that the total volume is given by

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\,dx,$$

and we are back to the problem of finding an anti-derivative!

Finding the limits of integration: left and right extremities of the solid. The only important ingredients missing are the limits of integration, a and b, and these are simply the positions of the left-most and right-most extremities of the solid. Sometimes these are specified as part of the description of the solid. However you often have to look at the geometry of the situation to work them out, like with computing the area between intersecting curves in the previous section.

One important detail is different for volumes: computing the area of a slice is not always as easy as computing the height of a strip. We deal with this for now by limiting ourselves to solids than can be slices into simple shapes like rectangles, circle and such. More general shapes are taken up in Calculus III, which concentrates on combining calculus with 3D geometry.

Example 2.2.1

1. Sketch the region under the curve $y = \sqrt{x}$ for $0 \le x \le 1$.
2. Indicate rotation about the x-axis on this sketch.
3. Sketch the solid obtained by rotating the above region about the x-axis.
4. Sketch a typical slice used to approximate the volume; either on the above sketch or separately.
5. Compute the volume of this solid.

□

In this very common case of a solid produced by rotation of curve $y = f(x)$ about the x-axis, the volume formula takes the special form

$$V = \int_a^b \pi r^2 dx = \pi \int_a^b [f(x)]^2\,dx$$

writing $r = f(x)$ as it is the "radius" of the solid at position x.

Example 2.2.2 Using the strategy developed above, show that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$. □

Slices do not have to be disks, and the range of x-values is not always given explicitly; sometimes you have to find the x limits, and it helps to examine a sketch in the x-y plane.

Example 2.2.3

1. Sketch the region bounded by $y = x$ and $y = x^2$.
2. Indicate rotation about the x-axis on this sketch.
3. Sketch the solid obtained by rotating this region about the x-axis.
4. Sketch a typical slice used to approximate the volume; describe how it differs from all our previous slices!
5. Compute the volume of this solid.

□

Slicing into washers (Annuli). In the example above, the solid is produced by rotation of an inner curve $y = g(x)$ and an outer curve $y = f(x)$, giving slices that are annuli ("washers") with

- outer radius $R = f(x)$,
- inner radius $r = g(x)$, and
- slice area $A = \pi(R^2 - r^2)$.

This gives volume

$$V = \int_a^b \pi[R^2 - r^2]dx = \pi \int_a^b \{[f(x)]^2 - [g(x)]^2\}\, dx.$$

Example 2.2.4 Slices do not have to be "circular" at all! Find the volume of the pyramid whose base is a square with side length L and whose height is h. □

Example 2.2.5 The rotation does not have to around a coordinate axis.

1. On the sketch of the plane region in Example 2.2.3, add the line $y = -1$ and indicate rotation about this line.
2. Find the volume of the solid obtained by rotating that region about the horizontal line $y = -1$.

□

Example 2.2.6 The rotation axis can be vertical instead of horizontal. But then we want functions of y instead of x.

1. Sketch the region bounded by $y = x^2$, $y = 8$ and $x = 0$.
2. Indicate rotation about the y-axis on this sketch.
3. Sketch the solid obtained by rotating this region about the y-axis.
4. Sketch a typical slice used to approximate the volume. Note that slice area depends on y, not x.
5. Compute the volume of this solid.

□

However, in the next section we see another strategy for solids created by rotation about vertical axes which avoids equation solving.

2.3 Volumes of Revolution: Cylindrical Shells

References.

- OpenStax Calculus Volume 2, Section 2.3[1].
- *Calculus, Early Transcendentals* by Stewart, Section 6.3.

Introduction. With some solids of revolution, computing the volume through slicing perpendicular to the axis of rotation involves solving equations to get a formula for the slice area, which can be difficult or even impossible.

This happens for example when the region is bounded above and below by curves described as functions of x and rotation is about the y-axis.

Volume Formula using Shells for Solids of Rotation about a Vertical Axis. In the previous section, we saw that when a plane region bounded above and below by curves is rotated about a *horizontal* axis, thin vertical strips of the region get rotated into disks or washers, whose approximate volumes $\pi r^2 \Delta x$ or $\pi(R^2 - r^2)\Delta x$ —we could then combine (integrate) to get the total volume.

[1] `openstax.org/books/calculus-volume-2/pages/2-3-volumes-of-revolution-cylindrical-shells`

When instead such a region is rotated about a *vertical* axis, the same thin vertical strips between the curves get rotated into "shells". The shells are still annuli like the washers in the previous section, but far taller and far thinner between the inner and outer radii.

We can again approximate the volume of each shell, so our basic strategy still works: *"sum" the shell volumes with an integral.*

One difference is that the shell position x is now the radius at which the shell is located from the rotation axis, so that its values are never negative: *we only need to rotate strips to one side of the rotation axis (say $x > 0$) to get the whole solid.*

Volume Formula with Shells from y-axis Rotation. If we start with a plane region lying

- vertically in $g(x) \le y \le f(x)$, and
- horizontally in $a \le x \le b$ with $a \ge 0$,

the region can be divided into thin vertical strips with

- height $f(x) - g(x)$, and
- width Δx.

When the region is rotated about vertical axis $y = 0$, rotating the strip at distance x from the y-axis sweeps out a *cylindrical shell* of height $f(x) - g(x)$, radius x, and thickness Δx. Each shell has volume ΔV of approximately the area of its outer surface times its thickness, which is "circumference times height times thickness", or

$$\Delta V \approx 2\pi x \times [f(x) - g(x)] \times \Delta x$$

for the shell of radius x, $0 \le a \le x \le b$.

In the limit $\Delta x \to 0$, this gives "infinitesimally thin shells" of "infinitesimal volume"

$$dV = 2\pi x\, [f(x) - g(x)]\, dx\ , 0 \le a \le x \le b,$$

and the total volume is the integral of these over the relevant values of radius x:

$$V = \int_{x=a}^{b} dV = \int_{x=a}^{b} 2\pi x\, [f(x) - g(x)]\, dx.$$

Example 2.3.1

1. Sketch the region between the curve $y = 2x^2 - x^3$ and the x-axis in the right half-plane only. (When rotating a region about an axis, we only need a region on one side of the rotation axis!)
2. Mark a typical vertical strip of width Δx within the region.
3. Indicate rotation of this strip about the y-axis on this sketch.
4. Sketch the typical cylindrical shell produce by rotation of the above strip.
5. Sketch the solid obtained by rotating the above region about the y-axis.
6. Find a definite integral expression for the volume of this solid, using cylindrical shells.
7. Compute the volume of this solid.

□

Volume Formula with Shells in $0 \le y \le f(x)$. In this case of a solid produced by rotation about the y-axis of the region between a curve $y = f(x) > 0$ and the x-axis, with the original region having x values in the range $a \le x \le b$, the volume is given by

$$V = \int_{x=a}^{b} 2\pi x\, f(x)\, dx.$$

As in previous sections, sometimes the relevant range of x-values is not stated explicitly, but comes naturally from the geometry of the situation.

Example 2.3.2

1. Sketch the region between the curves $y = x$ and $y = x^2$.
2. Mark a typical vertical strip in this region, noting the range of possible x-values where such strips lie occur, $a \leq x \leq b$.
3. Sketch the cylindrical shell produced by rotating this vertical strip about the y-axis.
4. Find a definite integral expression for the volume of this solid, and evaluate the volume.

□

Again, with practice, a single 2D sketch is enough, so the 3D sketching can be avoided.

The vertical axis of rotation need not always be the y-axis $x = 0$; it can be any vertical line $x = c$. Then the radius of each shell is $|x - a|$, but the absolute value should be eliminated from the formulas by working out the sign of $x - a$; — once again a sketch can help.

Example 2.3.3

1. Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.
2. Make a single 2D sketch of the curves, the region, the axis of rotation and a typical vertical strip, and label the strip with the radius of the shell produced by rotating it about the axis.

□

2.4 Arc Length of a Curve and Surface Area

References.

- OpenStax Calculus Volume 2, Section 2.4[1].
- *Calculus, Early Transcendentals* by Stewart, Section 8.1.

Note: We only address the topic of arc length here; surface areas are better handled in the context of parametric curves, in Section 7.2.

Introduction. Just as calculus was used to compute the areas of regions with curved boundaries, starting with an approximation by the area of a collection of polygons, we will now make sense of the length of a curve through approximating a curve by a collection of short straight line segments, and so approximate its area by the total length of that collection.

In this section we do this for curves in the plane that are the graph of a function: a curve C that is the set of points (x, y) given by $a \leq x \leq b$, $y = f(x)$.

Later, in Section 7.2, we will see how to consider the length of other curves, like circles and spirals.

Approximation by Polygonal Curves. The first step is familiar: divide the range of x values into n small intervals with values at spacing $\Delta x = (b - a)/n$: $x_0, x_1, \dots x_n$ with $x_i = a + i\Delta x$ so $x_0 = a$, $x_n = b$.

This then gives a sequence of points along the curve: $P_i(x_i, y_i)$ with $y_i = f(x_i)$, $0 \leq i \leq n$. The point $P_0(a, f(a))$ is called the *initial point* of the curve, the point $P_n(b, f(b))$ is called the *terminal point*, and both are called *endpoints*.

[1] openstax.org/books/calculus-volume-2/pages/2-4-arc-length-of-a-curve-and-surface-area

Joining these points with line segments gives a "polygonal curve", of length

$$L_n = \sum_{i=1}^{n} |P_{i-1}P_i| = \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

where $\Delta y_i = y_i - y_{i-1}$ is the vertical step on the i-th edge.

Curve Length as a Limit. The exact length of the curve C is then defined as the limit of these approximations:

$$L = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

As with areas and volumes, we want to turn this into a definite integral with the Fundamental Theorem of Calculus (FTC).

To get the factor Δx needed, rearrange as

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left[\frac{\Delta y_i}{\Delta x}\right]^2} \Delta x$$

Each difference quotient is approximately a value of f': using the Mean Value Theorem, we can get

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x \quad \text{[for suitable values .]}$$

Finally, the Fundamental Theorem of Calculus (Part 2) gives:

The Arc Length of a Curve. If $f'(x)$ is continuous on $[a, b]$ then the curve C of points $P(x, y)$ given by $a \le x \le b$, $y = f(x)$ has arc length

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{2.4.1}$$

Sometimes we are interested not just in the total length, but the length of the part of a curve from the initial point up to a certain x value.

This will be useful for example when the curve is describing motion in time, and we want to compute distance traveled as a function of time.

Changing the dummy variable of integration from x to t, the length of the curve from the initial point where $t = a$ to the point where $t = x$ is a function of x, the *arc length function*

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt \tag{2.4.2}$$

The rate of change of arc length as x increases is given by the Fundamental Theorem of Calculus (Part 1) as

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

The differential of the arc length function can be useful: it is

$$ds = \sqrt{1 + [f'(x)]^2}\, dx = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

which has a nicely mnemonic form if we square both sides:

$$(ds)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right](dx)^2 = (dx)^2 + (dy)^2$$

which is intuitively what the Pythagorean Theorem gives for the length of a small piece of the curve of horizontal extent dx, vertical extent dy.

Perhaps the best way to remember the arc length formulas is to first remember this intuitive differential formula and then integrate:

$$s(x) = \int_a^x ds = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt, \qquad L = s(b) = \int_a^b ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Study Guide. Study the part of Calculus Volume 2, Section 2.4[2] on *Arc Length of a Curve*; that is to Example 2.20 and Checkpoint 2.20; we omit the second topic of *Area of a Surface of Revolution*.

Do one or several exercises from each of the ranges 171–174 and 176–179, but if you cannot evaluate it explicitly, simply leave the result as a definite integral expresssion.

2.5 Physical Applications (omitted)

References.

- OpenStax Calculus Volume 2, Section 2.5[1].

2.6 Moments and Centers of Mass (omitted)

References.

- OpenStax Calculus Volume 2, Section 2.6[1].

2.7 Integrals, Exponential Functions, and Logarithms (omitted)

References.

- OpenStax Calculus Volume 2, Section 2.7[1].

2.8 Exponential Growth and Decay (omitted)

References.

- OpenStax Calculus Volume 2, Section 2.8[1].

[2] openstax.org/books/calculus-volume-2/pages/2-4-arc-length-of-a-curve-and-surface-area
[1] openstax.org/books/calculus-volume-2/pages/2-5-physical-applications
[1] openstax.org/books/calculus-volume-2/pages/2-6-moments-and-centers-of-mass
[1] openstax.org/books/calculus-volume-2/pages/2-7-integrals-exponential-functions-and-logarithms
[1] openstax.org/books/calculus-volume-2/pages/2-8-exponential-growth-and-decay

2.9 Hyperbolic Functions

References.

- OpenStax Calculus Volume 1, Section 1.5[1] for an introduction to the hyperbolic functions and their inverses.
- OpenStax Calculus Volume 2, Section 2.9[2] for their derivatives and some related integrals.
- *Calculus, Early Transcendentals* by Stewart, Section 3.11.

Introduction. The hyperbolic functions and their inverses are the last of the basic elementary functions. They arise in evaluating integrals (Chapter 3) and in solving differential equations (Chapter 4). The name comes from the fact that they are related to the hyperbola $x^2 - y^2 = 1$ in the same way that trigonometric functions (a.k.a. circular functions) are related to the circle $x^2 + y^2 = 1$. Almost every trig. identity (except ones about periodicity) and all the derivative formulas have hyperbolic counterparts, with at most a sign change.

2.9.1 Definitions

The first two definitions set things up so that everything else mimics trigonometry:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Then as you might guess, we define

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$
$$\operatorname{csch} x = \frac{1}{\sinh x} \qquad \coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}$$

2.9.2 Identities

$$\cosh(-x) = \cosh x \qquad \sinh(-x) = -\sinh x$$
$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

Checkpoint 2.9.1 Verify these identities.

2.9.3 Derivatives of the Hyperbolic Functions

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\sinh x) = \cosh x$$
$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Checkpoint 2.9.2 Verify these derivatives.

[1] openstax.org/books/calculus-volume-1/pages/1-5-exponential-and-logarithmic-functions
[2] openstax.org/books/calculus-volume-2/pages/2-9-calculus-of-the-hyperbolic-functions

Checkpoint 2.9.3 Practice using these derivatives in combination with the Chain Rule. For example, evaluate

$$\frac{d}{dx}(\cosh\sqrt{x}).$$

2.9.4 Domains, Ranges and Invertibility

sinh is defined everywhere and from the derivative above it is increasing, so it has an inverse. It can also be checked that it has the infinite limits

$$\lim_{x\to\pm\infty}\sinh x = \lim_{x\to\pm\infty}\frac{e^x - e^{-x}}{2} = \pm\infty.$$

So its range is all of $\mathbb{R}$, and it has inverse $\sinh^{-1} x$ that is also defined everywhere.

cosh is also defined everywhere, is even, is increasing for positive x, goes to infinity as $x \to \pm\infty$, and $\cosh(0) = 1$. Thus its range is $[1,\infty)$. It has no inverse on its whole domain (no even function can), but it has an inverse if the domain is restricted to $[0,\infty)$, and that inverse $\cosh^{-1} x$ has domain $[1,\infty)$, range $[0,\infty)$.

tanh is also defined everywhere because $\cosh x$ is never zero, and is odd and increasing, with

$$\lim_{x\to\pm\infty}\tanh x = \lim_{x\to\pm\infty}\frac{e^x - e^{-x}}{e^x + e^{-x}} = \pm 1.$$

Thus its range is $(-1,1)$, and it has inverse $\tanh^{-1} x$ with domain $(-1,1)$, range $\mathbb{R}$.

sech is even, defined everywhere, with range $(0,1]$, is decreasing on $[0,\infty)$, and its inverse is defined by restricting the domain to $[0,\infty)$.

csch is odd, defined everywhere except at zero, decreasing, with disjointed range $(-\infty,0)\cup(0,\infty)$, and has an inverse with the same disjointed range and domain $(-\infty,0)\cup(0,\infty)$.

coth is odd, defined everywhere except at zero, with disjointed range $(-\infty,-1)\cup(1,\infty)$, decreasing on each half of its domain, and has an inverse with disjointed domain $(-\infty,-1)\cup(1,\infty)$.

Checkpoint 2.9.4 Sketch the six hyperbolic functions, and their inverses.

2.9.5 Derivatives of the Inverse Hyperbolic Functions

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}(\text{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}} \qquad \frac{d}{dx}(\text{csch}^{-1} x) = -\frac{1}{|x|\sqrt{x^2+1}}$$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$$

Checkpoint 2.9.5 Verify the above derivatives.

Checkpoint 2.9.6 Find $\frac{d}{dx}[\tanh^{-1}(\sin x)]$.

2.9.6 Some Useful Integrals

From the above derivative results, we get some useful indefinite integrals:

$$\int \cosh x\,dx = \sinh x + C \qquad \int \sinh x\,dx = \cosh x + C$$

$$\int \text{sech}^2 x\,dx = \tanh x + C \qquad \int \text{csch}^2 x\,dx = -\coth x + C$$

$$\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1} x + C \qquad \int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + C$$

The last two can be combined with $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$ to handle many integrals involving the square roots of quadratics, as we will see in Section 3.3.

There are several more in Section 2.9[3] of OpenStax Calculus Volume 2, and in the Tables of Integrals[4] in that book, but the above are by far the ones that arise most often.

Study Guide. Study the latter part of Calculus Volume 1, Section 1.5[5], from *Hyperbolic Functions* onward, and Calculus Volume 2, Section 2.9[6]. In particular

- note the derivative and integral formulas, and how they differ from those for their trigonometic cousins,
- study Examples 47 to 50,
- work Checkpoints 47 to 50,
- and do one or several exercises from the following ranges: 385–390 and 405–409. (These are all derivatives; we will work with the integrals soon, in Chapter 3.)

Chapter Review. When reviewing this chapter, also look at the end of chapter review material in OpenStax Calculus Volume 2, including Key Terms[7], Key Equations[8] and Key Concepts[9].

[3] openstax.org/books/calculus-volume-2/pages/2-9-calculus-of-the-hyperbolic-functions
[4] openstax.org/books/calculus-volume-2/pages/a-table-of-integrals
[5] openstax.org/books/calculus-volume-1/pages/1-5-exponential-and-logarithmic-functions
[6] openstax.org/books/calculus-volume-2/pages/2-9-calculus-of-the-hyperbolic-functions
[7] openstax.org/books/calculus-volume-2/pages/2-key-terms
[8] openstax.org/books/calculus-volume-2/pages/2-key-equations
[9] openstax.org/books/calculus-volume-2/pages/2-key-concepts

Chapter 3

Techniques of Integration

References.

- OpenStax Calculus Volume 2, Chapter 3.[1]
- *Calculus, Early Transcendentals* by Stewart, Chapter 7.

3.1 Integration by Parts

References.

- OpenStax Calculus Volume 2, Section 3.1[1].
- *Calculus, Early Transcendentals* by Stewart, Section 7.1.

Introduction. The method of *Integration by Parts* adapts the Product Rule for derivatives in a way similar to how the Chain Rule for derivatives leads to the method of *Substitution* in Section 1.5. The key fact is the formula:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx \tag{3.1.1}$$

or with the short-hands $u = f(x)$ and $v = g(x)$,

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx \tag{3.1.2}$$

Along with Substitution, this is one of the main tools that help us to integrate many products of functions.

It is often used in the more concise "differential" form

$$\int u\,dv = uv - \int v\,du \tag{3.1.3}$$

derived below.

We will see that this is particularly useful for integrating functions that are products of polynomials, sines, cosines, exponentials, logarithms and inverse trigonometric functions.

A First Exploration. There are many ways to express a given integral $\int f(x)dx$ in the product form seen above; let us explore some possibilities, loking for hints about how to choose the factors u and dv/dx so that the above formula is not only *true* but also *useful*, in that the new integral at right is easier to evaluate.

[1] openstax.org/books/calculus-volume-2/pages/3-introduction

[1] openstax.org/books/calculus-volume-2/pages/3-1-integration-by-parts

Checkpoint 3.1.1 Evaluating the integral $\int x \cos x \, dx$.

1. First, use $u = x$ and thus $dv/dx = \cos x$.
2. Then try with some other combinations, like
   (a) $u = \cos x$, $dv/dx = x$
   (b) $u = x \cos x$, $dv/dx = 1$
   (c) $u = 1$, $dv/dx = x \cos x$

   and think about why the original choice works best.
3. It is often worthwhile to check an indefinite integral evaluation by differentiating the result, so do this.

Deriving From the Product Rule for Derivatives. The formula (3.1.1) comes from the Product Rule for differentiation. Starting from $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$ and integrating gives

$$\int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx = f(x)g(x) + C.$$

Subtracting off the first integral on each side

$$\int f(x)g'(x)\,dx = f(x)g(x) + C - \int f'(x)g(x)\,dx$$

But the indefinite integral at right already includes an arbitrary constant, so the "$+C$" is not needed; discarding that gives (3.1.1) above.

Deriving The Other Forms. The formulas in Equations (3.1.2) and (3.1.3) come from (3.1.1) in two steps. First, setting

$$f(x) = u,\ g(x) = v,\ \text{so that } f'(x) = du/dx \text{ and } g'(x) = dv/dx$$

and inserting these into (3.1.1) gives (3.1.2) ; then noting the differentials

$$du = f'(x)\,dx = (du/dx)dx \text{ and } dv = g'(x)\,dx = (dv/dx)dx$$

and inserting these pieces into either of those equations gives (3.1.3).

Using the Differential Form $\int u\,dv = uv - \int v\,du$. A useful strategy for doing Integration by Parts is this:

1. Choose a function u $[= f(x)]$ that is a factor of the integrand. (No need to introduce the name $f(x)$ though; just use u.)
2. Give the name dv to all the rest of the integral, including the differential dx, so that the original integral is now in the form $\int u\,dv$.
3. Integrate the dv part, to get $v = \int dv$ $[= g(x)]$. (Again, no need to introduce the name $g(x)$.)

   Note that no constant of integration is needed: *any* choice of v with the correct derivative will do.
4. Differentiate $u = f(x)$ to get a differential expression $du = f'(x)dx$.
5. Insert u, v and dv into Equation (3.1.3) to get an expression for

$$uv - \int v\,du, = f(x)g(x) - \int g(x)f'(x)\,dx$$

6. Note that you still have to evaluate the new integral at right!

Example 3.1.2 Evaluating $\int x \cos x \, dx$, again, this time using the differential approach.

- In the first exercise, we could start with choice $u = x$, so that $dv = \cos x \, dx$.
- Then the "partial integration" of $dv = \cos x \, dx$ gives $v = \sin x$, and differentiation of $u = x$ gives $du = dx$.
- Thus the new integral to be evaluated is $-\int v \, du = -\int \sin x \, dx, = \cos x + C$, so the full answer is $uv - \int v \, du = x \sin x + \cos x + C$.

□

Choosing u (and thus dv). How does one choose the part u to differentiate, which then forces the choice of the part $dv = g'(x)dx$ to be integrated? Some guidelines are:

1. Try to integrate as much as possible in the first integration, which gives v, so put as much as possible into dv, and as little as possible into the term u that gets differentiated.
2. Try to choose u so that its derivative is "simpler" than u, or at least not more complicated. (For example, positive integer powers of x are good candidates for u.)

Sometimes, there is no factor that you know how to integrate, and yet integration by parts can still help, by differentiating everything at the first step. That is, *use $v = x$, $dv = dx$, so u is the whole integrand.*

This is useful for integrating the inverses of common functions like inverse trigonometric, inverse hyperbolic, and logarithmic.

Checkpoint 3.1.3 Evaluate $\int \sinh^{-1} x \, dx$**.**

Repeated Application. Sometimes the next integration needed also requires integration by parts; this is useful for example when one factor is a positive integer power of the variable.

Checkpoint 3.1.4 Evaluate $\int t^2 e^t \, dt$**.** At each step, discuss the possible choices for u and for dv.

Note that when repeating, never "backtrack" by integrating the result of differentiation in the previous step and vice verca. Instead, a second stage should use a new u coming from the derivative of the previous choice of u.

Repeated application with a twist. Finally, sometimes repeated integration by parts almost takes you full circle even if you avoid backtracking, but the solution comes from some equation solving; this happens mainly in the special but important case of exponential-trigonometric products:

Checkpoint 3.1.5 Evaluate $\int e^x \cos 2x \, dx$**.**

Section Study Guide. Study Calculus Volume 2, Section 3.1[2]; in particular

- Theorems 1 and 2,
- all Examples and Checkpoints,
- one or two exercises from the ranges 1–5 and several from 6–37.

[2] openstax.org/books/calculus-volume-2/pages/3-1-integration-by-parts

3.2 Trigonometric Integrals

References.

- OpenStax Calculus Volume 2, Section 3.2[1].
- *Calculus, Early Transcendentals* by Stewart, Section 7.2.

Introduction. One recurring challenge with integration is handling products. Integration by Parts helps with many products of common elementary functions; here we learn how to deal with products and powers of trigonometric functions, with the most fundamental cases being products of sines and cosines.

Some key tools for "eliminating products" are

Integration by Substitution

$$\int f(u(x))\frac{du}{dx}\,dx = \int f(u)\,du$$

The Trig. Identities

$$\cos^2(ax) = \frac{1}{2}[1+\cos(2ax)] \tag{3.2.1}$$

$$\sin^2(ax) = \frac{1}{2}[1-\cos(2ax)] \tag{3.2.2}$$

$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x) \tag{3.2.3}$$

Checkpoint 3.2.1 Evaluate $\int \sin^3 x\,dx$ **and** $\int \sin^2 x \cos^7 x\,dx$.

Note that in each case above, the substitution is the complementary function to the one appearing as an odd power.

If instead both sine and cosine appear as even powers, we can convert to a case with one of them an odd power by (possibly repeated) use of the Half-Angle Formulas; using these identities lowers the powers of the trig. functions in the formula, so if used often enough one power must become odd, and then the previous substitution strategy works.

Checkpoint 3.2.2 Evaluate $\int \cos^2(3x)\,dx$ **and** $\int \sin^2 x \cos^2 x\,dx$.

Strategy for Evaluating $\int \sin^m(ax)\cos^n(ax)\,dx$ **(for** $m, n \geq 0$ **mostly).**

a) If the power of cosine is odd and positive, use

$$\cos^2(ax) = 1 - \sin^2(ax)$$

to reduce to a single factor of $\cos(ax)$, and then use the substitution

$$u = \sin(ax),\; du = a\cos(ax)\,dx.$$

b) If the power of sine is odd and positive, use

$$\sin^2(ax) = 1 - \cos^2(ax)$$

to reduce to a single factor of $\sin(ax)$, and then use the substitution

$$u = \cos(ax),\; du = -a\sin(ax)\,dx.$$

[1] openstax.org/books/calculus-volume-2/pages/3-2-trigonometric-integrals

c) If both powers are even and non-negative, use the half-angle identities (3.2.1) and (3.2.2) to half both powers (and double a).

   Repeat as necessary until one power is odd; then one can use method (a) or (b) above.

Sometimes it is also convenient to use the formula $\sin(ax)\cos(ax) = \frac{1}{2}\sin(2ax)$ mentioned above.

Products with Other Trig. Functions: $\tan x$**,** $\sec x$**, etc..** To integrate products involving the other four trig. functions, it sometimes works to express them in terms of sines and cosines; this works so long as you end up with an odd positive power of one of the two.

Checkpoint 3.2.3 Evaluate $\int \tan^3 x\,dx$**.**

Products of Powers of $\tan x$ **and** $\sec x$**.** If the above strategy fails because both powers are even and at least one is negative, it sometimes helps to rewrite as

$$\int \tan^j(ax)\sec^k(ax)\,dx.$$

Then if the power k of secant is even and positive, $k = 2p+2$, use the substitution

$$u = \tan(ax),\; du = a\sec^2(ax)dx$$

by reserving a factor $\sec^2(ax)$ for the differential and rewriting the remaining factor
$\sec^{k-2}(ax) = [\sec^2(ax)]^p$ as $[1+\tan^2(ax)]^p$.

Checkpoint 3.2.4 Evaluate $\int \sec^4 x\,dx$**.**

Note that converting to $\int \frac{1}{\cos^4 x}dx$ is not so useful because the half-angle formula method then gives $\int \frac{4}{(1+\cos(2x))^2}$.

Eliminating Products of Sines and Cosines. Products of sines and cosines (possibly with different frequencies) can be eliminated in favor of sums, using the **Product-to-Sum Identities for Trig. Functions**:

$$\cos A\cos B = \frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right] \tag{3.2.4}$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A-B)+\sin(A+B)\right] \tag{3.2.5}$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right] \tag{3.2.6}$$

Checkpoint 3.2.5 Evaluate $\int \cos(3x)\cos(5x)\,dx$**.**

Other Cases: Some Useful Integrals. Some cases are not covered by any of the above methods, and so require further experimentation with trig. identities, integration by parts and such.

Then sometimes the methods to be seen in Section 3.4 can help, when negative powers are present so that substitutions give rational functions instead of polynomials.

The most commonly encountered “outlier” is

$$\int \sec x\,dx = \ln|\sec x+\tan x| + C \tag{3.2.7}$$

We will soon be able to evaluate this, using ideas from Section 3.3 and Section 3.4; meanwhile:

Checkpoint 3.2.6 The above integral can be easily *verified*, so do that.

Section Study Guide. Study Calculus Volume 2, Section 3.2[2]; ou main focus is the first case of sine-cosisse products to in particular look at

- The *Problem Solving Stategy* for sine-cosine products
- Examples 8 to 13
- Checkpoints 5 to 10
- and work one or several exercises from each of the groups: 69–72, (73, 74 & 76), 79–86, 97–101, and 103–106.

3.3 Trigonometric Substitution

References.

- OpenStax Calculus Volume 2, Section 3.3[1].
- *Calculus, Early Transcendentals* by Stewart, Section 7.3.

Substitution Revisited. So far, substitution has been done by rearranging the integrand in $\int f(x)\,dx$ as a product $f(x) = a(g(x))g'(x)$, so that the substitution $u = g(x)$, $du = g'(x)dx$ gives

$$\int f(x)\,dx = \int a(g(x))g'(x)\,dx = \int a(u)\frac{du}{dx}\,dx = \int a(u)\,du.$$

The main challenge is finding a choice of g that allows *everything* to be put in terms of the new variable u, with no remaining appearance of the original variable x.

Inverse Substitution. Often, a better approach is "inverse substitution": instead of choosing $u = g(x)$, we specify $x = h(t)$; *the original variable as a function of a new one.* (The function h is the inverse of the function g used in normal substitution.)

Then $dx = h'(t)dt$, and inserting this and $x = h(t)$ *automatically* converts everything to the new variable t:

$$\int f(x)\,dx = \int f(h(t))h'(t)\,dt \tag{3.3.1}$$

An important question is (as always with substitution, and with integration by parts) *Is this new integral easier to evaluate that the original one?*

Inverse Substitution for Definite Integrals. For definite integrals, there is an extra step of solving for the new limits of integration:

$$\int_{x=a}^{b} f(x)\,dx = \int_{t=c}^{d} f(h(t))h'(t)\,dt \tag{3.3.2}$$

where $a = h(c)$ and $b = h(d)$, so some equation solving is required.

[2] openstax.org/books/calculus-volume-2/pages/3-2-trigonometric-integrals
[1] openstax.org/books/calculus-volume-2/pages/3-3-trigonometric-substitution

Example: The Area of a Circle. An example of how this can be useful is computing the area of the circle radius R by evaluating

$$A = \int_{-R}^{R} 2\sqrt{1-x^2}\,dx$$

This will be done in two parts: first the indefinite integral, and then showing two ways to insert the limits of integration and evaluate the definite integral.

Example 3.3.1 Evaluate $\int \sqrt{R^2-x^2}\,dx$. A substitution that works is $u = \arcsin(x/R)$, but that is far from obvious!

The inverse substitution form $x = R\sin\theta$, $-\pi/2 \leq \theta \leq \pi/2$ makes things easier:

$$dx = R\cos\theta\,d\theta$$

and so

$$\int \sqrt{R^2-x^2}\,dx = \int \sqrt{R^2-R^2\sin^2\theta} R\cos\theta\ d\theta = R^2\int \sqrt{1-\sin^2\theta}\cos\theta\ d\theta = R^2\int \cos^2\theta\,d\theta$$

(Here we used the domain of θ to ensure that $\sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$; no minus sign.)

This integral is one that can be done with the methods of the previous section:

$$R^2\int \cos^2\theta\,d\theta = R^2\int \frac{1}{2}+\frac{1}{2}\cos 2\theta\,d\theta = R^2\left(\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right)+C.$$

Next we need to express everything in terms of the original variable x, which requires inverting the substitution. In this case we can solve $x = R\sin\theta$ to get $\theta = \arcsin(x/R)$, so

$$\int \sqrt{R-x^2}\,dx = R^2\left(\frac{1}{2}\arcsin(x/R)+\frac{1}{4}\sin(2\arcsin x)\right)+C.$$

However there is a far nicer form, using the identity $\sin(2\theta) = 2\sin\theta\cos\theta$ and the facts that

$\sin\theta = x/R$ and $\cos\theta = \sqrt{R^2-x^2}/R$ as seen above.

That gives $\sin(2\theta) = 2x\sqrt{R^2-x^2}/R^2$ so

$$\int \sqrt{R-x^2}\,dx = \frac{R^2}{2}\arcsin(x/R)+\frac{1}{2}x\sqrt{R^2-x^2}+C.$$

□

Example 3.3.2 Use the above indefinite integral to compute the area of a circle. One method would be to use the above indefinite integral with the Fundamental Theorem of Calculus to insert the limits of integration. However an alernative approach is to work directly with the definite integral, avoiding the need to convert back to the original variable x.

$$A = 2\int_{x=-R}^{R} \sqrt{R^2-x^2}\,dx = 2R^2\left[\frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)\right]_{\theta=-\pi/2}^{\theta=\pi/2} = \pi R^2.$$

□

Three Square Root Forms and Recommended Substitutions. We have seen one of three basic square root forms that can occur in integrals, and for each there is an inverse substitution that often helps.

The three cases are where the integrand is a product of an integer power of x with an integer power of one of the following square root forms:

Table 3.3.3

With	$\sqrt{a^2 - x^2}$	$\sqrt{a^2 + x^2}$	$\sqrt{x^2 - a^2}$
use	$x = a\sin\theta$	$x = a\tan\theta$	$x = a\sec\theta$
giving	$\sqrt{a^2 - x^2} = a\cos\theta$	$\sqrt{a^2 + x^2} = a\sec\theta$	$\sqrt{x^2 - a^2} = a\tan\theta$
and	$dx = a\cos\theta\, d\theta$	$dx = a\sec^2\theta\, d\theta$	$dx = a\tan\theta\sec\theta\, d\theta$

The new integrals will be expressible as products of powers of $\sin\theta$ and $\cos\theta$, or alternatively of $\tan\theta$ and $\sec\theta$; the methods of the previous section are then useful.

Converting the Integral Back From θ to x. After using these substitutions and integrating, the answer is in terms of θ and various trig. functions of θ. This needs to be converted back into terms of the original variable.

Using $x = a\sin\theta$ for integrals involving $\sqrt{a^2 - x^2}$:

$$\cos\theta = \frac{\sqrt{a^2 - x^2}}{a},\ \sin\theta = \frac{x}{a},\ \tan\theta = \frac{x}{\sqrt{a^2 - x^2}},\ \theta = \sin^{-1}\frac{x}{a}.$$

Using $x = a\tan\theta$ for integrals involving $\sqrt{a^2 + x^2}$:

$$\cos\theta = \frac{a}{\sqrt{a^2 + x^2}},\ \sin\theta = \frac{x}{\sqrt{a^2 + x^2}},\ \tan\theta = \frac{x}{a},\ \theta = \tan^{-1}\frac{x}{a}.$$

Using $x = a\sec\theta$ for integrals involving $\sqrt{x^2 - a^2}$:

$$\cos\theta = \frac{a}{x},\ \sin\theta = \frac{\sqrt{x^2 - a^2}}{x},\ \tan\theta = \frac{\sqrt{x^2 - a^2}}{a},\ \theta = \sec^{-1}\frac{x}{a}.$$

Checkpoint 3.3.4 It is often easiest to work these formulas out from drawing an appropriate right triangle with angle θ, so draw the following three right triangles

1. For $x = a\sin\theta$: opposite side x, hypotenuse a, giving $\sin\theta = x/a$.
2. For $x = a\tan\theta$: opposite side x, adjacent side a, giving $\tan\theta = x/a$.
3. For $x = a\sec\theta$: hypotenuse x, adjacent side a, giving $\sec\theta = x/a$.

Use the Pythagorean Theorem to get the third side,

and then read off the sine, cosine, etc.There can be more than one way to approach a problem, so *always try the easiest methods first!*

Checkpoint 3.3.5 Evaluate $\int x\sqrt{1 - x^2}\, dx$.

Checkpoint 3.3.6 Compute the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Never neglect simplifications: after a trig.{} substitution, explore trig. identities to simplify the new integrand.

Checkpoint 3.3.7 Evaluate $\int \frac{\sqrt{9 - x^2}}{x^2}\, dx$.

More on simplification: after a trig. substitution, expressing in terms of sines and cosines often helps.

Checkpoint 3.3.8 Evaluate $\int \frac{1}{x^2\sqrt{x^2 + 4}}\, dx$.

Checkpoint 3.3.9 Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{x^2-2}}\,dx$**.** What is a?

Integer powers of the square root terms can also be handled; particularly *odd powers of a root*, since an even power eliminates the root, giving just a polynomial or rational function. (The latter case will be handled in Section 3.4.)

Checkpoint 3.3.10 Evaluate $\displaystyle\int \frac{x^2}{(1-x^2)^{3/2}}\,dx$**.**

Roots of quadratics: eliminating the bx term. Roots of other quadratics ax^2+bx+c need one more simplification first: complete the square to get a form like

$$ax^2+bx+c = \pm a(x-b/(2a)) \pm d^2,$$

and then use the substitution

$$u = x - b/(2a) \quad \text{(or !)}$$

Checkpoint 3.3.11 Evaluate $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$**.**

Section Study Guide. Study Calculus Volume 2, Section 3.3[2]; in particular

- learn the right-triangle diagrams for each of the three main cases
- review the three *Problem Solving Strategies*, one for each of those three cases
- study all Examples and Checkpoints
- and do one or several exercises from each of the groups 131–133, 134–142, 146 & 147, and 160–164.

3.4 Partial Fractions

References.

- OpenStax Calculus Volume 2, Section 3.4[1].
- *Calculus, Early Transcendentals* by Stewart, Section 7.4.

Introduction. The integration of rational functions can be based on a few examples that we already know how to handle:

$$\int \frac{1}{x-a}dx = \ln|x-a| + C \tag{3.4.1}$$

$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\frac{x}{a} + C \tag{3.4.2}$$

$$\int \frac{x}{x^2+a^2}dx = \frac{1}{2}\ln(x^2+a^2) + C \tag{3.4.3}$$

$$\text{and for } n>1: \tag{3.4.4}$$

$$\int \frac{1}{(x-a)^n}dx = \frac{-1}{n-1}\frac{1}{(x-a)^{n-1}} + C \tag{3.4.5}$$

$$\int \frac{x}{(x^2+a^2)^n}dx = \frac{-1}{2(n-1)}\frac{1}{(x^2+a^2)^{n-1}} + C. \tag{3.4.6}$$

$$\text{using substitution } u = x^2+a^2. \tag{3.4.7}$$

[2] openstax.org/books/calculus-volume-2/pages/3-3-trigonometric-substitution
[1] openstax.org/books/calculus-volume-2/pages/3-4-partial-fractions

$$\int \frac{1}{(x^2+a^2)^n} dx \ldots \qquad \text{Use substitution } x = a\tan\theta. \tag{3.4.8}$$

Simplifying to a Sum of the Above Forms: Proper Rational Functions. The rest of the task is simplifying any rational function to a sum of terms of the above forms.

The first stage of simplification is writing a rational function as the sum of a *proper rational function* and a polynomial. A **Proper Rational Function** is a ratio of polynomials, $\frac{P(x)}{Q(x)}$, with the top, $P(x)$, of lower degree than the bottom, $Q(x)$. This simplification is done by **synthetic division of polynomials**.

Checkpoint 3.4.1 $\int \frac{2x^3+2x+3}{x^2+1} dx$**.**

- Simplify $\frac{2x^3+2x+3}{x^2+1}$ to a sum of a polynomial and a proper rational function.
- Use this to evaluate $\int \frac{2x^3+2x+3}{x^2+1} dx$.

The rest of this section deals with integrating the resulting proper rational function.

Case I: Partial Fractions Expansion With All Distinct Linear Factors Below. The next step is to factorize the denominator, if possible.

We will deal first with the case that the denominator can be factorized, with no factor repeated. That is, the rational function is $f(x) = \frac{P(x)}{Q(x)}$ with denominator $Q(x)$ of degree n, and $Q(x) = C(x-r_1)(x-r_2)\cdots(x-r_n)$, with all the roots r_i different, C a constant.

Then the wonderful fact is that for certain numbers $A_1, \ldots A_n$, the function has a **Partial Fractions Expansion**: it can be put in the form

$$f(x) = \frac{P(x)}{Q(x)} = \frac{A_1}{x-r_1} + \frac{A_2}{x-r_2} + \cdots + \frac{A_n}{x-r_n} \tag{3.4.9}$$

and then each piece can be integrated using Equation (3.4.1)

But we need to find those coefficients A_i

The first step is to first *clear the denominator* in the partial fractions expansion of Equation (3.4.9) by multiplying both sides by all of the factors $(x-r_1)\ldots(x-r_n)$.

Then several methods can be used.

My favorite is **The Method of Strategic Substitution**: successively evaluate for each of the roots: $x = r_1$, $x = r_2$, $\ldots$, $x = r_n$. This gives a string of simple equations, each giving one of the coefficients $A_1, \ldots, A_n$.

An alternatve is **The Method of Equating Coefficients**, which however requires solving simultaneous equations. This approach is to expand both sides of the equation into powers of x and require that the coefficients of each power are the same on both sides.

It gives n simultaneous equations to be solved for the n unknown coefficients $A_1 \ldots A_n$.

Checkpoint 3.4.2 $\int \frac{7x+1}{x^2-4} dx$**.**

- Derive the partial fractions expansion of $\frac{7x+1}{x^2-4}$.
- Use this to evaluate $\int \frac{7x+1}{x^2-4} dx$.

The above strategy fails when a factor in denominator $Q(x)$ is repeated: for example, we cannot express $\dfrac{1}{(x-1)^2}$ as $\dfrac{A_1}{x-1}+\dfrac{A_2}{x-1}$.

But nor do we need to, because we can integrate $\dfrac{1}{(x-1)^2}$, and indeed any term $\dfrac{1}{(x-r)^n}$, using Equation (3.4.4). So we can work with negative powers of $(x-r)$:

Case II: Partial Fractions Expansion With All Linear Factors, Some Repeated. For $Q(x) = C(x-r_1)^{p_1}(x-r_2)^{p_2}\cdots(x-r_m)^{p_m}$ we modify the partial fractions form to have a term for each power of $\dfrac{1}{x-r_i}$ up to the one appearing in $Q(x)$, $\dfrac{1}{(x-r_i)^{p_i}}$:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-r_1)^{p_1}} + \cdots + \frac{A_{p_1}}{(x-r_1)} + \cdots \tag{3.4.10}$$

For example, the partial fractions form for $\dfrac{2x^2+4x+1}{(x-2)^2(x+3)}$ is

$$\frac{2x^2+4x+1}{(x-2)^2(x+3)} = \frac{A}{(x-2)^2} + \frac{B}{x-2} + \frac{C}{x+3}.$$

Determining the Coefficients in the Partial Fractions Expansion with Repeated Linear Factors. As before, the first step is always to *clear the denominator* in the partial fractions expansion by multiplying both sides by all of the factors $(x-r_1)^{p_1}\ldots(x-r_m)^{p_m}$. However, the method of successively evaluating for each of the roots only gets some of the coefficients: one for each different root, so m of them.

Determining the Rest of the Coefficients with Repeated Linear Factors. To determine the other coefficients, one can substitute these values into the expansion, and then use one of several methods:

1. Insert $n-m$ other simple values for x like $x=0$, $x=1$, $x=-1$ etc. to get $n-m$ simultaneous equations for the remaining $n-m$ coefficients.
2. Use the Method of Equating Coefficients described above: expand into various powers of x, and require that the coefficient of each power is the same on each side.

   This again gives $n-m$ simultaneous equations for the remaining coefficients.

In fact, these strategies work for all cases, including those described next.

Checkpoint 3.4.3 $\displaystyle\int \frac{2x^2+4x+1}{x^3-x^2-x+1}dx$.

- Derive the partial fractions expansion of $\dfrac{2x^2+4x+1}{x^3-x^2-x+1}$.
- Use this to evaluate $\displaystyle\int \frac{2x^2+4x+1}{x^3-x^2-x+1}dx$.

Case III: Partial Fractions Expansion With Irreducible Quadratic Factors, None Repeated.

- Sometimes the denominator $Q(x)$ cannot be written completely in terms of linear factors given by roots (at least not using only real numbers); for example, if $Q(x) = x^2+1$.
- But again the integrals in Equations (3.4.2) and (3.4.3) should help.
- So we seek a partial fractions form using terms like this.

Irreducible Quadratic Factors: $x^2 + bx + c$**,** $b^2 < 4c$**.**

- This situation arises when factorizing the the denominator leads to a quadratic factor $x^2 + bx + c$ that has no real roots because it has negative **discriminant**, $b^2 - 4c < 0$.
- The quadratic formulas then involves the square root of this negative quantity. (Note that there is no need for a coefficient on x^2; the denominator and each factor can always be written with coefficient 1 on the highest power of x.)
- Another way to characterize such quadratics is that when you complete the square, the form is $(x+k)^2 + a^2$: in fact $k = b/2$, $a = \sqrt{c - b^2/4}$.
- The new rule for finding the partial fractions form is that for each irreducible quadratic factor x^2+bx+c, the form has a factor
$$\frac{Ax + B}{x^2 + bx + c} \tag{3.4.11}$$
in addition to the terms as above for each linear factor.

Determining the Coefficients: Same as Above! As mentioned above, the coefficients can then be determined by the same methods as in Case II of all linear factors, some repeated.

Checkpoint 3.4.4 Partial fractions form. Find the partial fractions *form* for

$$\frac{2x^2 + 3x + 1}{(x+1)(x-2)^2(x^2+3)(x^2+x+1)}.$$

Case IV: Anything Goes (Repeated Irreducible Quadratic Factors, etc.). Finally, we can deal with the case where anything can happen, including repeated irreducible quadratic factors $(x^2 + bx + c)^p$, $b^2 < 4c$.

For each such factor, the partial fractions form uses terms that combine the ideas seen above for repeated (linear) factors and irreducible factors:

$$\frac{A_1 x + B_1}{x^2 + bx + c} + \cdots + \frac{A_p x + B_p}{(x^2 + bx + c)^p} \tag{3.4.12}$$

Note that, *as always*, the number of coefficients (here $2p$) is the same as the degree of the factor that they go with: here, $(x^2 + bx + c)^p, = x^{2p} + \cdots$.

Integrating The Irreducible Quadratic Terms: Substitution $u = x + k = x + b/2$**.** The new type of terms arising in Equation (3.4.12) can be integrated with the help of the substitution $u = x + k$ where $k = b/2$.

This turns the denominator into the form $u^2 + a^2$, $a = \sqrt{c - b^2/4}$, while the numerators stay linear, so all terms are of the forms in Equations (3.4.2), (3.4.3), (3.4.5) and (3.4.6).

Summary of the Terms Used in the Partial Fractions Form. To summarize, the partial fractions form for a proper rational function $\dfrac{P(x)}{Q(x)}$ is a sum of pieces determined by the factors of the denominator $Q(x)$ as follows:

- For a factor $x - r$ appearing only once, a term of the form
$$\frac{A}{x - r}$$
- For a repeated factor $(x-r)^p$, p terms of the form
$$\frac{A_1}{(x-r)^p} + \cdots + \frac{A_p}{x - r}$$

- For an irreducible quadratic factor $x^2 + bx + c$ $[4c > b^2]$ appearing only once, two coefficients in a term of the form
$$\frac{Ax + B}{x^2 + bx + c}$$
- For an irreducible quadratic factor $x^2 + bx + c$ appearing as $(x^2 + bx + c)^p$, $2p$ coefficients in p terms of the form
$$\frac{A_1 x + B_1}{(x^2 + bx + c)^p} + \cdots + \frac{A_p x + B_p}{x^2 + bx + c}$$

Recap of Methods for Determining the Values of The Coefficients, and Thus the Partial Fractions Expansion.

- Once you have the Partial Fractions Form, multiply both sides through by the denominator $Q(x)$, to get an equation with polynomials on each side.
- I then recommend substituting in turn each root of $Q(x)$ as the value of x, which easily determines the values of at least some of the coefficients.
- If this does not determine all the coefficients, put these values into the above polynomial equation, to reduce the number of unknown coefficients remaining.
- Insert various other simple values for x like 0, 1, -1: one x value for each coefficient still to be determined, or
- expand out the polynomials on each side as a sum of different powers of x, and write the equations that equate each power of x on the two sides.
- Either way, then solve these linear equations for the remaining coefficients.
- These coefficients can then be inserted into the partial fractions form (the rational function form before clearing the denominator) to express the original integrand $\frac{P(x)}{Q(x)}$ as a sum of simpler rational functions, each of which can be integrated.

Section Study Guide. Study Calculus Volume 2, Section 3.4[2]; in particular

- *The Problem Solving Strategy* under heading *The General Method*
- Examples 28–34
- All Checkpoints
- and one or several exercises from each of the following groups: 182–185, 186& 187, 188–191, 196–200, 202–204, and a few from 207, 209, 210, 211 and 212.

3.5 Other Strategies for Integration

References.

- OpenStax Calculus Volume 2, Section 3.5[1].
- *Calculus, Early Transcendentals* by Stewart, Section 7.6.

[2] openstax.org/books/calculus-volume-2/pages/3-4-partial-fractions
[1] openstax.org/books/calculus-volume-2/pages/3-5-other-strategies-for-integration

Strategy for Evaluating Indefinite Integrals. Here is a basic strategy for combining the various integration methods from this Chapter and Chapter 1. After any step that does not give the final answer, start again from the top on each piece.

1. Use known integrals (basic anti-drivatives): build a list of them in your notes. Also use lists of integrals like those in Appendix A of OpenStax Calculus Volume 2[2], which gives 113 such integrals.
2. *Simplify* with algebra, trig., and such, break up into a sum of simpler integrals, and move any *constant* common factors outside an integral.
3. Try *Substitution*, $u = g(x)$: often u is the term inside a composition. *Do not forget to convert the differential too:* $du = g'(x)dx$.
4. Try Integration by Parts, $\int u\,dv = uv - \int v\,du$. When arranging as $\int u\,dv$, try to put as much as possible into dv while still being able to integrate it to get $v = \int dv$.
5. Finally, some important special cases:
   (a) For functions involving the square root of a quadratic, or powers of such a root and powers of x,try the inverse trigonometric substitutions of Section 3.3.
   (b) For products of trigonometric functions, try the special substitutions and half-angle formulas of Section 3.2.
   (c) For rational functions, use partial fractions expansions (and synthetic division if necessary), as in Section 3.4.

Using Reduction Formulas. When using tables of integrals such as in Appendix A of OpenStax Calculus Volume 2[3] one very useful and flexible case is **Reduction Formulas**, which express an integral in terms of a somehow simpler integral, and which are typically used repeatedly or in combination until the original integral is in terms of one that can be evaluated directly.

This reduction process is seen with integration by parts of functions like $x^3 e^x$.

Using $dv = e^x dx$, $u = x^3$, one gets $\displaystyle\int x^3 e^x\,dx = x^3 e^x - 3\int x^2 e^x\,dx$ which gives a very similar integral where integration by parts with $dv = e^x dx$ again gives $\displaystyle\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx$.

Yet another integration by parts with $dv = e^x dx$ gives $\displaystyle\int x e^x\,dx = x e^x - \int e^x\,dx$.

Finally the integral needed is a known one:

$$\int e^x\,dx = e^x + C. \tag{3.5.1}$$

Putting all these together, $\displaystyle\int x^3 e^x\,dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C$.

But the three integrations by parts above are very similar, so doing them all seems redundant and would be far more so if we needed to evaluate something like $\int x^{17} e^x\,dx$. Instead we can roll all these integrations into one more general *Reduction Formula:*

$$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1} e^x\,dx \tag{3.5.2}$$

which can be verified by a single integration by parts with $dv = e^x\,dx$, $u = x^n$.

Using this successively with $n = 3$, then $n = 2$, then $n = 1$ and then using (3.5.1) gets the above result.

[2] openstax.org/books/calculus-volume-2/pages/a-table-of-integrals
[3] openstax.org/books/calculus-volume-2/pages/a-table-of-integrals

In fact, this formula is one of the reduction formulas in that Appendix A[4]: number 43, though there it is in a slightly more general form

$$\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$$

so we also need to specify $a = 1$ when using it. (We could even do the final step using number 3 from there.)

Any indefinite integral that can be evaluated in terms of elementary functions can be reduced to using a combination of formulas in that list through Substitution, Integration by Parts, and various algebraic and trigonometric simplifications like those seen in this Chapter.

Evaluating Definite Integrals. Often the best approach for a definite integral $\int_a^b f(x)\,dx$ is to first evaluate the indefinite integral $F(x) + C$ and then use the Fundamental Theorem of Calculus:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

However with substitutions and inverse substitutions, it sometimes saves effort to convert the definite integral to the new variable, to avoid having to convert back to the original variable:

Evaluating Definite Integrals with Substitutions.

- For substitution $u = g(x)$

$$\int_{x=a}^{x=b} f(g(x))g'(x)\,dx = \int_{u=c}^{u=d} f(u)\,du,\ c = g(a),\ d = g(b).$$

- For inverse substitution $x = h(t)$

$$\int_{x=a}^{x=b} f(x)\,dx = \int_{t=c}^{t=d} f(h(t))h'(t)\,dt,$$

  where now you have to solve equations to get c from a and d from b: $h(c) = a$, $h(d) = b$.

Section Study Guide. Read the first part of Calculus Volume 2, Section 3.5[5], about *Tables of Integrals*.

There is only one worked example there, so work a selection from Exercises in each of the ranges 240–260, 279–284, and 285–288.

3.6 Numerical Integration

References.

- OpenStax Calculus Volume 2, Section 3.6[1].
- *Calculus, Early Transcendentals* by Stewart, Section 7.7.

Definite Integrals cannot always be evaluated using anti-derivatives and the Fundamental Theorem of Calculus.

One reason is that not all elementary functions have anti-derivatives that can be expressed in terms of elementary functions: two important examples are $\int e^{-x^2}dx$ and $\int \frac{\sin x}{x}dx$.

[4] openstax.org/books/calculus-volume-2/pages/a-table-of-integrals
[5] openstax.org/books/calculus-volume-2/pages/3-5-other-strategies-for-integration
[1] openstax.org/books/calculus-volume-2/pages/3-6-numerical-integration

Another situation is when a function $f(x)$ is only known through values at a finite collection of x values, for example when working with data from experiments or computer simulation.

One solution is already familiar: get an approximate numerical value using a Riemann Sum as in Section 1.2, and often the best simple choice or Riemann sum is the **Midpoint Rule** seen there. It is simplest to use N intervals of equal width $h = \Delta x = \dfrac{b-a}{N}$, which gives *the N-point Midpoint Rule Approximation*

$$\int_a^b f(x)\,dx \approx M_N = h\sum_{n=1}^{N} f(\bar{x}_n) = h\sum_{n=1}^{N} f(a+(n-1/2)h), \tag{3.6.1}$$

where

- $x_n = a + nh$ ($x_0 = a, x_1 = a+h$, etc.) and
- $\bar{x}_i = \dfrac{x_{i-1}+x_i}{2} = a + (i-1/2)h$

so $\bar{x}_1 = a + \dfrac{h}{2}, \bar{x}_2 = a + \dfrac{3h}{2}$, etc.

The Mid-point Rule on a TI Calculator. This sum can be evaluated on a TI-84 or similar calculator with

```
sum(seq(f(x),x,a+h/2,b-h/2,h))*h
```

or preferably (because it avoids occasional wrong answers with the above!)

```
sum(seq(f(a+(n-1/2)*h),n,1,N))*h
```

When using these formulas, f is replaced by a formula, and a, b, h and N by actual numbers, but x or n are still letter names.

The Trapezoid Rule. Sometimes the Midpoint Rule cannot be used as one does not have values of f at the needed points; it is common instead with $\int_a^b f(x)\,dx$ to have measurements at equally spaced values starting at a and ending at b: in other words, as the points x_n above that divide the intervals on which the rectangles in the Riemann Sum are based.

Then a natural (and usually accurate) approach is to approximate the function on each interval $[x_{n-1}, x_n]$ by the average of its values at the two endpoints: $\dfrac{f(x_{n-1})+f(x_n)}{2}$.

The area of each rectangle is then $\dfrac{f(x_{n-1})+f(x_n)}{2}h$ which is also the area of a trapezoid on the interval $[x_{n-1}, x_n]$ of height $f(x_{n-1})$ at $x = x_{n-1}$ and height $f(x_n)$ at $x = x_n$; in other words, touching the curve $y = f(x)$ at each x_n.

With N intervals of equal width again, we get *the N-point Trapezoid Rule Approximation*

$$\begin{aligned}\int_a^b f(x)dx \approx T_N &= h\sum_{n=1}^{N}\frac{f(x_{n-1})+f(x_n)}{2}\\ &= \frac{h}{2}\left[f(x_0) + 2\sum_{n=1}^{N-1} f(x_n) + f(x_N)\right]\\ &= h\left[\frac{f(a)}{2} + f(x_1) + \cdots + f(x_{N-1}) + \frac{f(b)}{2}\right]\end{aligned}$$

This can be evaluated on a TI-84 or similar calculator with

```
( f(a) + f(b) + 2*sum(seq(f(x),x,a+h,b-h,h)) )*h/2
```

or by the safer method as above,

```
( f(a) + f(b) + 2*sum(seq(f(a+n*h),n,1,N-1)) )*h/2
```

Errors and Error Bounds. An approximation is useless without at least some idea of how accurate it is, like the number of significant digits. Thus we want to know something about the error in the above methods, defined as the difference between the exact result and the approximation.

$$E_M = \int_a^b f(x)\,dx - M_N \qquad E_T = \int_a^b f(x)\,dx - T_N$$

For a variety of examples where the exact value of the integral is known (not the case where we would usually be using an approximation!) two patterns are seen, at least when N is reasonably large:

- Increasing the number N of intervals 10-fold reduces the error about 100-fold.
- The error for the Midpoint Rule is about half as large as that for the Trapezoid Rule, and of opposite sign.

These patterns are partially explained by the following theoretical results (not proved here, but in a Numerical Methods course like Math 245.) Suppose that $|f''(x)| \leq K$ for $a \leq x \leq b$. Then

$$|E_M| \leq \frac{K(b-a)^3}{24N^2}, \qquad |E_T| \leq \frac{K(b-a)^3}{12N^2}.$$

If we have some idea of a value K, we can use these results to put an upper limit on the error in a Midpoint Rule or Trapezoid Rule approximation. Better yet, we can use them to guarantee a needed degree of accuracy by choosing a sufficiently large number of intervals N.

Checkpoint 3.6.1 Accuracy of M_4 for $\int_2^4 \frac{dx}{x}$.

1. If we approximate $\int_2^4 \frac{dx}{x}$ using the Midpoint Rule with 100 intervals, what is the worst that the error can be?
2. If we wish the value to be accurate to eight decimal places, how many intervals are needed?

Simpson's Rule. The last result is a bit disappointing when you need high accuracy: four decimal places can be got rather cheaply with the Midpoint Rule (about $N = 100$), but the full fifteen decimal places that a typical computer's hardware is capable of takes a lot of work (about $N = 10^7$.)

In situations calling for high accuracy, a far more efficient method is **Simpson's Rule**. For this, the idea is to replace the rectangle and trapezoid approximations of the curve by a collection of quadratics, and then compute the area under each of these, and sum those areas.

Fortunately this leads to a fairly simple formula in terms of the same function values $f(x_i)$ used in the Trapezoid Rule, so we do not need to actually find the quadratics or integrate them; The only catch is that there must be an even number N of intervals, as each quadratic spans two intervals.

With n intervals of equal width, Simpson's Rule is

$$\int_a^b f(x)\,dx \approx S_N = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots 2f(x_{N-2}) + 4f(x_{N-1}) + f(x_N)\right]$$

Perhaps the simplest way to evaluate S_{2N} is to first evaluate M_N and T_N as above and then use the surprising formula

$$S_{2N} = \frac{2M_N + T_N}{3}$$

Error Bound For Simpson's Rule. Suppose that $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$. Then the error $E_S = \int_a^b f(x)\,dx - S_N$ satisfies

$$|E_S| \leq \frac{K(b-a)^5}{180N^4}.$$

The good news here is the trend that a 10-fold increase in the number of intervals reduces error by a factor of about 10,000, instead of 100 as for the previous methods. Thus, *for large enough N*, Simpson's Rule is usually far more accurate than the previous two methods.

Section Study Guide. Study Calculus Volume 2, Section 3.6[2]; in particular

- the formulas in Theorems 3, 4 and 6,
- Examples 39–42, 45 and 46,
- Checkpoints 22–24 and 26,
- and one or several exercises from each of the range 299–306.

3.7 Improper Integrals

References.

- OpenStax Calculus Volume 2, Section 3.7[1].
- *Calculus, Early Transcendentals* by Stewart, Section 7.8.

Some physical problems are naturally described in terms of an integral of all real values of a variable, or all positive values. For example, the variable might represent speed, with (in classical physics) no upper limit, leading to integrals like $E = \displaystyle\int_0^\infty ve^{-v^2}dv$.

How do we make sense of the infinite value for the upper limit? The basic idea is to first approximate by integrating over "most" of the domain, up to some large but finite value M:

$$E_M := \int_0^M ve^{-v^2}dv$$

and then define the exact value to be the limit of the ever-improving approximations given by increasing M:

$$E = \int_0^\infty ve^{-v^2}dv := \lim_{M\to\infty} E_M = \lim_{M\to\infty}\int_0^M ve^{-v^2}dv.$$

Definition 3.7.1 Improper Integral, Type I: Infinitely Wide Domains.

a) The improper integral $\displaystyle\int_a^\infty f(x)\,dx$ is defined as

$$\int_a^\infty f(x)\,dx := \lim_{M\to\infty}\int_a^M f(x)\,dx$$

if this limit exits; and then the integral is called **convergent**.

If the limit does not exit, this integral is called **divergent**. This includes the case of an infinite limit.

b) Likewise we define $\displaystyle\int_{-\infty}^a f(x)\,dx := \lim_{M\to-\infty}\int_M^a f(x)\,dx.$

[2] openstax.org/books/calculus-volume-2/pages/3-6-numerical-integration
[1] openstax.org/books/calculus-volume-2/pages/3-7-improper-integrals

c) If both the above integrals exist, we can define

$$\int_{-\infty}^{\infty} f(x)\,dx := \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

It can be checked that any choice of the value a gives the same answer.

◊

Checkpoint 3.7.2 Determine whether $\displaystyle\int_1^\infty \frac{dx}{x}$ or $\displaystyle\int_1^\infty \frac{dx}{x^2}$ is convergent, and find the values if they exist.

Checkpoint 3.7.3 Verify that $\displaystyle\int_{-\infty}^{0} xe^x\,dx$ is convergent and evaluate it.

Checkpoint 3.7.4 Evaluate $\displaystyle\int_0^\infty ve^{-v^2}\,dv$, or show that it is divergent.

Checkpoint 3.7.5 Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dt}{1+t^2}$, or show that it is divergent.

Checkpoint 3.7.6 Determine for which values of p the improper integral $\displaystyle\int_1^\infty \frac{dx}{x^p}$ is convergent and for which it is divergent, and evaluate for all cases where it converges.

Improper Integrals of Type II: Regions of Infinite Height. There is a second case where the definition of the "true" integral fails, but we can make sense of the area under the curve: regions that go to infinite height at some point, due to a vertical asymptote, like $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$.

A similar strategy to above works. First approximate the domain of integration by leaving out a bit near the bad point:

$$I_t = \int_t^1 \frac{dx}{\sqrt{x}},\ t > 0.$$

Then take the limit as the modified end-point goes back to where it should be:

$$I = \int_0^1 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+} I_t = \lim_{t\to 0^+} \int_t^1 \frac{dx}{\sqrt{x}}.$$

Note that this must be a one-sided limit, as only values $t > 0$ avoid the vertical asymptote and give a proper integral.

Definition 3.7.7 Improper Integral, Type II: Vertical Asymptotes.

a) If f is continuous on $[a, b)$ but discontinuous (including undefined) at b,

$$\int_a^b f(x)\,dx := \lim_{t\to b^-} \int_a^t f(x)\,dx.$$

b) Likewise, if f is continuous on $(a, b]$ but discontinuous at a,

$$\int_a^b f(x)\,dx := \lim_{t\to a^+} \int_t^b f(x)\,dx$$

Note that in each case, the modified endpoint t must always be "inside" the original interval, so as to exclude the vertical asymptote.

c) If the integrand is continuous everywhere on interval $[a, b]$ except at c, $a < c < b$, then we chop it into a sum of two improper integrals as above:

$$\int_a^b f(x)\,dx := \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

◊

Checkpoint 3.7.8

a) Evaluate $\displaystyle\int_{-1}^{2} \frac{dx}{\sqrt[3]{x}}$ or show that it is divergent.

b) Do the same for $\displaystyle\int_{-1}^{2} \frac{dx}{x^2}$.

l'Hôpital's Rule (see Section 4.8 of OpenStax Calculus Volume 1[2]) is often useful for evaluating the limits arising in improper integrals:

Checkpoint 3.7.9 Evaluate $\displaystyle\int_0^1 \ln x\,dx$ or show that it is divergent.

A Comparison Theorem: When Showing Convergence is Enough. Often with complicated integrands, it is not possible to find the indefinite integrals and limits needed to evaluate as above, but it is enough to show that an improper integral is convergent (or not). (The actual numerical value might then be approximated using methods like those in the previous section.)

One can show convergence by showing that the integrand is "smaller" than another function for which you know convergence, or conversely:

Theorem 3.7.10 Comparison of Improper Integrals of Non-Negative Functions. *Suppose that for $a \geq 0$, f and g are both non-negative and continuous and f is bigger than g: $f(x) \geq g(x) \geq 0$.*

a) If the "larger" integral $\displaystyle\int_a^\infty f(x)\,dx$ converges, then the "smaller" one $\displaystyle\int_a^\infty g(x)\,dx$ also converges, and so

b) if the "smaller" integral $\displaystyle\int_a^\infty g(x)\,dx$ diverges, the "larger" one $\displaystyle\int_a^\infty f(x)\,dx$ also diverges.

In either case, when the improper integral of a non-negative function diverges, the limit in its definition is infinite, and allowing this infinite value for such an integral, we can always say that

$$0 \leq \int_a^\infty g(x)dx \leq \int_a^\infty f(x)dx.$$

Checkpoint 3.7.11 Show that $\displaystyle\int_0^\infty \frac{1+\sin(x^3)}{e^x}dx$ converges.

Do not try to evaluate it, but think about how you might approximate its value using numerical integration methods.

Section Study Guide. Study Calculus Volume 2, Section 3.7[3]; in particular

- The two Definitions and Theorem 3.7,
- Examples 47, 48, 50, 51 and 52–56,
- all Checkpoints,

[2] openstax.org/books/calculus-volume-1/pages/4-8-lhopitals-rule
[3] openstax.org/books/calculus-volume-2/pages/3-7-improper-integrals

- and one or several exercises from each of the following groups: (348, 350, 351, 352), 347 & 349, 353 & 354, 355–371.

Chapter Review. When reviewing this chapter, also look at the end of chapter review material in OpenStax Calculus Volume 2, including Key Terms[4], Key Equations[5] and Key Concepts[6].

[4]openstax.org/books/calculus-volume-2/pages/3-key-terms
[5]openstax.org/books/calculus-volume-2/pages/3-key-equations
[6]openstax.org/books/calculus-volume-2/pages/3-key-concepts

Chapter 4

Introduction to Differential Equations

The topics in this chapter are optional, and we will cover only a part of it, depending on time available. The notes are only complete to Section 4.3, on Separable Equations.

References.

- OpenStax Calculus Volume 2, Chapter 4.[1]
- *Calculus, Early Transcendentals* by Stewart, Chapter 9.

4.1 Basics of Differential Equations

References.

- OpenStax Calculus Volume 2, Section 4.1[1].

Introduction. Perhaps the most common use of calculus is in describing mathematical and scientific problems as *differential equations:* equations involving the derivatives of a function. A very simple example is when position x as a function of time t is related the the known velocity $v = f(t)$ by

$$\frac{dx}{dt} = f(t). \tag{4.1.1}$$

There are many solutions to this, depending on where the object is initially, so to determine the position function completely, we also need to know the position at one time. For example if at an initial time time $t = t_0$ the position is known to be x_0, we have the *initial condition*

$$x(t_0) = x_0. \tag{4.1.2}$$

General Differential Equations. We start by defining the main new concept of this Chapter:

Definition 4.1.1 A **differential equation** is an equation involving an unknown function $y = F(x)$ and one or more of its derivatives. A solution to a differential equation is any function $y = F(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation. ◊

[1] openstax.org/books/calculus-volume-2/pages/4-introduction
[1] openstax.org/books/calculus-volume-2/pages/4-1-basics-of-differential-equations

Example 4.1.2

$$\frac{dy}{dx} = \cos x \tag{4.1.3}$$

with solutions $y = F(x) = \sin x + C$ for any constant C. □

Example 4.1.3 The above is just integration, with the possible solutions being all anti-derivatives of the right-hand side $\cos x$. More generally, for any continuous function $f(x)$, the differential equation

$$\frac{dy}{dx} = f(x) \tag{4.1.4}$$

has the **family of solutions** consisting of all the antiderivatives of $f(x)$; that is, the general indefinite integral $y = F(x) = \int f(x)\,dx$. □

Example 4.1.4 If we ask for a solution if the above differential equation (4.1.4) with the extra information that $y(x_0) = y_0$ for some known quantities x_0 and y_0, we can work out the constant of integration: there is then a unique solution

$$y = F(x) = y_0 + \int_{x_0}^{x} f(t)\,dt$$

□

Example 4.1.5

$$\frac{dy}{dx} = y \tag{4.1.5}$$

with solutions $y = F(x) = Ce^x$ for any constant C.

So there is again a family of solutions with an arbitrary constant, *but it is no longer just "$+C$".* □

Example 4.1.6 If we again add the fact that $y(x_0) = y_0$, there is again a unique solution of the differential equation (4.1.5):

$$y = F(x) = y_0 e^{x-x_0}, = (y_0/e^{x_0})e^x$$

That is, $C = y_0/e^{x_0}$. □

Example 4.1.7

$$\frac{d^2y}{dt^2} = -g \text{ (a constant)} \tag{4.1.6}$$

with solutions

$$y = F(x) = -\frac{g}{2}t^2 + Ct + D$$

this time with *two* arbitrary constants, C and D. □

Example 4.1.8

$$\frac{d^2y}{dt^2} = -y \tag{4.1.7}$$

with solutions

$$y = F(t) = C\cos(t) + D\sin(t)$$

again with *two* arbitrary constants. □

Checkpoint 4.1.9 Verify each of the solutions stated above.In each case, compute dy/dx for the claimed solution (and also d^2y/dx^2 if that appears in the differential equation) and insert the formulas for y, dy/dx, etc. into the differential equation.

Note that most of the examples above involve on the first derivative, but (4.1.6) and (4.1.7) involve second derivatives. To talk about this feature, we define

Definition 4.1.10 The **order** of a differential equation is the highest order of any derivative of the unknown function that appears in the equation. ◊

In the examples above, (4.1.6) and (4.1.7) are second order while all others are first order.

In this chapter we focus on **first order differential equations**, which have the general form

$$\frac{dy}{dx} = g(x, y) \tag{4.1.8}$$

General and Particular Solutions. We have seen that a differential equation typically has an infinite collection of solutions, often called its **family of solutions**.

Furthermore, there is often a single formula describing them all, with the help of one of more constants that can be chosen to get a specific solution like Ce^x for (4.1.5) or $-\frac{g}{2}t^2 + Ct + D$ for (4.1.6): such a formula is called a **general solution** of the differential equation.

For example, the general solution of (4.1.4) is the indefinite integral of the right-hand side function $f(x)$.

Finally, once we have somehow selected one solution from the family (such as $2e^x$ for (4.1.5) or $-\frac{g}{2}t^2 + 2t + 3$ for (4.1.6)), this is called a **particular solution** of the differential equation.

In the case $dy/dx = g(x)$, any one anti-derivative of $g(x)$ is a particular solution, and finding this is very useful for finding all the solutions: just add an arbitrary constant to get the general solution. Similarly, we will see that it often helps to start by finding any one particular solution, and then get from there to a general solution; however as seen above, ***this second step is usually not done by just adding a constant!***

Initial-Value Problems. One very common way that a problem involving a differential equation leads to a unique solution is when we also have some information about the value of the function (and maybe of some of its derivatives) at one value of the argument x.

Example 4.1.11 $\frac{dy}{dx} = y$, $f(0) = 2$ has the unique solution $y = f(x) = 2e^x$. □

Example 4.1.12 $\frac{d^2y}{dt^2} = -y$, with $f(0) = 1$, $f'(0) = 0$ has the unique solution $y = f(t) = \cos t$.

The same differential equation with $f(0) = 0$, $f'(0) = 1$ has the unique solution $y = f(t) = \sin t$. □

Checkpoint 4.1.13 Show that the more general initial value problem

$$\frac{d^2y}{dt^2} = -y, \quad y(0) = y_0,\ y'(0) = y'_0 \tag{4.1.9}$$

has the particular solution

$$y = y_0 \cos(t) + y'_0 \sin(t). \tag{4.1.10}$$

In many physical applications, the independent variable (the argument of the solution F is time; hence the use of argument t instead of x in some examples above. Then the extra information is about what we know at one time; often the starting time so that the solution to the differential equation then describes what happens at later times.

Because of this, the combination of a differential equation with such side conditions, like

$$\frac{dy}{dx} = g(x, y), \quad y(x_0) = y_0 \tag{4.1.11}$$

or (4.1.9) is called an **Initial Value Problem**.

Study Guide. Study Calculus Volume 2, Section 4.1[2]; in particular

- The definitions of **differential equation**, **general solution**, **particular solution** and **initial value problem**;
- all Examples and Checkpoints;
- and one or several exercises from each of the following groups: 1–7, 8–17, 18–27, 28–37, 38–42. (You could use Desmos[3] to graph solutions).

4.2 Direction Fields and Numerical Methods (omitted)

References.

- OpenStax Calculus Volume 2, Section 4.2[1].

This section will be omitted.

4.3 Separable Equations

References.

- OpenStax Calculus Volume 2, Section 4.3[1].

Prelude. We have seen that the special kind of differential equation

$$\frac{dy}{dx} = f(x) \tag{4.3.1}$$

can be solved by simply integrating; the general solution is given by the indefinite integral

$$y = \int f(x)\, dx. \tag{4.3.2}$$

A variant of this can be applied with the right-hand side instead depends only on the unknown, y

$$\frac{dy}{dx} = g(y) \tag{4.3.3}$$

One way to do this is to divide through by the right-hand side and then integrate:

$$\int \frac{1}{g(y)} \frac{dy}{dx} dx = \int 1\, dx.$$

The left hand-side is as for substitution, with y in the role of u; that is, we can effectively cancel the two occurences of dx, getting

$$\int \frac{1}{g(y)} dy = x + C \tag{4.3.4}$$

A complication is that this then gives an equation involving y rather than a direct formula for y, so one might have to do some equation solving.

[2]openstax.org/books/calculus-volume-2/pages/4-1-basics-of-differential-equations
[3]www.desmos.com/calculator
[1]openstax.org/books/calculus-volume-2/pages/4-2-direction-fields-and-numerical-methods
[1]openstax.org/books/calculus-volume-2/pages/4-3-separable-equations

Example 4.3.1 Find the general solution of

$$\frac{dy}{dx} = y. \tag{4.3.5}$$

The procedure above gives

$$\int \frac{1}{y}\frac{dy}{dx}dx = \int \frac{1}{y}dy = \int 1\,dx$$

so that

$$\ln|y| = x + C.$$

Solving for $|y|$ first gives $|y| = e^{x+C} = e^x e^C$ so the two options for y are $y = (\pm e^C)e^x$

The factor $k = \pm e^C$ can be any non-zero constant, so we have the solutions

$$y = ke^x, \ k \neq 0. \tag{4.3.6}$$

In fact, $k = 0$ also works, giving $y = 0$; we lost that possibility along way because dividing by y was not possible in that special case. □

Separation of Variables. The two special cases above can be combined into a more general form

$$\frac{dy}{dx} = f(x)g(y) \tag{4.3.7}$$

called **separable** because the two variables can be separated into an integral in x on one side and an integral in y on the other.

To do this, start much as with (4.3.3): divide by $g(y)$ to get

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x) \tag{4.3.8}$$

and then integrate and use the substitution rule as above to "cancel" the occurences of dx at left:

$$\int \frac{1}{g(y)}\frac{dy}{dx}dx = \int \frac{1}{g(y)}dy = \int f(x)\,dx \tag{4.3.9}$$

Integrating each side gives an equation connecting y to x; as above, you might need to solve this equation for y as a function of x; however, sometimes this equation connecting the two variables is enough information!

Example 4.3.2 Find the general solution of

$$\frac{dy}{dx} = -\frac{x}{y}. \tag{4.3.10}$$

Multiplying by y on both sides gives

$$y\frac{dy}{dx} = -x$$

and integrating,

$$\int y\frac{dy}{dx}dx = \int y\,dy = -\int x\,dx$$

so that $y^2/2 = -x^2/2 + C$ or

$$x^2 + y^2 = 2C.$$

The solution is always (part of) a circle, which is often all one needs to know.

Solving for y explicitly in terms of x, being careful with the two possibilities for the square root, one gets

$$y = \pm\sqrt{2C - x^2}$$

so to be precise, the solution is either the top half or the bottom half of a circle. □

Checkpoint 4.3.3 Solve the initial value problem for the above differential equation (4.3.10) with $y(x_0) = y_0$; that is, find the solution that passes through the point (x_0, y_0).

The above example shows an interesting possibility with first order differential equations: they can sometimes be viewed as describing the slope dy/dx at each point (x, y) on a curve (like a circle), with the solution being given as an equation for the curve, and maybe the solution curve is not describable explicitly as the graph of a function $y = F(x)$.

Applications of Separation of Variables. We omit this, but read this subsection in Openstax Calculus if you are interested in some applications to physics and physical chemistry, or just see a few more worked examples.

Study Guide. Study OpenStax Calculus Volume 2, Section 4.3[2]; in particular

- Problem-Solving Strategy,
- Examples 10 and 11, and the corresponding Checkpoints,
- and one or several exercises from each of the ranges 123–132 and 133–142.

4.4 The Logistic Equation (a very brief introduction)

References.

- OpenStax Calculus Volume 2, Section 4.4[1].

These notes just introduce this example of a scientifically interesting equation that can be solved by the methoids introuduced in Section 4.3, leaving further discussion and examples to the reference above.

Here I just state the equation itself: the **Logistic Equation** is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) \tag{4.4.1}$$

where

- t is time,
- $P = P(t)$ is the population size at that time,
- r is the **natural growth rate**: when the population is small, it behaves approximately as with $dP/dt = rP$), and
- K is the **carrying capacity**, which as will be seen is the natural equilibrium size of the polulation; the size at which resources are just sufficient to sustain the population.

This is separable, and indeed *autonomous* (there is no explicit dependence on the independent variable, t), so it can be solved by integration:

$$\int \frac{dP}{P\left(1 - \frac{P}{K}\right)} = \int r dt, = r(t - t_0) \tag{4.4.2}$$

but the integration takes some care, due to the zeros of the denominator for $P = 0$ and $P = K$.

[2] openstax.org/books/calculus-volume-2/pages/4-3-separable-equations
[1] openstax.org/books/calculus-volume-2/pages/4-4-the-logistic-equation

4.5 First-order Linear Equations

References.

- OpenStax Calculus Volume 2, Section 4.5[1].

Introduction. A first order differential equation is **linear** if it has the form $a(x)\frac{dy}{dx} = d(x) + c(x)y$, more usually rewritten with $b(x) = -d(x)$ and putting all the unknown quantities at left

$$a(x)\frac{dy}{dx} + b(x)y = c(x) \tag{4.5.1}$$

where $a(x)$, $b(x)$ and $c(x)$ are continuous functions on some interval of x values.

Note that we already know how to solve such an equation in the special case where $c(x) = 0$; then it is *separable* and has the differential form

$$\frac{dy}{y} = -\frac{b(x)}{a(x)}dx.$$

We will build on this idea, but first, a little simplification.

Standard Form. The coefficient $a(x)$ of the derivative can be divided out, giving the **standard form**

$$\frac{dy}{dx} + p(x)y = q(x) \tag{4.5.2}$$

where $p(x) = b(x)/a(x)$ and $q(x) = c(x)/a(x)$; again we want these to be continuous functions.

Note: there is a problem if $a(x) = 0$ at some x-values in the domain; indeed solution can fail to exist at such points: consider $x\frac{dy}{dx} = 1$, with solutions $y = C\ln|x|$.

As noted above, the special case $q(x) = 0$ is separable; it is of so-called **homogeneous** form

$$\frac{dy}{dx} + p(x)y = 0 \tag{4.5.3}$$

which gives

$$\frac{1}{y}\frac{dy}{dx} = -p(x), \text{ assuming for now that } y \neq 0.$$

Taking $P(x)$ to be any antiderivative of $p(x)$, integrating gives

$$\ln|y| = -P(x) + k$$

and thus the family of solutions

$$y = Ce^{-P(x)}$$

Here $C = \pm e^k$, so it is strictly non-zero, which goes with the assumption above that $y \neq 0$. However, fortunately we can solve easily in that "forbidden" case, getting the special **stationary** solution of $y = 0$ for all x, which corresponds to the previously forbidden value $C = 0$.

To use this idea for the general case of Equation (4.5.2), start by defining $\mu(x) = e^{P(x)}$ and note that the quantity $\mu(x)y$ is constant in that case:

$$\frac{d}{dx}(\mu(x)y) = 0$$

[1] openstax.org/books/calculus-volume-2/pages/4-5-first-order-linear-equations

This very simple form suggests that the same transformation might help in the general case (4.5.2), and indeed

$$\frac{d}{dx}(\mu(x)y) = \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y = \mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)\left(\frac{dy}{dx} + p(x)y\right) = \mu(x)q(x)$$

which can be integrated to get

$$\mu(x)y = \int \mu(x)q(x)\,dx + C.$$

Here the constant of integration C is shown explicitly (even though that is a bit redundant) to show where it appears in the general solution after diving by μ:

$$y = \frac{1}{\mu(x)}\int (e^{P(x)}q(x))dx + Ce^{-P(x)} \tag{4.5.4}$$

However, rather than try to remember this collection of formulas, it is far better to describe a strategy for getting the solutions, using the very useful ideas of **integrating factors**, **particular solutions**, and first solving the simpler **homogeneous part** of the equation.

Integrating Factors. The key to getting the solution above was multiplying by a function $\mu(x)$, called an **integrating factor**. There, the choice of μ was just stated, but here is a strategy for choosing a suitable function.

The goal is to multiply equation (4.5.2) by $\mu(x)$ to get the form

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)q(x) \tag{4.5.5}$$

so that the left-hand side is just a derivative of a single quantity, via the product rule; that is, we want

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \frac{d}{dx}(f(x)g(x)), = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx}$$

Firstly, we can get the y terms by choosing $f(x) = y$, so that we want

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \frac{d}{dx}(g(x)y), = g(x)\frac{dy}{dx} + \frac{dg}{dx}y$$

Then the remaining factors match if $g(x) = \mu(x)$ and $dg/dx = \mu(x)p(x)$; that is

$$\frac{d\mu}{dx} = p(x)\mu$$

This gets us back to a separable equation, solved via $d\mu/\mu = p(x)dx$, so

$$\ln|\mu| = \int p(x)dx = P(x).$$

We only need any one suitable function $\mu(x)$, not every possibility, so can let μ be positive (to avoid the absolute values) and use any one anti-derivative $P(x)$, getting

$$\mu(x) = e^{P(x)} \tag{4.5.6}$$

as above as a suitable choice. Then solving for y proceeds as above, by integrating the transformed equation

$$\frac{d}{dx}(\mu(x)y) = \mu(x)q(x) \tag{4.5.7}$$

(with constant of integration included this time) and dividing by $\mu(x)$.

Particular Solutions and Solving the Homogeneous Part First. Note that the general solution (4.5.4) is the sum of two parts; the first is one possible solution; the second is the general solution of the homogeneous part, equation (4.5.3). This gives a convenient strategy for finding the general solution:

1. Find any one **particular solution** of the whole equation; here

$$\frac{1}{\mu(x)} \int (e^{P(x)} q(x)) dx$$

2. Add any multiple C of any one solution of the homogenous part (4.5.3); here

$$e^{-P(x)}$$

.

This puts aside any dealing with constants of integration and multiple possible solutions and anti-derivatives until the last step.s

Study Guide. Study Calculus Volume 2, Section 4.5[2]; in particular,

- the Problem-Solving Strategy,
- Examples 15–17,
- Checkpoints 15–17,
- and one or several exercises from each of the groups 213–217, 218–222 and 223–232.

Chapter Review. When reviewing this chapter, also look at the end of chapter review material in OpenStax Calculus Volume 2, including Key Terms[3], Key Equations[4] and Key Concepts[5].

[2] openstax.org/books/calculus-volume-2/pages/4-5-first-order-linear-equations
[3] openstax.org/books/calculus-volume-2/pages/4-key-terms
[4] openstax.org/books/calculus-volume-2/pages/4-key-equations
[5] openstax.org/books/calculus-volume-2/pages/4-key-concepts

Chapter 5

Sequences and Series

References.

- OpenStax Calculus Volume 2, Chapter 5.[1]
- *Calculus, Early Transcendentals* by Stewart, Chapter 11, Sections 1–7.

5.1 Sequences

References.

- OpenStax Calculus Volume 2, Section 5.1[1].
- *Calculus, Early Transcendentals* by Stewart, Section 11.1.

An **infinite sequence** or just **sequence** is an infinite list of of numbers with an order of first, second, third, etc.:

$$\{a_1, a_2, a_3, \dots, a_n, \dots\}$$

For example,

$$\{1, 1/2, 1/3, 1/4, \dots, 1/n, \dots\}$$

The subscript (here n) is the **index**.

A sequence can also be thought of as a function $a(n)$ whose domain is the natural numbers, so that the index n is the argument of this function.

The individual numbers in a sequence are called the **terms** of the sequence, and a_n is sometimes called the n-th term.

One compact notation for the above is $\{a_n\}_{n=1}^{\infty}$, and sometimes a sequence is specified by just a formula for the **general term** a_n. For example, the above sequence can be described by $a_n = 1/n$.

Other starting points. The index can also start at any integer n_0, and 0 is another common starting point:

$$\{a_n\}_{n=0}^{\infty} = \{a_0, a_1, a_2, a_3, \dots, a_n, \dots\}$$

Since starting at $n = 1$ is the most common case, this is also denoted as just $\{a_n\}$, but for any other case we must be explicit, using $\{a_n\}_{n=n_0}^{\infty}$ to denote a sequence that starts with term a_{n_0}.

[1] openstax.org/books/calculus-volume-2/pages/5-introduction
[1] openstax.org/books/calculus-volume-2/pages/5-1-sequences

For example the sequence

$$\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right\}$$

is most elegantly described as $\{1/2^n\}_{n=0}^{\infty}$. To specify this with a formula for the general term a_n, we now must specify the range of index values:

$$a_n = 1/2^n, \quad n \geq 0.$$

Further examples.

- The natural numbers themselves are a sequence:

$$\{a_n\} = \{n\} = \{n\}_{n=1}^{\infty} = \{1, 2, 3, \dots\}$$

- The sequence with terms given by $a_n = \frac{n}{n+1}$ can be written in the forms

$$\left\{\frac{n}{n+1}\right\} = \left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \frac{n}{n+1}, \dots\right\}$$

- The sequence of the digits of π starts out as $\{3, 1, 4, 1, 5, 9, 2, 6, 5, \dots\}$. This time there is not a simple direct formula for a_n, but there is a way of specifying how to compute the digits.

Checkpoint 5.1.1 Find an expression for the general term of a sequence that starts

$$\{2/5, -4/7, 8/9, -16/11, \dots\}$$

It helps to consider three sequences that are "ingredients" of the above one: $\{2, 4, 8, 16, \dots\}$, $\{5, 7, 9, 11, \dots\}$, and $\{1, -1, 1, -1, \dots\}$

Further examples: Recursion. Another case where there is not a direct formula for a_n, but there is a procedure for computing the terms in order, is the sequence with

$$a_1 = 1, a_2 = \frac{a_1 + 2/a_1}{2} = \frac{1 + 2/1}{2} = 1.5,$$
$$a_3 = \frac{a_2 + 2/a_2}{2} = \frac{1.5 + 2/1.5}{2} = 1.41\bar{6}, \dots, a_{n+1} = \frac{a_n + 2/a_n}{2}, \dots$$

Some more terms of this sequence, to ten decimal places, are

$$a_4 = 1.4142156863, a_5 = 1.4142135624, a_6 = 1.4142135624, \dots$$

The values seem to be approaching $\sqrt{2} = 1.414213562373\dots$.

This is indeed the case, and the above calculation is an example of an efficient way to compute square roots: *a_6 is already accurate to 10 decimal places!*

This is an example of a **recursively defined sequence**, a central concept in computer science and methods of accurate calculation of the numerical values of functions.

More recursion: The Fibonacci sequence. Probably the most famous example of a recursively defined sequence is the one devised by Fibonacci as a model of population growth (before calculus or exponential functions were known):

$$f_1 = 1, f_2 = 1, f_3 = f_2 + f_1 = 2, f_4 = f_3 + f_2 = 3, \dots, f_n = f_{n-1} + f_{n-2}, \dots$$

The final, general formula only works for $n \geq 3$, once the process has been initiated with the first two values.

We can compactly describe this as

$$f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 3.$$

The limit of a sequence. Sequences often arise as successively better approximations of some value, like the above recursive sequence with terms approaching $\sqrt{2}$. If the terms of a sequence $\{a_n\}$ approach a value L as n increases, this value is the **limit** of the sequence:

Definition 5.1.2 The Limit of a Sequence. If for every positive number ϵ, there is an integer N so that whenever $n > N$, we have $|a_n - L| < \epsilon$ then we say that *the sequence* $\{a_n\}$ *has limit* L, denoted

$$\lim_{n\to\infty} a_n = L, \quad \text{or} \quad a_n \to L \text{ as } n \to \infty.$$

◊

Compare to the precise definition of $\lim_{x\to\infty} f(x)$, as seen in Section 4.6 of OpenStax Calculus Volume 1[2]: the similarities will allow us to adapt many methods for computing the limit of a function at infinity (as with horizontal asymptotes) to computing the limits of sequences.

Infinite Limit of a Sequence. As with functions, sequences can also have infinite limits:

Definition 5.1.3 If for every number M there is an integer N so that whenever $n > N$, we have $a_n > M$ then we say that *the sequence* $\{a_n\}$ *has limit infinity*, denoted

$$\lim_{n\to\infty} a_n = \infty, \qquad \text{or} \qquad a_n \to \infty \text{ as } n \to \infty.$$

A limit of $-\infty$ is similarly defined. ◊

Note that sequences can have no limit at all, not even an infinite one.

Checkpoint 5.1.4 $\{(-1)^n\}_{n=0}^{\infty} = \{1, -1, 1, -1, \dots\}$

Sequence limits using limits at infinity of functions. Often the limit of a sequence is the same as the limit at infinity of a function, and can be computed as we compute values for horizontal asymptotes.

Theorem 5.1.5 $a_n f(x) a_n = f(n) \lim_{x\to\infty} f(x)$

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x).$$

The above fact allows many methods for limits of functions to be used with sequences, including l'Hôpital's Rule (see Section 4.8 of OpenStax Calculus Volume 1[3]). In fact, limits of sequences, along with improper integrals, are two of the most important uses of that Rule.

Example 5.1.6 For $a_n = \dfrac{\ln n}{n}$, we can use $f(x) = \dfrac{\ln x}{x}$, and

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{(\ln x)'}{(x)'} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

□

So strangely, it is effectively possible to differentiate in the variable n, though its values are originally only integers!

Limit laws: much as for functions. If $\{a_n\}$ and $\{b_n\}$ are convergent sequences with $a_n \to A$, $b_n \to B$, and c is a constant, then

$$a_n + b_n \to A + B$$

[2]openstax.org/books/calculus-volume-1/pages/4-6-limits-at-infinity-and-asymptotes
[3]openstax.org/books/calculus-volume-1/pages/4-8-lhopitals-rule

$$a_n - b_n \to A - B$$
$$a_n \cdot b_n \to A \cdot B$$
$$c \cdot a_n \to c \cdot A$$
$$\frac{a_n}{b_n} \to \frac{A}{B}, \quad \text{if } B \neq 0$$
$$a_n^p \to A^p, \quad \text{if } p > 0 \text{ and all } a_n > 0$$

And we again get a Squeeze Theorem:

Theorem 5.1.7 The Squeeze Theorem for Sequences. *If $a_n \leq b_n \leq c_n$ and the "outer" sequences have the same limit, then so does the "middle" sequence.*

That is, if $a_n \to L$ and $c_n \to L$, then $b_n \to L$ also.

This can be used to show that

Theorem 5.1.8 *If $|a_n| \to 0$, then $a_n \to 0$.*

Checkpoint 5.1.9 $\dfrac{(-1)^n}{n} \to 0$.

Checkpoint 5.1.10 For what values of r is the geometric sequence $\{r^n\}$ convergent, and what are the limits?

Look for both finite and infinite limits.

Definition 5.1.11 Increasing, Decreasing and Monotonic sequences.

- Sequence $\{a_n\}$ is **increasing** [a.k.a. *non-decreasing*] if for every index value n, $a_n \leq a_{n+1}$.

  Note that successive values do not have to be *larger* than the previous one; just *no smaller*; if further $a_n < a_{n+1}$ and we want to note that, the sequence is called **strictly increasing**.
- Sequence $\{a_n\}$ is **decreasing** [a.k.a. *non-increasing*] if for every index value n, $a_n \geq a_{n+1}$.
- A sequence is **monotonic** if it is either increasing or decreasing.

For the last two cases, there are also "strictly" versions. ◊

An important type of increasing sequence is those produced by successively adding positive values, like these recursively defined sequences:

- $s_0 = 1$, $s_n = s_{n-1} + 1/n!$ for $n \geq 1$ so $s_n = 1/0! + 1/1! + 1/2! + \cdots + 1/n!$, and
- $s_0 = 0$, $s_n = s_{n-1} + 1/2^n$ for $n \geq 1$ so $s_n = 1/2 + 1/4 + 1/8 + \cdots + 1/2^n$.

Definition 5.1.12 Sequences bounded above, or below, or both.

- Sequence $\{a_n\}$ is **bounded above** [by M] if $a_n \leq M$ for all index values n.
- Sequence $\{a_n\}$ is **bounded below** [by M] if $a_n \geq M$ for all index values n.
- A sequence is **bounded** if it is both bounded above and bounded below.

◊

It can be shown that each of the two increasing sequences above is also bounded above; even more obviously, each is bounded below, by zero. This helps to establish that each has a limit, because of:

Theorem 5.1.13 The Bounded Monotonic Sequence Theorem. *Every bounded monotonic sequence is convergent.*

This shows that each of the two previous examples has a limit; we will find the values of those limits later in this chapter.

Study Guide. Study Calculus Volume 2, Section 5.1[4]; in particular

- The definition of an **Infinite Sequence**
- The definitions of the **Limit** of an Infinite Sequence (a lot like a horizontal asymptote!) and of **Convergence/Divergence**
- The definition of **Bounded**, **Increasing/Decreasing** and **Monotone** sequences
- Theorem 1, connecting limits of sequences back to limits at infinity of functions (horizontal asymptotes again!)
- Theorem 2, which shows that limits of sequences have a lot of properties in common with limits of functions
- Theorems 3 and 4; more properties in common with limits of functions
- The **Monotone Convergence Theorem** (Theorem 6)
- Examples 1, 2, 3, 4, 5, 6
- Checkpoints 1, 2, 3, 4, 5, 6
- and one or several exercises from each of the following groups: 1–4, 8–11, 13, 14&15, 27–29, 31–34.

5.2 Infinite Series

References.

- OpenStax Calculus Volume 2, Section 5.2[1].
- *Calculus, Early Transcendentals* by Stewart, Section 11.2.

Given a sequence $\{a_n\}_{n=1}^{\infty}$ we can easily sum the first N terms, giving

$$S_N := a_1 + a_2 + \cdots + a_N = \sum_{n=1}^{N} a_n \tag{5.2.1}$$

(Note that we have to use a *dummy index* n to indicate each term in the sum that is different from the index N indicating the number of terms in that sum.)

This is analogous to a definite integral $\int_1^N f(x)\,dx$.

It can also be useful to make sense of summing all the terms of an infinite sequence, to get

$$S = a_1 + a_2 + \cdots + a_n + \cdots$$

which is analogous to an improper integral $\int_1^\infty f(x)\,dx$. To do this, we use the same strategy as with improper integrals:

Definition 5.2.1 Infinite Sum, or Series. The infinite sum of sequence $\{a_n\}_{n=1}^{\infty}$ is

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{N} a_n, \tag{5.2.2}$$

if this limit exists with a finite value, and we then say that *the series (infinite sum) converges.*

If there is not a finite limit, we say that *the series (infinite sum) diverges.*

If there is an infinite limit, we sometimes say that *the series (infinite sum) diverges to infinity.*

[4] openstax.org/books/calculus-volume-2/pages/5-1-sequences
[1] openstax.org/books/calculus-volume-2/pages/5-2-infinite-series

The finite sum S_N up to term N of the sequence is called the N**-th partial sum** of the series. $\Diamond$

Checkpoint 5.2.2 For the sequence $\left\{\frac{1}{2^{n-1}}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right\}$

1. Show that the N-th partial sum $S_N = \sum_{n=1}^{N} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{N-1}} = 2 - \frac{1}{2^{N-1}}$ (Aside: the sequence of partial sums is thus bounded above.)
2. Show that the series (infinite sum) converges, with value $\sum_{n=1}^{\infty} \frac{1}{2^n} = 2$.

Geometric sequences and series. The above example is one case of a **geometric series**, one of the few situations where we can get an easily evaluated expression for the sum of an infinite sequence. A geometric *sequence* is one in which the ratio of every pair of consecutive terms has the same value r. Thus calling the first term a, the remaining terms are given recursively by $a_n = ra_{n-1}$ and this leads to the explicit form

$$a_n = ar^{n-1}, n \geq 1.$$

For $r \neq 1$, the partial sums can be shown to be $S_N = a\frac{1-r^N}{1-r}$. Then for $|r| < 1$, these partial sums have a limit, giving this geometric *series* the value

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \frac{a}{1-r}, \ |r| < 1.$$

Geometric series: divergence for $|r| \geq 1$. What happens if instead $|r| \geq 1$? The sum diverges, but in a variety of different ways:

- For $r > 1$, the partial sums grow without bound and have no [finite] limit: the series diverges to an infinite value; for example, with $a = 1$, $r = 2$ gives the sum form $1 + 2 + 4 + 8 + \cdots$ so the partial sums are going towards ∞, and the series *diverges to infinity*.
- For $r < -1$, the partial sums oscillate ever more wildly, and diverge without even an infinite limit; with $a = 1$, $r = -2$ gives the sum form $1 - 2 + 4 - 8 + \cdots$ so the partial sums do not even have an infinite limit.
- For $r = 1$, $a_n = a$ so $S_N = aN$, and the series again diverges to infinity (except in the trivial case $a = 0$, where $S_N = 0$.)
- Finally, for $r = -1$, the sequence is $\{a, -a, a, -a, \dots\}$, so the successive partial sums are $a, 0, a, 0, \dots$; again not converging to any value (except if $a = 0$).

Other starting points for sums. It is useful to define infinite sums for other starting points too:

$$\sum_{n=b}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=b}^{N} a_n.$$

For example, a geometric sequence is more elegantly described with indices starting at 0:

$$\{ar^n\}_{n=0}^{\infty} = \{ar^0, ar^1, ar^2, \dots\} = \{a, ar, ar^2, \dots\}$$

and the most elegant way to state the result for convergent geometric series is

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \ |r| < 1.$$

The Harmonic Series: the terms converge to zero, but the series diverges! The **harmonic series** is

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

and is unusual in that it is important enough to have a name, yet it does not have a value, because it diverges [to infinity]. It is also an important cautionary example: our first example where the sequence of terms has limit zero [$1/n \to 0$], but the infinite sum of these terms diverges. One way to see that the sum diverges is to group terms like this:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

Each term in parentheses has value of at least $1/2$, so the partial sums are at least $1/2 + 1/2 + 1/2 + \cdots$ and increase to arbitrarily large values. We will see another way to show this divergence in Section 5.3

Terms going to zero is necessary *but not sufficient* for series convergence. There seems to be *some* connection between the terms of a sequence going to zero and its infinite sum converging, though the harmonic series shows that terms going to zero in not always enough to guarantee convergence of the infinite sum.

There is a guarantee in the other direction though:

Theorem 5.2.3 *If a series $\sum a_n$ converges, then its terms must converge to zero: $a_n \to 0$,* ***but not conversely!***

Rules for sums and constant multiples. Series have some familiar properties, akin to rules for integrals, limits of sequences, and derivatives:

$$\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$$

$$\sum_{n=1}^{\infty} (a_n + b_n) = \left(\sum_{n=1}^{\infty} a_n\right) + \left(\sum_{n=1}^{\infty} b_n\right)$$

$$\sum_{n=1}^{\infty} (a_n - b_n) = \left(\sum_{n=1}^{\infty} a_n\right) - \left(\sum_{n=1}^{\infty} b_n\right)$$

and of course the sum can instead start at any value on n, not just at $n = 1$.

The first few terms do not matter for convergence/divergence. Just as with limits of sequences, whether a series converges does not depend on the first few terms—where by "few", I mean any finite number of them!

If you add any number of extra terms at the start of a series, or remove any number of terms, or change the values of any finite number of the terms, that can only change the value of the series (sum) by a finite amount; it cannot convert a convergent series into a divergent one, or vice versa.

Study Guide. Study Calculus Volume 2, Section 5.2[2]; in particular

- The Definition in terms of **Partial Sums** and *Limits*
- The Definition and properties of **Geometric Series**
- Theorem 7, which shows that series have a lot of properties in common with improper integrals
- Examples 7 [parts (a) and (b)], 8, 9
- Checkpoints 7, 8, 9
- and one or several exercises from each of the following groups: 67–70, 71–74, 79 & 80, 83–86, 87–92, 93–96.

[2]openstax.org/books/calculus-volume-2/pages/5-2-infinite-series

5.3 The Divergence and Integral Tests

References.

- OpenStax Calculus Volume 2, Section 5.3[1].
- *Calculus, Early Transcendentals* by Stewart, Section 11.3.

Introduction. Often the first question to be answered about a series $\sum a_n$ is whether it even converges, giving a numerical value.

Then, since we will usually get values only via the approximation of summing a finite nunber of terms and so using a partial sum $S_N = \sum_{n=1}^{N} a_n$ as an approximation of the infinite sum, the next important question is to be able to say somethig about how accurate this approximation is.

In this section we first see a test that is fairly simple, but can only give the "negative" result that a series does not converge; then we meet our first method for showing that some series do converge, and also for saying something about the accuracy of partial sums as approximations.

5.3.1 The Divergence Test

There is an important point worth repeating from the previous section: for an infinite sum to converge, it is *required* that the individual terms go to zero, but this alone is not enough, as shown by the Harmonic Series $\sum_{n=1}^{\infty} 1/n$, for which the terms $a_n = 1/n \to 0$ as $n \to \infty$, but the sum diverges ("is infinite").

Theorem 5.3.1 *If a series $\sum a_n$ converges, then its terms must converge to zero: $a_n \to 0$,* ***but not conversely!***

That is, failure to have $a_n \to 0$ proves ***divergence****.*

The name refelcts the fact that this test can only show *divergence*; its result is always either

"the series diverges" or "I do not know".

5.3.2 The Integral Test and Estimates of Sums

Most often, the value of a series (infinite sum) $S = \sum a_n$ will be computed accurately but not exactly by computing a partial sum S_N for a sufficiently large N.

This leaves two important and related questions:

a. *Does the series converge to a number $S = \sum_{n=1}^{\infty} a_n$?*

b. If so, *how accurate is a given approximation $S_N = \sum_{n=1}^{N} a_n$?* Or if in pursuit of som eneeded degree of accuracy, you might ask *how large does N need to be to meet a given accuracy requirement $|S - S_N| \leq \epsilon$?*

These questions might seem familiar; we will see that they are closely related to the questions of whether an improper integral $I = \int_1^{\infty} f(x)\,dx = \lim_{N\to\infty} \int_1^N f(x)\,dx$ converges, and of how close those proper "partial" integrals $I_N = \int_1^N f(x)\,dx$ are to the value of the improper integral I.

Question A, and The Integral Test. Both questions can be answered for a sequence of positive numbers given by a decreasing function, $a_n = f(n)$, so that the sequence itself is also decreasing.

Loosely speaking, the main observation is this:

$$\int_1^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} a_n \leq a_1 + \int_1^{\infty} f(x)\,dx \tag{5.3.1}$$

so that either

[1] openstax.org/books/calculus-volume-2/pages/5-3-the-divergence-and-integral-tests

- both the integral and the sum are convergent (finite values for all three terms above), or
- both are divergent (infinite values for all terms above).

This is seen by considering the sum as the are of a sequence or rectangles along the x-axis of height a_n, width 1 on the interval $[n, n+1]$, and comparing this to the areas represented by the improper integral.

Theorem 5.3.2 The Integral Test. *If $a_n = f(n)$ for f a positive valued and decreasing function defined for $x \geq 1$, then*

- *If $\displaystyle\int_1^\infty f(x)\,dx$ converges, so does $\displaystyle\sum_{n=1}^{\infty} a_n$.*
- *If $\displaystyle\int_1^\infty f(x)\,dx$ diverges, so does $\displaystyle\sum_{n=1}^{\infty} a_n$.*

Checkpoint 5.3.3 Determine which of the following sums converges, using the above test.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

More generally, what about $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ for various values of the constant p?

Such sums are called *p-series*, and are one of our two archetypical types of examples, along with Geometric Series.

Question B, and The integral remainder estimate. If the Integral Test has shown that a sum converges, we can move onto the second question above, of how close a partial sum S_N is to the exact value S of the series.

The basic fact is a modification of (5.3.1) above, using integrals and sums starting at N and $N+1$ instead of at 1, along with the observation that the error or **remainder** in S_N as an approximation of S is

$$R_N = S - S_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n = \sum_{n=N+1}^{\infty} a_n.$$

We get

$$\int_{N+1}^{\infty} f(x)\,dx \leq R_N \leq \int_N^{\infty} f(x)\,dx. \tag{5.3.2}$$

The second part is probably the most useful, since it gives a upper bound on the size of the error, if we can evaluate the improper integral.

Checkpoint 5.3.4 For the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

1. How accurate is the approximation $\displaystyle\sum_{n=1}^{100} \frac{1}{n^2}$?

2. What about $\displaystyle\sum_{n=1}^{10^6} \frac{1}{n^2}$?
3. If we want a result accurate to within 10^{-10}, how many terms should we sum?

Study Guide. Study Calculus Volume 2, Section 5.3[2]; in particular

- The Definition of **p-series**
- Theorem 8 *The Divergence Test*
- Theorems 9 *The Integral Test* and 10 *Remainder Estimate from the Integral Test*
- Examples 13, 14, 15, 16
- Checkpoints 12, 13, 14, 15
- and one or several exercises from each of the following groups: 138–147, 152–157, 158–161, 169–172, 178 & 179.

5.4 Comparison Tests

References.

- OpenStax Calculus Volume 2, Section 5.4[1].
- *Calculus, Early Transcendentals* by Stewart, Section 11.4.

In the previous section, we used comparisons of series with positive terms to improper integrals to do two things:

- show that a series converges by comparing to a convergent improper integral, and
- get an upper limit on the error in a partial sum as an approximation of the value of a series by comparing to an improper integral whose value we know.

We can do similar things by comparing one series to another instead of to an improper integral:

- showing that a series converges by comparing to another series already known to converge, and
- getting an upper limit on the error in a partial sum as an approximation of the value of a series by comparing to the known value of another series (often geometric!).

The basic idea is to compare a new series $\sum a_n$ to another series $\sum b_n$ whose convergence or divergence is already known:

Theorem 5.4.1 The [Basic] Comparison Test. *If sequences $\{a_n\}$ and $\{b_n\}$ have all terms non-negative and $a_n \leq b_n$ for all terms, then* Convergence of $\sum b_n$ ensures convergence of $\sum a_n$, with sum no larger.

That is,

$$\text{If } 0 \leq a_n \leq b_n \text{ for all } n, \text{ then } 0 \leq \sum a_n \leq \sum b_n.$$

In the other direction, divergence of $\sum a_n$ implies divergence of $\sum b_n$.

Common choices for the "known" series $\sum b_n$ include *geometric series* $\sum ar^n$ and *p-series* $\displaystyle\sum \frac{1}{n^p}$.

[2] openstax.org/books/calculus-volume-2/pages/5-3-the-divergence-and-integral-tests
[1] openstax.org/books/calculus-volume-2/pages/5-4-comparison-tests

Accuracy of partial sums by comparisons. The above idea can also be applied to the remainder $R_N = (\sum a_n) - S_N = \sum_{n=N+1}^{\infty} a_n$ to get an upper limit on the error in the partial sum $S_N = \sum_{n=1}^{N} a_n$ and thus an upper bound on the exact value of the series:

If $0 \leq a_n \leq b_n$ for $n > N$ then

$$0 \leq \left(\sum a_n\right) - S_N = \sum_{n=N+1}^{\infty} a_n \leq \sum_{n=N+1}^{\infty} b_n.$$

For example, if $0 \leq a_n \leq ar^n$, $0 < r < 1$, then

$$0 \leq \left(\sum a_n\right) - S_N \leq \frac{ar^{N+1}}{1-r}.$$

Ignoring the first few terms and comparing with ratios. Whether a series converges or not does not depend on the first few terms, even if "few" means "billion": convergence only depends on what happens in the long run.

Loosely speaking: *Do the terms individual terms a_n approach zero "fast enough"?*

One useful measure of "fast enough" is *at least as fast as a constant multiple of some convergent series:*

The Limit Comparison Test. If sequences $\{a_n\}$ and $\{b_n\}$ have all terms positive, and for some positive number c

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c, > 0,$$

then either both the series $\sum a_n$ and $\sum b_n$ converge, or both diverge.

Also, if $a_n \leq cb_n$ for all n and some constant c, and $\sum b_n$ converges, then $\sum a_n$ converges too.

The Limit Comparison Test is often more convenient than the (basic) Comparison Test. In particular, when a series has slightly larger terms than a similar convergent series, or slightly smaller terms than a similar divergent series, the inequalities are the wrong way around for the Comparison Test to say anything, but the similarities can be enough for Limit Comparison to work.

On the other hand, the Limit Comparison Test does not give an accuracy result for partial sums like the one given above for the original Comparison Test.

Study Guide. Study Calculus Volume 2, Section 5.4[2]; in particular

- Theorem 11 *The Comparison Test*, a.k.a. *The **Direct** Comparison Test*
- Theorem 12 *The **Limit** Comparison Test*
- Examples 17 and 18
- Checkpoints 16 and 17
- and one or several exercises from each of the groups 194–200 and 207–211.

5.5 Alternating Series (and Conditional vs. Absolute Convergence)

References.

- OpenStax Calculus Volume 2, Section 5.5[1].

[2]openstax.org/books/calculus-volume-2/pages/5-4-comparison-tests
[1]openstax.org/books/calculus-volume-2/pages/5-5-alternating-series

- *Calculus, Early Transcendentals* by Stewart, Section 11.5.

An Alternating Series is one whose terms a_n are alternately positive and negative:

$$b_1 - b_2 + b_3 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

where the b_n are non-negative. That is, $a_n = (-1)^{n-1} b_n$, $b_n = |a_n|$.

As is often the case, indexing from zero can be more elegant:

$$\sum_{n=0}^{\infty} (-1)^n b_n = b_0 - b_1 + b_2 - \cdots \qquad b_n \geq 0$$

Examples. One example we have already seen is a geometric series with negative ratio $r = -s$

$$\sum_{n=0}^{\infty} (-1)^n s^n = 1 - s + s^2 - s^3 + \cdots = \frac{1}{1+s}, \qquad 0 < s < 1$$

Another important example is the **Alternating Harmonic Series**:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

which as we will soon see is convergent, unlike the Harmonic Series. (In fact the value is $\ln(2)$, as will be seen in Section 6.3)

We will also see in Section 6.3 that the "almost but not quite geometric" series

$$\sum_{n=0}^{\infty} \frac{(-1)^n r^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-r^2)^n}{(2n)!} = 1 - \frac{r^2}{2!} + \frac{r^4}{4!} - \frac{r^6}{6!} + \cdots$$

converges for any number r. (In fact the value is $\cos(r)$, as will be seen in Section 6.3)

Checkpoint 5.5.1 A Diverging Alternating Series. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is alternating, but diverges.

Convergence of *alternating* series with terms that *decrease* in size to *zero*. Under two simple conditions, we can both show that an alternating series converges, and also rather easily get upper and lower bounds on the value of its sum, making such series very convenient for practical calculations:

Theorem 5.5.2 The Alternating Series Test for convergence. *If the alternating series*

$$\sum_{n=0}^{\infty} (-1)^n b_n = b_0 - b_1 + b_2 - \cdots$$

satisfies the two conditions that

1. *the terms are decreasing in magnitude:* $b_{n+1} \leq b_n$*, and*
2. *the terms converge to zero:* $b_n \to 0$ *(as is always needed for convergence.)*

then the series converges.

Measuring the accuracy of partial sums. This is much easier to do than with the comparison to improper integrals seen in Section 5.3

Theorem 5.5.3 The Alternating Series Estimation Theorem. *If an alternating series satisfies the above two conditions, its value S lies between any two consecutive partial sums, and the error in any partial sum S_N is no larger than the first term not used in that partial sum, and so smaller than b_N, the last term used:*

$$|R_N| = |S - S_N| < b_{N+1}, \leq b_N$$

That is:

- *As you successively add terms, after each addition of a positive value the running total S_N is more than the exact value S of the series, and after each addition of a negative term it is less.*
- *The size of the last term added gives an upper limit on error in the partial sum.*

Absolute Convergence and Conditional Convergence. To move beyond the special cases of series with all positive terms or ones with alternating signs, it helps to start by looking at just the magnitudes of the terms:

Definition 5.5.4 Absolutely Convergent. The series $\sum a_n$ is called **absolutely convergent** if the positive-term series $\sum |a_n|$ is convergent. ♢

Checkpoint 5.5.5 Verify that the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ is absolutely convergent.

Not surprisingly:

Theorem 5.5.6 *If a series is **absolutely convergent**, it is also convergent.*

But the opposite is not true!

Checkpoint 5.5.7 Verify that the **Alternating Harmonic Series** $\displaystyle\sum \frac{(-1)^{n-1}}{n}$ is convergent but not absolutely convergent.

Checkpoint 5.5.8 Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent.Use both the idea of absolute convergence and then a comparison test from Section 5.4.

Definition 5.5.9 Conditionally Convergent. A series that is **convergent** but not **absolutely convergent** is called **conditionally convergent**. ♢

As a rule of thumb, conditionally convergent series are far less useful and when we seek to express the solution to a problem as the sum of a series, we greatly prefer to find an absolutely convergent series that does the job.

Checkpoint 5.5.10 Verify directly that the alternating p-series $\displaystyle\sum \frac{(-1)^{n-1}}{n^2}$ is both convergent and absolutely convergent.

Checkpoint 5.5.11 Verify that the **alternating p-series** $\displaystyle\sum \frac{(-1)^{n-1}}{n^p}$ is

- absolutely convergent for $p > 1$.
- conditionally convergent for $0 < p \leq 1$.
- divergent for $p \leq 0$.

Checkpoint 5.5.12 Verify that any convergent *geometric* series is also absolutely convergent.

(But note that this is not true for all series!)

Study Guide. Study Calculus Volume 2, Section 5.5[2]; in particular

- The definition of an **Alternating Series**
- The following proof that the **Alternating Harmonic Series** converges, even though the original **Harmonic Series** diverges
- The **Alternating Series Test** in Theorem 13, noting the extra requirement that the terms are *decreasing in magnitude*.
- Theorem 14 about how accurate a partial sum is at approximating the infinite sum, including whether it is an underestimate or an over-estimate.
- The definitions of **Absolute Convergence** and **Conditional Convergence**, and Theorem 15
- Examples 19, 20, 21
- Checkpoints 18, 19, 20
- and one or several exercises from each of the groups 250–257 and 280–283.

5.6 Ratio and Root Tests

References.

- OpenStax Calculus Volume 2, Section 5.6[1].
- *Calculus, Early Transcendentals* by Stewart, Section 11.6.

The Ratio Test for Absolute Convergence. Often the best way to show that a series converges is to show absolute convergence, by comparison to a geometric series with positive ratio r.

One way to do this is to note that for a geometric series, with $a_n = ar^n$, one has $\left|\dfrac{a_{n+1}}{a_n}\right| = |r|$, and so check if a series behaves roughly like this for large n:

Theorem 5.6.1 (The Ratio Test, part I: Absolute Convergence).

A. *If the limit* $\rho = \lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$ *exists, and* $\rho < 1$*, then* $\sum a_n$ *is absolutely convergent.*

The basic idea is that for any r with $\rho < r < 1$, the convergent geometric series $\sum r^n$ is "bigger", so limit comparison gives convergence of $\sum |a_n|$.

Theorem 5.6.2 The Ratio Test, concluded: divergence and *the possibility of no answer*.

B. *If the above limit* ρ *exists but with* $\rho > 1$*, the series diverges. (In fact it fails the* n*-th term test dramatically, because* $|a_n| \to \infty$*.)*

C. *If the limit* $\rho = 1$*, or the limit defining* ρ *does not exist,* this test gives no answer. *For example, all* p*-series have* $\rho = 1$*, but some converge while others diverge.*

Checkpoint 5.6.3 Test the series $\displaystyle\sum_{n=0}^{\infty} \frac{n^3}{3^n}$ for convergence.

Checkpoint 5.6.4 Test the series $\displaystyle\sum_{n=0}^{\infty} \frac{n^n}{n!}$ for convergence.

[2] openstax.org/books/calculus-volume-2/pages/5-5-alternating-series
[1] openstax.org/books/calculus-volume-2/pages/5-6-ratio-and-root-tests

Checkpoint 5.6.5 Test the series $\sum_{n=0}^{\infty} \frac{r^n}{n!}$ for convergence.

Note that this is actually an infinite family of series, one for each choice of the value r, so you must answer for each value of r.

The n-th Root Test for Absolute Convergence. A cousin of the Ratio Test compares to geometric series in a different way, using the fact that for a geometric series, $\sqrt[n]{|a_n|} \to |r|$.

Theorem 5.6.6 The Root Test.

a) *If the limit $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists, and $\rho < 1$, then $\sum a_n$ is absolutely convergent.*

b) *If the above limit ρ exists but with $\rho > 1$, the series diverges. (Again, $|a_n| \to \infty$.)*

c) *If the limit $\rho = 1$, or the limit does not exist, again* this test gives no answer. *(Again, all p-series have $\rho = 1$, but some converge while others diverge.)*

Which of these two tests to use?

- The two tests work quite similarly; for example, *if the Ratio Test works, then the Root Test also works*, and gives the same value for ρ.
- *The Ratio Test is often easier to use*, so it is usually best to try it first.
- On the other hand, *sometimes the Ratio Test fails but the Root Test succeeds*, so it is good to have as a backup.

Study Guide. Study Calculus Volume 2, Section 5.6[2] that relate to the **Ratio Test**; in particular

- Theorem 6 (stating the **Ratio Test**).
- Example 23.
- Checkpoint 21.
- Review the *Problem-Solving Strategy*, which summarizes ideas from the last several sections, and then do Exercise 25 and Checkpoint 23.
- Do one or several exercises from the ranges 317–327 and 364–367.

Chapter Review. When reviewing this chapter, also look at the end of chapter review material in OpenStax Calculus Volume 2, including Key Terms[3], Key Equations[4] and Key Concepts[5].

[2] openstax.org/books/calculus-volume-2/pages/5-6-ratio-and-root-tests
[3] openstax.org/books/calculus-volume-2/pages/5-key-terms
[4] openstax.org/books/calculus-volume-2/pages/5-key-equations
[5] openstax.org/books/calculus-volume-2/pages/5-key-concepts

Chapter 6

Power Series

References.

- OpenStax Calculus Volume 2, Chapter 6.[1]
- *Calculus, Early Transcendentals* by Stewart, Chapter 11, Sections 8–11.

Introduction. The goal of this chapter is to learn how to construct series (infinite sums!) whose values solve various problems of interest. For example, I have claimed that we can evaluate the exponential e^x for any value x by summing the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$$

This is not a single series, but a family of them, one for each value of x, with that quantity x appearing only in natural number powers. Thus it resembles a polynomial, but with an infinite number of terms. Such series are called **Power Series** (or sometimes "infinite polynomials").

6.1 Power Series and Functions

References.

- OpenStax Calculus Volume 2, Section 6.1[1].
- *Calculus, Early Transcendentals* by Stewart, Chapter 11, Sections 8 & 9.

Example 6.1.1 Geometric Series. We have already seen one important type of a power series: the geometric series

$$\sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + \cdots + ax^n + \cdots$$

and determined that

- it converges for some values of x ($|x| < 1$, so $-1 < x < 1$) but not others, and
- for x values giving convergence, the value is $\frac{a}{1-x}$.

□

[1]openstax.org/books/calculus-volume-2/pages/6-introduction
[1]openstax.org/books/calculus-volume-2/pages/6-1-power-series-and-functions

These two questions will arise for other power series:

1. for which x values does the series converge, and
2. when it does converge, what is the value of the sum? That is, what function does the power series give?

Definition 6.1.2 Power Series. A **Power Series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

where x is some number, and the c_n are constants: that is, they do not depending on x.

More generally, powers of $x - a$ can be used for some constant a, so the most general power series is of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

The constant a is called the **center** of the series. ♢

Example 6.1.3 For which values of x does one get convergence of the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots$$

The sum converges for $-1 \le x < 1$ and diverges otherwise. □

Example 6.1.4 For which values of x does one get convergence of the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

The sum converges for all x; that is $|x| < \infty$. □

Example 6.1.5 For which values of x does one get convergence of the series

$$\sum_{n=0}^{\infty} n! x^n = 1 + x + 2x^2 + 6x^3 + \cdots$$

The sum converges only for $x = 0$; that is "$|x| \le 0$". □

Two patterns are worth noting in the above examples:

1. One primarily gets convergence for $|x|$ "small enough", divergence for sufficiently large $|x|$.
2. Convergence and divergence is shown primarily by the Ratio Test (or the Root Test).
3. Exceptions to the previous two observations occur at the two borderline points where $|x|$ has the largest value for which convergence might occur.
4. These two borderline x values give $\rho = 1$ in the Ratio Test (or the Root Test), so those tests give no answer; thus to determine convergence we must use some other method, like the Alternating Series Test or one of the comparison tests.

The above patterns are in fact universal:

Theorem 6.1.6 *For a given power series* $\sum_{n=0}^{\infty} c_n (x-a)^n$ *one of the following is true*

1. *There is a positive number R such that the series converges (absolutely) for $|x-a|<R$, and diverges for $|x-a|>R$.*
2. *The series converges (absolutely) for all x: that is, for $|x-a|<\infty$, or $x\in(\infty,\infty)$. (Informally, "$R=\infty$".)*
3. *The series converges only for $x=a$: that is, for $|x-a|\leq 0$, or $x\in[a,a]$. ("$R=0$").*

The first case is silent on two values of x: $a+R$ and $a-R$. At each of these, one can have either convergence or divergence: see for example the "50-50" case of Example 6.1.3 above.

The number R in case (i) is called the **Radius of Convergence**. In fact, we can make sense of a radius of convergence in every case:

- in case (ii), we say the radius of convergence is $R=\infty$;
- in the boring case (iii), we say the radius of convergence is $R=0$.

Also, in every case the x values giving convergence form an interval, which we call the **Interval of Convergence**:

- In case (i), the interval of convergence can be $(a-R,a+R)$, $[a-R,a+R)$, $(a-R,a+R]$, or $[a-R,a+R]$.
- In case (ii) it is $(-\infty,\infty)$
- In case (iii) it is just $[a,a]$.

Example 6.1.7 Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty}(-2)^n\sqrt{n}(x-2)^n.$$

The root test shows that the sum converges for $|x-2|<1/2$ and diverges for $|x-2|>1/2$, so the radius of convergence is $R=1/2$. Then testing the end cases $x=3/2,5/3$, where $|x-2|=1/2$, gives divergence in each case, so the interval of convergence is $(3/2,5/2)$. □

Example 6.1.8 Find the radius of convergence and interval of convergence for the series

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

This defines the **Bessel function of the first kind and order 0**, denoted $J_0(x)$, which solves the differential equation

$$\frac{d^2J_0}{dx^2}+\frac{1}{x}\frac{dJ_0}{dx}+J_0=0.$$

Bessel functions are involved in the solution of a variety of physical problems, from planetary motions to the vibrations in a disk, such as a drum-head. For more details, see this Wikipedia entry[2].

The series for $J_0(x)$ converges for all x, so the radius of convergence is $R=\infty$, and so with no extra work needed, the interval of convergence is $(-\infty,\infty)$. □

Study Guide. Study Calculus Volume 2, Section 6.1[3]; in particular

- The defintions of a **Power Series**, its **Center** and its **Radius of Convergence**.
- Theorem 1 about the possibilites for which x values give convergence.
- Examples 1, 2 and 3 (focus on "radius" more than "interval")
- Checkpoints 1 and 3

[2]en.wikipedia.org/wiki/Bessel_function
[3]openstax.org/books/calculus-volume-2/pages/6-1-power-series-and-functions

- and one or several exercises from each of the following ranges: 1–4, 5 and 6, 13–16, 23–26, 29–32.

6.2 Properties of Power Series

References.

- OpenStax Calculus Volume 2, Section 6.2[1].
- *Calculus, Early Transcendentals* by Stewart, Chapter 11, Sections 8 & 9.

Combining Power Series. Power series can be combined to fomer ne ones more or less like polynomials, with the one new detail that one needs to keep track of the radius of convergence. The results il be stated for th basic case with center $a = 0$, but they all carry over to other centers in an intutive way.

Theorem 6.2.1 *Consider any two power series $\sum_{n=0}^{\infty} c_n x^n$ and c, which resepctively converge to $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with radius of convergence R_1 and $g(x) = \sum_{n=0}^{\infty} d_n x^n$ with radius of convergence R_2. Then*

1. *Their sums $\sum_{n=0}^{\infty}(c_n + d_n)x^n$ converges to $f(x) + g(x)$, with radius of convergence at least the minimum of R_1 and R_2.*
2. *Similarly, their difference $\sum_{n=0}^{\infty}(c_n - d_n)x^n$ converges to $f(x) - g(x)$.*
3. *Any scalar multiple $\sum_{n=0}^{\infty} bc_n x^n$ converges to $bf(x)$.*
4. *Muliplying by x^m for any natural number m gives $\sum_{n=0}^{\infty} bc_n x^{n+m}$ which converges to $x^m f(x)$.*
5. *Composing with any monomial bx^m for any natural number m gives $\sum_{n=0}^{\infty} c_n(bx^m)^n = \sum_{n=0}^{\infty}(b^n c_n)x^{mn}$, a power seried that converges to the composition $f(bx^m)$.*

The results above can be used to multiply a power series by a polynomial, and one can in fact go further and multiply two power series, but as with derivatives and integrals, handling products of functions is a bit more involved than sums, differences and constant multiples!

$$\begin{aligned} f(x)g(x) &= \left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} c_n x^n\right) \\ &= (c_0 + c_1 x + c_2 x^2 + \cdots)(d_0 + d_1 x + d_2 x^2 + \cdots) \\ &= c_0 d_0 + (c_0 d_1 + c_1 d_0)x + (c_0 d_2 + c_1 d_1 + c_2 d_0)x^2 + \cdots \end{aligned}$$

The pattern is that the coefficient of x^n is the sum of all coefficients $c_i d_j$ with $i + j = n$; that is $f(x)g(x) = \sum_{n=0}^{\infty} p_n x^n$ with $p_n = \sum_{i=0}^{n} c_i d_{n-i}$.

The radius of convergence is the minimum of those for the two series being multiplied together.

Calculus with Power Series. The derivatives and integrals of power series work just a for polynomials, term be term, and the radius of convergence is unchanged:

Theorem 6.2.2 *For a power series giving $f(x) = \sum_{n=0}^{\infty} c_n x^n$ in radius of convergence R,*

1. *$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \sum_{n=1}^{\infty} nc_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)c_{n+1}x^n$ with this series having the same radius of convergence R.*
2. *$\displaystyle\int f(x)\,dx = C + c_0 x + \frac{c_1}{2}x^2 + \frac{c_2}{3}x^3 + \cdots = C + \sum_{n=1}^{\infty} \frac{c_{n-1}}{n} c_n x^n$ with this series again having the same radius of convergence R.*

[1] openstax.org/books/calculus-volume-2/pages/6-2-properties-of-power-series

The second of these in particular will be extremely useful: it allow us to evaluate integrals like $\int e^{-x^2}dx$ and $\int \frac{\sin x}{x}dx$ that cannot be handled by any of the methods seen in Chapter 3 or a previous calculus course.

Uniqueness of the Coefficients. One final, intuitive observation: if there is a power series for a function, then it is unique. That is, there can ony be one possible set of values for the coefficients $c_0, c_1, c_2, \ldots$. One way to verify this leads into the topic of the next section, Section 3, *Taylor and Maclaurin Series*.

Theorem 6.2.3 *If two powers series $\sum_{n=0}^{\infty} c_n(x-a)^n$ and $\sum_{n=0}^{\infty} b_n(x-a)^n$ both converge to the same function $f(x)$ for some x values $|x-a| < R$ near a, then their coefficients are the same: $c_n = d_n$ for all $n \geq 0$, so they are actually the same series.*

Proof. Subtracting the two series shows that

$$\sum_{n=0}^{\infty} e_n(x-a)^n = e_0 + e_1(x-a) + e_2(x-a)^2 + \cdots = 0 \tag{6.2.1}$$

where $e_n = c_n - d_n$; thus it suffices to show that all the $e_n = 0$ in this case.

Evaluating Equation (6.2.1) for $x = a$ gives $e_0 + e_1 \cdot 0 + e_2 \cdot 0^2 + \cdots = e_0 = 0$,

Then the derivative of Eq. (6.2.1) gives

$$e_1 + 2e_2(x-a) + 3e_3(x-a)^2 \cdots = 0 \tag{6.2.2}$$

and evaluating this for $x = a$ gives $e_1 + 2e_2 \cdot 0 + 3e_3 \cdot 0^2 + \cdots = e_1 = 0$

Differentiating again similarly gives $2e_2 + 3 \cdot 2e_3 \cdot 0 + 4 \cdot 3e_4 \cdot 0 = 2e_2 = 0$, so $e_2 = 0$, and so on: all coefficients $e_n = c_n - d_n$ are zero as claimed, so $c_n = d_n$ and the two series for $f(x)$ are actually the same. ■

Study Guide. Study Calculus Volume 2, Section 6.2[2]; in particular

- Theorems 2, 4, 5
- Examples 4, 6, 9, 10
- Checkpoints 4, 6, 8, 9
- and one or several exercises from each of the following groups: 63 and 64, 69–71, 87 and 88, 89 and 90.

Note that we de-emphasize *interval of convergence*, so when that is asked for, it is sufficient to determine the *center* and the *radius of convergence*.

Also, we will not do much with products of power series: integrals of power series are by far the most important new idea in this section.

6.3 Taylor and Maclaurin Series

References.

- OpenStax Calculus Volume 2, Section 6.3[1].
- *Calculus, Early Transcendentals* by Stewart, Chapter 11, Section 10.

The last main ingredient in using power series is a more systematic way to find such a series for a particular function, like $\cos x$ or $\int e^{-x^2}dx$.

We will follow a familiar strategy of

[2]openstax.org/books/calculus-volume-2/pages/6-2-properties-of-power-series
[1]openstax.org/books/calculus-volume-2/pages/6-3

- devising a basic method,
- using this to get power series for the main elementary functions, and then
- learning how to combine these to get series for many other functions more easily, to avoid starting from scratch each time.

Consider a function $f(x)$, and let us try to find the coefficients c_0, c_1 etc. giving a power series form

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots , \tag{6.3.1}$$

at least for small enough $|x|$.

Evaluating at $x = 0$ gives

$$c_0 = f(0)$$

To continue, differentiate, getting

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots$$

so again evaluating at $x = 0$ gives

$$c_1 = f'(0)$$

The MacLaurin Series. The pattern can continue of alternately differentiating and then evaluating at $x = 0$: we get

$$f^{(n)}(x) = c_n n! + c_{n+1}(n+1)(n)\cdots(2)x + c_{n+2}(n+2)(n+1)\cdots(3)x^2\cdots \tag{6.3.2}$$

so that $f^{(n)}(0) = c_n n!$ and

$$c_n = \frac{f^{(n)}(0)}{n!} \tag{6.3.3}$$

Thus, it *seems* that $f(x)$ is given by the power series

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 \cdots + \frac{f^{(n)}(0)}{n!}x^n \cdots \tag{6.3.4}$$

and this series is called **the Maclaurin Series for function** f.

Example 6.3.1 Find the Maclaurin series for $f(x) = e^x$, and its radius of convergence. □

The Taylor Series. Changing to the general power series with center a and using derivatives at $x = a$ instead gives **the Taylor Series for function** f **at center** a:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a)\cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n\cdots \tag{6.3.5}$$

But we must answer two questions:

- For which x values does such a series converge?
- When is the value that it converges to equal to $f(x)$, as we would hope?

Taylor Polynomials and Their Remainders. To determine whether a Taylor series converges, and whether it gives the expected value, we look at partial sums and the remainders for them, which are now functions of x.

The N-th partial sum for a Taylor series is the **Taylor Polynomial**

$$T_N(x) = f(a) + f'(a)(x-a)\cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N. \tag{6.3.6}$$

The Taylor Series has a radius of convergence R (possibly 0 or ∞!); the error we care about is the difference between the N-th Taylor polynomial and the function $f(x)$.

This is called the **remainder of the Taylor series**; the function

$$R_N(x) = f(x) - T_N(x). \tag{6.3.7}$$

Is the Remainder Small Enough? Taylor's Theorem and Taylor's Inequality. We hope that as $N \to \infty$, $T_N(x) \to f(x)$, which is the same as the condition $R_N(x) \to 0$.

The main tool for checking this is to look at the size of the terms in the series through the size of the derivative values $f^{(n)}(a)$: it first helps to know that *the remainder $R_N(x)$ looks almost like the first term not used in the partial sum $T_N(x)$:*

Theorem 6.3.2 Taylor's Theorem. *For a Taylor polynomial $T_n(x)$ for f with center a, and any number x in the domain of f, there is a number c between x and a such that*

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1} \tag{6.3.8}$$

Then if the N-th derivatives do not get too big as N increases, we are done:

Theorem 6.3.3 Taylor's Inequality, and convergence to $f(x)$, *sometimes*. *If for some numbers M and $d > 0$, $|x-a| \le d$ ensures that $|f^{(n)}(x)| \le M$, for all n, then the remainder of the Taylor series for f with center a satisfies*

$$|R_N(x)| \le \frac{M}{(N+1)!}|x-a|^{N+1} \quad \text{for } |x-a| \le d. \tag{6.3.9}$$

If this is true, then for $|x-a| \le d$, $|R_N(x)| \to 0$ as $N \to \infty$, so $T_N(x) \to f(x)$, so that the value of the Taylor Series is $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \cdots \tag{6.3.10}$$

A Warning: This theorem does not guarantee that the series has value $f(x)$ for all x in its interval of convergence: this might be true only for some d less than R.

As an extreme case, the Taylor series for $f(x) = e^{-1/x^2}$ (defining $f(0) = 0$ to make it continuous there) is $0 + 0 \cdot x + 0 \cdot x^2 + \cdots = 0$.

The good news is that, as we will soon see, the Taylor Series for any of our favorite elementary functions $f(x)$ does converge to that function, for all x values in the interval of convergence of the series.

Checkpoint 6.3.4 Prove that e^x is equal to its Maclaurin series for all x. That is: for any $d > 0$, find a suitable M to use in Taylor's inequality.

Checkpoint 6.3.5 Find the Taylor series for $f(x) = \ln x$ with center $a = 1$.

Then verify that the series converges to $\ln x$ for all values x in its interval of convergence.

Building new Taylor Series from old. As much as possible we wish to avoid working directly from the formula for the Taylor series, instead we seek to build new power series form ones already known:

Example 6.3.6 Find the MacLaurin series for $f(x) = \cos x$, by differentiating the above MacLaurin series for $\sin x$. □

Checkpoint 6.3.7 Find a power series for $f(x) = \dfrac{\sin x}{x}$ (with value $f(0) = 1$ for continuity), again using the above MacLaurin series for $\sin x$.

Checkpoint 6.3.8 Find a power series for $\sinh x$, using the above MacLaurin series for e^x.

Checkpoint 6.3.9 Find the first three nonzero terms of the MacLaurin series for $e^x \sin x$.

Square Roots. Having dealt with the most important transcendental functions, we need to deal with algebraic functions, and in particular roots: we start with the square root.

Checkpoint 6.3.10 A power series for the square root.

- Find the Taylor series for $\sqrt{x}$ with center $a = 1$,
- verify that its radius of convergence is $R = 1$,
- and verify that for $|x - 1| < 1$, the series converges to $\sqrt{x}$.

Study Guide. Study Calculus Volume 2, Section 6.3[2]; in particular

- The definitions of **Taylor series** and **Maclaurin series** and **Taylor polynomials**
- Theorem 6 on the uniqueness of Taylor series
- Theorem 7 on Taylor's Theorem with remainder
- Theorem 8 on the Convergence of Taylor series
- Examples 11, 12, 13, 14, 15, 16
- Checkpoints 10, 11, 12, 13, 14, 15
- and one or several exercises from each of the following groups: 116–123, 130 and 131, 132–135.

6.4 Working with Taylor Series

References.

- OpenStax Calculus Volume 2, Section 6.4[1].
- *Calculus, Early Transcendentals* by Stewart, Chapter 11, Sections 10 & 11.

The Binomial Series. The Taylor series for the square root seen in exercise Checkpoint 6.3.10 in the previous section can be viewed as giving a fractional power $x^{1/2}$; in fact, a similar calculation leads to a Taylor series for any power function.

However, most powers x^r will not have a Maclaurin series (one with center $a = 0$), due to the function being undefined for $x < 0$, which is why center $a = 1$ was used above.

To get a simpler form in powers of x, we shift the function over, and look for the Maclaurin series of $(1 + x)^r$. For r a positive integer, this will give a case of the familiar binomial expansion

$$(a + b)^r = a^r + ra^{r-1}b + \frac{r(r-1)}{2}a^{r-2}b^2 \cdots + b^r \tag{6.4.1}$$

Theorem 6.4.1 The Binomial Series. *For any real number r and $|x| < 1$,*

$$(1 + x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots$$

where

$$\binom{r}{n} := \frac{r}{1}\frac{r-1}{2}\cdots\frac{r-n+1}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}$$

[2] openstax.org/books/calculus-volume-2/pages/6-3

[1] openstax.org/books/calculus-volume-2/pages/6-4-working-with-taylor-series

Noe that the formula for $\begin{pmatrix} r \\ n \end{pmatrix}$ is the same as for the combinatorial quantity denoted ${}^rC_n$ or ${}_rC_n$, giving the number of ways of selecting r objects from a collection of n objects, except that now we now allow r to have any real value.

Checkpoint 6.4.2

1. Verify that the result is as expected for $r = -1$ and for $r = 2$ (or any positive integer).
2. Give the Maclaurin series for $\sqrt[3]{1+x}$ and show that for $0 < x < 1$ this series is eventually "alternating and decreasing". (What about for $x < 0$?)

Common Functions Expressed as Taylor Series.

Function	Maclaurin Series	Interval and Radius of Convergence
$f(x) = \dfrac{1}{1-x}$	$\displaystyle\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$, $R = 1$
$f(x) = e^x$	$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$, $R = \infty$
$f(x) = \ln(1+x)$	$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$	$-1 < x \le 1$, $R = 1$
$f(x) = \sin(x)$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$-\infty < x < \infty$, $R = \infty$
$f(x) = \cos(x)$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$-\infty < x < \infty$, $R = \infty$
$f(x) = \arctan(x)$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$	$-1 \le x \le 1$, $R = 1$
$f(x) = (1+x)^r$	$\displaystyle\sum_{n=1}^{\infty} \begin{pmatrix} r \\ n \end{pmatrix} x^n$	$-1 < x < 1$, $R = 1$ (see note)
	where $\begin{pmatrix} r \\ n \end{pmatrix} = \dfrac{r(r-1)\cdots(r-n+1)}{n!}$	

Note: the Binomial Series for $(1+x)^r$ also converges for all x if r is a natural number (where the series is just a polynomial), and also can converge at the endpoints $x = \pm 1$ for some other values of r.

Evaluating Nonelementary Integrals. Evaluating integrals for which we do not know an explicit anti-derivative formula is one important use of power series.

Example 6.4.3 The **Error Function**, very important in statistics is defined as

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt; \tag{6.4.2}$$

that is, the anti-derivative of $P(x) = \dfrac{2}{\sqrt{\pi}} e^{-x^2}$ with the constant of integration chosen to have $\mathrm{Erf}(0) = 0$.

There is no expression for this integral in terms of elementary functions, but a power series can be derived for it, starting from the Maclaurin seried for e^x,

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$$

First, the substitution $x \to (-x^2)$ gives

$$\begin{aligned} e^{-x^2} &= 1 + (-x^2) + \frac{1}{2}(-x^2)^2 + \frac{1}{3!}(-x^2)^3 + \cdots + \frac{1}{n!}(-x^2)^n + \cdots \\ &= 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{3!}x^6 + \cdots + \frac{(-1)^n}{n!}x^{2n} + \cdots \end{aligned}$$

Next, this can be integrated term-by-term to get the indefinite integral

$$\int e^{-x^2} dx = C + x - \frac{1}{3}x^3 + \frac{1}{5 \cdot 2}x^4 - \frac{1}{7 \cdot 3!}x^6 + \cdots + \frac{(-1)^n}{(2n+1)n!}x^{2n} + \cdots$$

Finally, multiplying by that factor $\frac{2}{\sqrt{\pi}}$ and using the fact that $\mathrm{Erf}(0) = 0$ to show that the constant is $C = 0$,

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\left(x - \frac{1}{3}x^3 + \frac{1}{5 \cdot 2}x^4 - \frac{1}{7 \cdot 3!}x^6 + \cdots + \frac{(-1)^n}{(2n+1)n!}x^{2n} + \cdots\right)$$

As an example of using this we can approximate the value of the Error Function for $x = 1$ accurate two decimal places. First

$$\mathrm{Erf}(1) = \frac{2}{\sqrt{\pi}} - \frac{2}{3\sqrt{\pi}} + \frac{1}{5\sqrt{\pi}} - \frac{1}{21\sqrt{\pi}} + \frac{1}{108\sqrt{\pi}} + \cdots + \frac{(-1)^n 2}{(2n+1)n!} + \cdots$$

This can be verified to meet all conditions the Alternating Series Theorem, and the fifth term is smaller than 0.01, so the fifth partial sum

$$\frac{2}{\sqrt{\pi}} - \frac{2}{3\sqrt{\pi}} + \frac{1}{5\sqrt{\pi}} - \frac{1}{21\sqrt{\pi}} + \frac{1}{108\sqrt{\pi}} = 0.843449\ldots$$

is accurate enough, and we also know that thisis an over-estimate. (Note: this is $p_9(1)$; evaluating the degree 9 Maclaurin polynomial.) In fact, that fifth term bounds the error in the fourth partial sum (which is $p_7(1)$; from the degree 7 Maclaurin polynomial), so

$$\frac{2}{\sqrt{\pi}} - \frac{2}{3\sqrt{\pi}} + \frac{1}{5\sqrt{\pi}} - \frac{1}{21\sqrt{\pi}} = 0.838225\ldots$$

is already accurate enough, and an under-estimate: we have shown that $0.838 \le \mathrm{Erf}(1) \le 0.844$, so to two decimal places, $\mathrm{Erf}(1) \approx 0.84$. □

Checkpoint 6.4.4

1. Find a power series for $\displaystyle\int \frac{\sin x}{x} dx$.
2. Use this series to evaluate $\displaystyle\int_{-\pi}^{\pi} \frac{\sin x}{x} dx$ to within 0.01.

Evaluating "$0/0$" Limits Using Power Series. Sometimes, limits $\lim_{x\to a} f(x)/g(x)$ that lead to the indeterminate form "$0/0$" can be evaluated by expressing the top and bottom in terms of power series with center a: there will be a common factor $x - a$ top and bottom to cancel out.

Checkpoint 6.4.5 Evaluate $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x - x^2/2}{x^2 \sin x}$.

(Note: This would involve third derivatives if done with l'Hôpital's Rule.)

Study Guide. Study Calculus Volume 2, Section 6.4[2]; in particular

- all about The Binomial Series
- Examples 17, 18, 22
- Checkpoint 16 (Hint: as usual, first rewrite using a negative power), 17, 21
- and one or several exercises from each of the following ranges: 174–177, 186 and 187, 194 and 195, 210–213.

Note that we omit the topic of *Solving Differential Equations with Power Series*.

Chapter 6 Review. When reviewing this chapter, also look at the end of chapter review material in OpenStax Calculus Volume 2, including Key Terms[3], Key Equations[4] and Key Concepts[5].

[2] openstax.org/books/calculus-volume-2/pages/6-4-working-with-taylor-series
[3] openstax.org/books/calculus-volume-2/pages/6-key-terms
[4] openstax.org/books/calculus-volume-2/pages/6-key-equations
[5] openstax.org/books/calculus-volume-2/pages/6-key-concepts

Chapter 7

Parametric Equations and Polar Coordinates

References.

- OpenStax Calculus Volume 2, Chapter 7, Sections 1–4.[1]
- *Calculus, Early Transcendentals* by Stewart, Chapter 10.
- The Desmos online graphing calculator.[2]

Introduction. Many interesting curves in the plane cannot be described as the graph of a function $y = f(x)$; the circle $x^2 + y^2 = 1$ is a very familiar example.

Relatedly, many curves describe the position of a moving object as a function of time; an object moving around the above circle might have coordinates at time t given by $x(t) = \cos t, y(t) = \sin t$. This is an example of a **parametric** description of a curve, with the new, "auxilliary" variable t called a **parameter**.

Note that this parametric form can convey more information that the equation $x^2 + y^2 = 1$ for the curve, because of the information about time and place — this is important for example for computing the velocity of a moving object.

7.1 Parametric Equations

References.

- OpenStax Calculus Volume 2, Section 7.1[1].
- *Calculus, Early Transcendentals* by Stewart, Section 10.1.

Introduction. Many interesting curves in the plane cannot be described as the graph of a function $y = f(x)$; the circle $x^2 + y^2 = 1$ is a very familiar example.

Relatedly, many curves describe the position of a moving object as a function of time; an object moving around the above circle might have coordinates at time t given by $x(t) = \cos t, y(t) = \sin t$. This is an example of a **parametric** description of a curve, with the new, "auxilliary" variable t called a **parameter**.

[1] openstax.org/books/calculus-volume-2/pages/7-introduction
[2] www.desmos.com/calculator
[1] openstax.org/books/calculus-volume-2/pages/7-1-parametric-equations

Note that this parametric form can convey more information that the equation $x^2 + y^2 = 1$ for the curve, because of the information about time and place—this is important for example for computing the velocity of a moving object.

Parametric Equations and Their Graphs. Let us introduce the main new concepts in this chapter:

Definition 7.1.1 For two functions F and G defined on a common interval I, the pair of equations

$$x = x(t) = F(t),\ y = y(t) = G(t) \tag{7.1.1}$$

are the **parametric equations** of a curve.

The set of all points $x(t), y(t)$ for all $t \in I$ is called the **graph** of these equations; also known as a **parametric curve** or **plane curve**, and typically denoted C.

If the domain is a closed interval $I = [a, b]$ then the curve has **initial point** $(F(a), G(a))$ and **terminal point** $(F(b), G(b))$; these are are collectively called the **endpoints** of the curve.

Note however that that the interval could be open or semi-open and so lack one or both endpoints: it can even be infinite, like $(-\infty, \infty)$ or $[0, \infty)$. ♢

The parameter t often has the physical meaning of time, and will informally be referred to as "time" here.

Example 7.1.2 Circles. For any constant $R > 0$ and any point (c, d) in the plane

$$x = c + R\cos t,\ y = d + R\sin t,\ 0 \le t \le 2\pi$$

describes a circle of radius R, center (c, d). □

Checkpoint 7.1.3 Graphing this circle with Desmos. Graph the above parametric curve for the case of radius $R = 1$, center $(2, 3)$, using the Desmos online graphing calculator[2]: input the parametric equations as

(2 + cos(t), 3 + sin(t))

and then edit the limits of the t values (the value π can be entered by typing "pi").

Practice using the mouse/trackpad/finger to move around the graph and to zoom in and out.

Note that in this case, the initial and terminal points are the same; the right-most extremity, $(c + R, d)$:

Definition 7.1.4 A **closed curve** is one whose initial and final points are the same. ♢

Example 7.1.5 A spiral. The curve

$$x = t\cos t,\ y = t\sin t,\ 0 \le t < \infty$$

describes one kind of spiral; at time t, the point is at distance t from the origin and as the parameter increases, the position rotates around the origin infinitely often. Its initial point is the origin, but it has no terminal point. □

Checkpoint 7.1.6 Graph this spiral with Desmos. One catch is that Desmos[3] cannot handle an infinite interval of t values, and anyway the whole spiral is infinitely large; thus, experiment with graphing a couple of turns; say $0 \le t \le 4\pi$.

Example 7.1.7 An exponential spiral. The curves

$$x = e^{at}\cos t,\ y = e^{at}\sin t,\ -\infty < t < \infty$$

describe another kind spiral; this time the point is at distance e^{at} from the origin at time t.

[2] `www.desmos.com/calculator`
[3] `www.desmos.com/calculator`

This has no initial or terminal point; however it makes sense to say that (for $a > 0$)

$$\lim_{t \to -\infty} (x(t), y(t)) = (0, 0)$$

so informally, it starts at the origin. □

Checkpoint 7.1.8 Visualizing exponential spirals. The above exponential spiral grows rather fast so to visualize, it is best to keep the parameter a small. Thus, start by look at a case like $x = e^{(t/4)} \cos t$, $y = e^{(t/4)} \sin t$; Desmos[4] input can be done as (exp(t/4) cos(t), exp(t/4) sin(t))

(You can also experiment with inputing exponents, to get the notation $e^{t/4}$.)

Again the infinite interval has to be reduced; start with one turn on either side $t = 0$ with $-2\pi \leq t \leq 2\pi$. Then larger t intervals can be visualized with the help of zooming in and out.

As a further experiment with Desmos, include the parameter k with the form

(exp(k t) cos(t), exp(k t) sin(t)),

and see how Desmos allows setting up sliders for parameters.

Eliminating the Parameter. Sometimes the parameter can be eliminated, getting back to an equation of the form $y = F(x)$ or $x = G(y)$, or just a more general equation form $F(x, y) = 0$ like the equation for a circle. However, we will soon see that this is not always possible, and even when it is, some useful information can be lost.

Example 7.1.9 Circles again. Consider the parametric equations

$$x = \cos t,\ y = \sin t,\ 0 \leq t \leq 3\pi/2$$

with initial point $(1, 0)$ and terminal point $(0, -1)$. We can use a very familiar trig. identity to get

$$x^2 + y^2 = 1,$$

which looks like the equation of a circle.

However, three things are lost here:

1. Information about where the point is at a give time t,
2. the fact that this only covers three-quarters of the circle, due to the limits on the parameter values, and
3. the "function" form that will allow us to do calculus with curves in the next section, like computing their slopes.

Also, we needed a bit of luck here, with the trig. identity; this strategy often fails, as seen with the example of cycloids below. □

Example 7.1.10 Part of a parabola. For the parametric curve

$$x = \cos^2(t) + 2,\ y = \cos t,\ 0 \leq t \leq 2\pi$$

we can to some extent do better than above, getting a function describing this curve: substituting the second equation into the first gives

$$x = y^2 + 2$$

This equation describes a side-ways parabola, but it hides two facts:

1. The y values are only in the interval $-1 \leq y \leq 1$
2. The curve both starts and ends at the point $(1, 3)$, in between traveling to $(-1, 3)$ and then back-tracking.

□

[4] www.desmos.com/calculator

Checkpoint 7.1.11 Graph this parametric curve $x = \cos^2(t) + 2,\ y = \cos t$. Note that "$\cos^2 t$" can be typed into Desmos[5] as either "cos^2(t)" or "cos(t)^2".

Cycloids and Other Parametric Curves. A very useful example of a parametric curve are the **cycloids**

$$x = a(t - \sin t),\ y = a(1 - \cos t) \tag{7.1.2}$$

because this cannot be written as the graph of a function in any useful way, and yet we can answer all kinds of questions about it, like computing the slope at a point on it, the length along the curve between points on this curve, and related areas under the curve.

The origin of this curve is that it describes the trajectory of a point on a wheel of radius a as that wheel rolls along, starting on the ground at point $(0, 0)$ at time $t = 0$.

Note that no domain for t is specified above; this curve can be consider as defined for all time. However it is also use to resrict to a single rotation

$$x = a(t - \sin t),\ y = a(1 - \cos t),\ 0 \le t \le 2\pi \tag{7.1.3}$$

which goes from initial point $(0, 0)$ to terminal point $(2\pi a, 0)$ with $y > 0$ in between, looking like an "arch"; for times before and after that, the curve "repeats" with copies of that arch shifted left and righ by multiples of $2\pi a$.

Checkpoint 7.1.12 Graph two arches of a cycloid. Set the scale as $a = 1$, so the Desmos[6] input can be (t-sin(t), 1-cos(t)); use interval $0 \le t \le 4\pi$.

Note the special behavior at the points where $y = 0$, and zoom in on the point $(2\pi, 0)$ given by $t = 2\pi$.

If you wish to explore further the capablities of Desmos, use the full form

(a(t-sin(t)), a(1-cos(t))) with parameter a,

and see how it allows setting up a slider for it.

Checkpoint 7.1.13 Another Desmos experiment: Prolate and Curtoid Cycloids. If you instead look at a point on the edge of the overhanging flange of a train wheel, so that the flange has radius $b > a$, one gets a **prolate cycloid**

$$x = at - b \sin t,\ y = a - b \cos t \tag{7.1.4}$$

If instead $b < a$, this describes the motion of a point on the wheel that is closer to the center, and is called a **curtate cycloid.**

Look at these with Desmos[7], using input (a t - b sin(t), a - b cos(t)) and setup sliders for both parameters.

Note the special behavior at the points where $t = 0, 2\pi$, etc.

Study Guide. Study Calculus Volume 2, Section 7.1[8]; in particular

- The Definition of **parametric curves** and **parameters**
- everything about Cycloids
- Examples 1 and 2
- Checkpoints 1 and 2
- and one or several exercises from each of the groups: 1–4, 6–9, 51–53. Of these, 6–9 should be done with Desmos[9] or similar software.

[5] www.desmos.com/calculator
[6] www.desmos.com/calculator
[7] www.desmos.com/calculator
[8] openstax.org/books/calculus-volume-2/pages/7-1-parametric-equations
[9] www.desmos.com/calculator

7.2 Calculus of Parametric Curves

References.

- OpenStax Calculus Volume 2, Section 7.2[1].
- *Calculus, Early Transcendentals* by Stewart, Section 10.2.

Derivatives of Parametric Equations. Often we are interested in the slope of a parameteric curve $(x, y) = (F(t), G(t))$, meaning intuitively $m = dy/dx$, but we do not have an explicit formula $y = f(x)$. Fortunately, rather than having to solve for f by eliminating th parameter t, the needed derivative and slope can be computed by *implicit differentiation*: if $y = G(t) = f(x) = f(F(t))$ then

$$\frac{dy}{dt} = \frac{dG}{dt} = f'(F(t))F'(t) = f'(x)F'(t) = \frac{dy}{dx}\frac{dx}{dt};$$

which is the familiar intuitive pattern of the Chain Rule. This can than be solved to get

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \tag{7.2.1}$$

so long as the denominator dx/dt is non-zero.

Note that this will be a formulas in terms of t, not x!

Example 7.2.1 Slope of the spiral $(x, y) = (t\cos(t), t\sin(t))$.

First,

$$dx/dt = \cos(t) - t\sin(t) \text{ and } dy/dt = \sin(t) + t\cos(t),$$

so the slope at the point given by any value of the parameter t is given by

$$\frac{dy}{dx} = \frac{\sin(t) + t\cos(t)}{\cos(t) - t\sin(t)}$$

□

Example 7.2.2 Slope of the cycloid $(x, y) = (t - \sin(t), 1 - \cos(t))$.

$dx/dt = 1 - \cos(t)$ and $dy/dt = \sin(t)$, so the slope at the point given by any value of the parameter t is given by

$$\frac{dy}{dx} = \frac{\sin(t)}{1 - \cos(t)},$$

except where the denominator is zero; that happens when $\cos(t) = 1$, which is for t an integer multiple of 2π; $t = 2n\pi$, where also $\sin(t) = 0$.

This is the points $(x, y) = (2n\pi, 0)$ where the cycloid "touches down", and where the graphs done in Section 7.1 suggested that something strange was happening. □

Second-Order Derivatives. Once one has a formula for the first derivative dy/dx (albeit in terms of t), computing second and higher derivatives is relatively strightforward; no furhter implicit differentition is needed. First,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$$

Next, use the above Equation (7.2.1) with y replaced by dy/dx:

$$\frac{d^2y}{dx^2} = \frac{d(dy/dx)/dt}{dx/dt}$$

[1] openstax.org/books/calculus-volume-2/pages/7-2-calculus-of-parametric-curves

Since the method above gives dy/dx as a function of t, we can evaluate the two derivatives here.

Checkpoint 7.2.3 The concavity of the cycloid. Compute the second derivative $\frac{d^2y}{dx^2}$ of the above cycloid.

Graphs suggest that this curve is always concave down, so check that.

Integrals Involving Parametric Equations: Area Under a Curve. If a curve $y = f(x)$, $a \leq x \leq b$ also has a parametric form $x = F(t)$, $y = G(t)$, $\alpha \leq t \leq \beta$, with f an increasing function and $y = g(t) \geq 0$, then it lies over a region $a \leq x \leq b$ with $x(\alpha) = a$, $x(\beta) = b$, and it makes sense to talk of the area between this curve and the x-axis.

If we could eliminate the parameter and describe the curve as $y = F(x)$, this area would be $A = \int_a^b F(x)\,dx$, but in fact, we do not need to get an explicit formulas for $F(x)$! Instead, use the (inverse) substitution $x = f(t)$ to get

$$A = \int_{x=a}^{b} y\,dx = \int_{x=a}^{b} f(x)\,dx = \int_{t=\alpha}^{\beta} f(x(t))\frac{dx}{dt}dt = \int_{t=\alpha}^{\beta} f(x(t))F'(t)dt = \int_{t=\alpha}^{\beta} y\frac{dx}{dt}dt. \tag{7.2.2}$$

where we use the fact that $y = F(x(t))$ and also $y = g(t)$.

That is, we get the intuitive change of variables

$$A = \int_{x=a}^{b} y\,dx = \int_{t=\alpha}^{\beta} y\frac{dx}{dt}dt. \tag{7.2.3}$$

As always, note how the limits of integration change when the dummy variable is changed by substitution!

Checkpoint 7.2.4 Compute the area under one arch of the cycloid $(x, y) = (a(t - \sin(t)), a(1 - \cos(t)))$

Arc Length of a Parametric Curve. The formula for arc-length in Section 2.4 can also be converted when $y = f(x)$ also has a parametric form, using the same substitution $x = F(t)$ as for areas:

$$L = \int_{x=a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{t=\alpha}^{\beta} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\frac{dx}{dt}\,dt = \int_{t=\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\frac{dx}{dt}\right)^2}\,dt = \int_{t=\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

As in Chapter 2, a good intuitive way to see this is that each "infinitesimally" small piece of the curve has length

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \tag{7.2.4}$$

sometimes called the **arc length differential**; the arc length is then the "sum" or integral of these infinitesimal lengths: $L = \int ds$.

This idea can be used as in Section 2.4 to show that in fact for any parametric curve with $f'(t)$ and $g'(t)$ continuous for $\alpha \leq t \leq \beta$, the arc length is, as above,

$$L = \int ds = \int_{t=\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \tag{7.2.5}$$

The curve does not have to be in the form of the graph of a function, and in particular, *x need not be an increasing function of the parameter.*

Note that when $x = t$—so the curve is just the graph of $y = g(x)$— this gives the same formula as in Equation (2.4.1) of Section 2.4. So if you only learn one arc length formula, it should be the one here!

Checkpoint 7.2.5 Compute the circumference of a circle or radius R, parameterized as $(x, y) = (R\cos(t), R\sin(t))$

Checkpoint 7.2.6 Compute the length of one arch of the cycloid $(x, y) = (a(t - \sin(t)), a(1 - \cos(t)))$

Surface Area Generated by a Parametric Curve (Omitted). This topic is not covered in this course, but I include this brief introduction; it is discussed further in Section 7.2 of the OpenStax Calculus text.[2]

The area of the surface produced by rotating a parametric curve about the x-axis can be computed, and the most intuitive way to see the result is to work with a **surface area differential** dS, much as the **arc length differential** ds was used above.

When an infinitesimal part of the parametric curve $x = F(t)$, $y = G(t)$ of arc length ds is rotated about the x-axis, it produces an angled strip of width ds, radius y, circumference $2\pi y$, and thus with infinitesimal area given by the **surface area differential** $dS = 2\pi y\, ds$.

Thus, with appropriate limits of integration, the surface area is

$$S = \int dS = \int 2\pi y\, ds = \int_{t=\alpha}^{b} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{7.2.6}$$

Checkpoint 7.2.7 Compute the area of the "football shaped" surface produced by rotating one arch of the cycloid $(x, y) = (a(t - \sin(t)), a(1 - \cos(t)))$ about the x-axis.

Study Guide. Study Calculus Volume 2, Section 7.2[3]; in particular

- Theorems 1 2, 3
- Examples 4, 5, 6, 7, 8
- Checkpoints 4, 5, 6, 7, 8
- and one or several exercises from each of the following groupls: 62–65, 66–70, 71–74, 75–77, 88–90, 104–107 (areas under curves), 108–112 (arclengths: no need to evaluate for 112, just setup the integral).

Note that we omit the final topic of *Surface Area Generated by a Parametric Curve*

7.3 Polar Coordinates

References.

- OpenStax Calculus Volume 2, Section 7.3[1].
- *Calculus, Early Transcendentals* by Stewart, Section 10.3.

Defining Polar Coordinates. Polar coordinates describe the location of a point P in the plane in terms of

- the **polar distance** r from a reference point O, the **pole**, and
- the **polar angle** θ, describing the direction of motion from O to P relative to a direction considered horizontal.

In comparison to standard cartesian coordinates (x, y) for P, using the pole as the origin and with the positive x-axis as the horizontal direction, the polar distance is easily described:

$$r = |OP| = \sqrt{x^2 + y^2}. \tag{7.3.1}$$

[2] openstax.org/books/calculus-volume-2/pages/7-2-calculus-of-parametric-curves
[3] openstax.org/books/calculus-volume-2/pages/7-2-calculus-of-parametric-curves
[1] openstax.org/books/calculus-volume-2/pages/7-3-polar-coordinates

The angle is measured from the positive x-axis to the ray $\overrightarrow{OP}$, going in the direction towards the positive y-axis (so "anti-clockwise").

It is not so simple to give a formula for it in terms of x and y, so it helps to go the other way first:

Cartesian Coordinate Values from Polar Coordinate Values. The point P with cartesian coordinates (x, y) lines an the circle of radius r, center $(0, 0)$, so the angle θ determines its coordinates to be

$$x = r\cos\theta,\ y = r\sin\theta. \tag{7.3.2}$$

Polar Coordinate Values from Cartesian Coordinate Values. Getting r from x and y was easy. To get θ, first divide, getting $\tan\theta = \dfrac{y}{x}$, if $x \neq 0$

There are two problems: the case $x = 0$ and the multiple angles with the same tangent.

We can of course restrict the allowed θ values to "one rotation"; two favorite choices are

- $-\pi < \theta \leq \pi$ to keep the size of θ small (with a bias to positive values) and
- $0 \leq \theta < 2\pi$ to keep θ positive and still as small as possible.

However that still leaves two possible values for θ, differing by π, and using $\theta = \arctan(y/x)$ does not always give the correct value: it always gives an angle $-\pi/2 < \theta < \pi/2$ and so a point in the right half-plane.

Polar Coordinate Values from Cartesian Coordinate Values: A Solution. Often we prefer the smallest magnitude for θ, and use the value in $(-\pi, \pi]$. (Excluding $\theta = -\pi$ as it would be redundant.) Here is one way to do that:

$$\theta = \begin{cases} \arctan(y/x) & \text{if } x > 0 \text{, so } -\pi/2 < \theta < \pi/2 \\ \arctan(y/x) - \pi & \text{if } x < 0 \text{ and } y < 0 \text{, so } -\pi < \theta < -\pi/2 \\ \arctan(y/x) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \text{, so } \pi/2 < \theta \leq \pi \\ \pi/2 & \text{if } x = 0 \text{ and } y > 0 \\ -\pi/2 & \text{if } x = 0 \text{ and } y < 0 \end{cases} \tag{7.3.3}$$

This still omits the case of the pole, where $x = y = 0$ (so $r = 0$): there the polar angle is ill-defined, but the good news is that any value of θ is acceptable, in that the equations (7.3.2) give the correct cartesian coordinates.

Simpler Equations for Getting from Cartesian to Polar Coordinates. Often we can avoid these complications by working with the following simpler equations whenever possible:

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x} \text{ when } x \neq 0, \tag{7.3.4}$$

with special handling of points with $x = 0$: these are on the y-axis, so we can use

- $\theta = \pi/2$ if $y > 0$
- $\theta = -\pi/2$ or $3\pi/2$ if $y < 0$
- any θ value we want if also $y = 0$, so we are at the origin where the polar angle makes no sense.

Also, since both these equations (7.3.4) and Equations (7.3.2) giving cartesian coordinates in terms of polar coordinates make sense for all real values of r and θ, we sometimes do not restrict to $r \geq 0$, $-\pi < \theta \leq \pi$.

This more flexible approach will help below to produce elegant descriptions of some interesting **Polar Curves**.

Polar Curves. Many curves have rotational features that make them easiest to describe in terms of polar coordinates, with equations in terms of r and θ, like

$$r = f(\theta).$$

Note well: we will graph them as curves in the cartesian plane, with axes x and y!

The cartesian coordinates are given with the help of the equations (7.3.2) as

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta \tag{7.3.5}$$

so that they are a type of parametric curve, with polar angle θ as the parameter.

The most basic example is the equation $r = C$, for C a positive constant. This is a circle of radius C. Since the equation says nothing about the angle θ, it can take any value, and with the standard range of values $(-\pi, \pi]$, the angle θ becomes a convenient parameter giving a parametric description of the curve by inserting $r = C$ into the Equations (7.3.2):

$$x = C\cos\theta,\ y = C\sin\theta.$$

Another example is $r = e^\theta$, which is the exponential spiral seen in Example 7.1.7.

$$x = e^\theta\cos\theta, \quad y = e^\theta\sin\theta.$$

Example 7.3.1 The Graph of the Polar Equation $\theta = c$, c a constant. The simple equation $\theta = c$ requires a little more care to graph, and r must be used as the parameter instead of θ.

Equation (7.3.4) gives $\tan\theta = \tan c = y/x$, so $y = (\tan c)x$. This looks like the equation of a straight line through the origin, and even the cases where $\tan c$ does not exist make sense: they give the vertical line $x = 0$.

However, if we restrict to $r \geq 0$, the curve is actually only part of this line: it is the ray starting at the origin and going in the direction specified by the angle c.

The moral is that, as always with graphs and functions, ***we must specify the domain***: do we want to allow all r (and get a line), or $r \geq 0$ (and get a ray)? For example, with $r \geq 0$,

- $\theta = \pi/2$ is the positive y-axis,
- $\theta = -\pi/2$ is the negative y-axis.

□

Tangents to Polar Curves. Since any polar curve given by an equation $r = f(\theta)$ is a parametric curve as in Equation (7.3.5), there is nothing really new here, but it is worth noting the formulas for the tangent slope dy/dx:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Evaluation of this is a good case of the strategy I have been using recently, of evaluating key pieces and then gathering them is stages: the term $dr/d\theta$ should be evaluated and simplified before inserting, to avoid duplicated effort.

Symmetry in Polar Coordinates. (Omitted)

Study Guide. Study Calculus Volume 2, Section 7.3[2]

The main content relevant for us is up to Example 13 and Checkpoint 13 (we skip the topic of symmetry, but I suggest reading it).

Do one or several exercises from each of the ranges 136–141, 142–148, 154–157, and 158–160.

[2] openstax.org/books/calculus-volume-2/pages/7-3-polar-coordinates

7.4 Area and Arc Length in Polar Coordinates

References.

- OpenStax Calculus Volume 2, Section 7.4[1].
- *Calculus, Early Transcendentals* by Stewart, Section 10.4.

Areas of Regions Bounded by Polar Curves. A polar curve $r = f(\theta)$ typically encloses a region *inside* the curve, $r < f(\theta)$ rather than *below* it. That is, we are often interested in the region between the curve and the pole (origin), rather than between the curve and a horizontal axis.

Thus, the strategy for finding area as the integral of infinitesimal fragment or area differential dA will be based on looking at the thin region that lies between the curve and the pole over a narrow range of polar angle values $d\theta$: a very thin sector of radius $r = f(\theta)$ and angular extent $d\theta$

A circular sector of radius r covering angle θ has area $\frac{1}{2}r^2\theta$; thus this infinitesimal sector has area described by the area differential

$$dA = \frac{1}{2}r^2 d\theta = \frac{1}{2}[f(\theta)]^2 d\theta \tag{7.4.1}$$

If you prefer using approximations with small finite pieces, the range of angles $\theta_i \le \theta \le \theta_i + \Delta\theta$ gives a region that is approximately a sector, with area approximation

$$\Delta A_i \approx \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta \quad \text{for some angle } \theta_i^* \text{ in } [\theta_i, \theta_i + \Delta\theta]$$

The transition from finite increments to infinitesimal ones turns this into the exact formulas seen above,

$$dA = \frac{1}{2}r^2 d\theta = \frac{1}{2}[f(\theta)]^2 d\theta.$$

The familiar argument with limits and the FTC then shows that the area of the "sector" inside the curve $r = f(\theta)$, $a \le \theta \le b$ is

$$A = \int dA = \frac{1}{2}\int_{\theta=a}^{b} r^2 d\theta = \frac{1}{2}\int_{\theta=a}^{b} [f(\theta)]^2 d\theta. \tag{7.4.2}$$

Arc Length in Polar Curves. The arc length of a polar curve $r = f(\theta)$, $a \le \theta \le b$ is given by simply using the results for a parametric curve applied to

$$x = r\cos\theta, \quad y = r\sin\theta.$$

The arc length differential for this polar curve,

$$ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta,$$

simplifies nicely when combined with the formulas for the derivatives

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta + r\sin\theta$$
$$\frac{dr}{d\theta} = \frac{dr}{d\theta}\sin\theta - r\cos\theta$$

to give

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta, \tag{7.4.3}$$

[1]openstax.org/books/calculus-volume-2/pages/7-4-area-and-arc-length-in-polar-coordinates

and thus

$$L = \int ds = \int_{\theta=a}^{b} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{7.4.4}$$

Study Guide. Study Calculus Volume 2, Section 7.4[2]; in particular

- Theorems 6 and 7
- Examples 16 and 18
- Checkpoints 15 and 17
- and one or several exercises from each of the following groups: 188–194, 201–206, 214–217, 218–222.

Chapter Review. When reviewing this chapter, also look at the end of chapter review material in OpenStax Calculus Volume 2, including Key Terms[3], Key Equations[4] and Key Concepts[5].

[2] openstax.org/books/calculus-volume-2/pages/7-4-area-and-arc-length-in-polar-coordinates
[3] openstax.org/books/calculus-volume-2/pages/7-key-terms
[4] openstax.org/books/calculus-volume-2/pages/7-key-equations
[5] openstax.org/books/calculus-volume-2/pages/7-key-concepts

Appendix A

Rules for Derivatives and Integrals

A.1 Rules for Derivatives

Sums, differences, constant factors.

$$\frac{d}{dx}(kf(x)) = k\frac{df}{dx}$$
$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{df}{dx} \pm \frac{dg}{dx}$$

Products, Quotients and Compositions.

$$\frac{d}{dx}(f(x)g(x)) = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx}$$
$$\frac{d}{dx}(f(x)/g(x)) = \frac{\frac{df}{dx}g(x) - f(x)\frac{dg}{dx}}{g^2(x)}$$
$$\frac{d}{dx}(f(g(x)) = f'(g(x))\frac{dg}{dx}$$

That is, with $u = g(x), y = f(u)$,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

A.2 Rules for Integrals

Sums, differences, constant factors.

$$\int kf(x)\,dx = k\int f(x)\,dx$$
$$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Substitution.

$$\text{If } \int f(x)\,dx = F(x) + C$$

then $\int f(g(x))g'(x)\,dx = F(g(x)) + C$

That is, with $u = g(x)$,

$$\int f(u(x))\frac{du}{dx}dx = \int f(u)\,du$$

Integration by Parts.

$$\int u(x)\frac{dv}{dx}dx = u(x)v(x) - \int v(x)\frac{du}{dx}dx$$

That is, $\int u\,dv = uv - \int v\,du$

Appendix B

Calculus Formula Checklists

These checklists are for checking that you can recall the most important derivatives and antiderivatives (indefinite integrals).

The strategy I recommend is:

- *Step 1* Attempt to complete the formulas below (in pencil?).
- *Step 2* Check all your answers, and make corrections.
- *Step 3* Note well any that you got wrong, and learn the correct formulas.

A few formulas related to "cot" and "csc" are marked with (†); these are less important to memorize, but note how they relate to the corresponding "tan" and "sec" cases.

In the following, k and a are any constants.

B.1 Derivatives Checklist

$$\frac{d}{dx}x^a = \qquad \frac{d}{dx}\sqrt{x} =$$

$$\frac{d}{dx}\sin x = \qquad \frac{d}{dx}\cos x =$$

$$\frac{d}{dx}\tan x = \qquad \frac{d}{dx}\sec x =$$

$$(\dagger)\frac{d}{dx}\cot x = \qquad (\dagger)\frac{d}{dx}\csc x =$$

$$\frac{d}{dx}e^x = \qquad \text{For } a > 0,\ \frac{d}{dx}a^x =$$

$$\text{For } x > 0,\ \frac{d}{dx}\ln x = \qquad \text{For } x \neq 0,\ \frac{d}{dx}\ln|x| =$$

$$\frac{d}{dx}\arcsin x = \frac{d}{dx}\sin^{-1} x = \qquad \frac{d}{dx}\arccos x = \frac{d}{dx}\cos^{-1} x =$$

$$\frac{d}{dx}\tan^{-1} x = \qquad \frac{d}{dx}\sec^{-1} x =$$

$$(\dagger)\frac{d}{dx}\cot^{-1} x = \qquad (\dagger)\frac{d}{dx}\csc^{-1} x =$$

$$\frac{d}{dx}\sinh = \qquad \frac{d}{dx}\cosh x =$$

$$\frac{d}{dx}\tanh = \qquad \frac{d}{dx}\text{sech}\, x =$$

$$(\dagger)\frac{d}{dx}\coth = \qquad (\dagger)\frac{d}{dx}\text{csch}\, x =$$

$$\frac{d}{dx}\sinh^{-1} x = \qquad \frac{d}{dx}\cosh^{-1} x =$$

$$\frac{d}{dx}\tanh^{-1} x = \qquad \frac{d}{dx}\text{sech}^{-1} x =$$

$$(\dagger)\frac{d}{dx}\coth^{-1} x = \qquad (\dagger)\frac{d}{dx}\text{csch}^{-1} x =$$

B.2 Integrals Checklist

$$\int k\, dx =$$

For $a \neq 1, \int x^a\, dx =$, but $\int \frac{1}{x}\, dx = \int x^{-1}\, dx =$

$$\int e^x\, dx = \qquad \int a^x\, dx =$$

$$\int \sin x\, dx = \qquad \int \cos x\, dx =$$

$$\int \tan x\, dx = \qquad \int \sec x\, dx =$$

$$(\dagger)\int \cot x\, dx = \qquad (\dagger)\int \csc x\, dx =$$

$$\int \sec^2 x\, dx = \qquad \int \sec x \tan x\, dx =$$

$$(\dagger)\int \csc^2 x\, dx = \qquad (\dagger)\int \csc x \cot x\, dx =$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx =$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \qquad \int \frac{1}{a^2 + x^2}\, dx =$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \qquad \int \frac{1}{x^2 - a^2} dx =$$

$$\int \sinh x\, dx = \qquad \int \cosh x\, dx =$$

$$\int \tanh x\, dx = \qquad \int \text{sech}\, x\, dx =$$

Appendix C

Reduction Formulas For Integrals

This is a collection of reduction formulas; for a more comprehensive list of integrals, see OpenStax Calculus Volume 2, Appendix A: Table of Integrals.[1]

C.1 Integrals Involving Exponential or Trigonometric Functions

$$\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du \tag{C.1.1}$$

$$\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1}u\cos u + \frac{n-1}{n}\int \sin^{n-2}u\,du \tag{C.1.2}$$

$$\int \cos^n u\,du = \frac{1}{n}\cos^{n-1}u\sin u + \frac{n-1}{n}\int \cos^{n-2}u\,du \tag{C.1.3}$$

$$\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1}u - \int \tan^{n-2}u\,du \tag{C.1.4}$$

$$\int \cot^n u\,du = \frac{-1}{n-1}\cot^{n-1}u - \int \cot^{n-2}u\,du \tag{C.1.5}$$

$$\int \sec^n u\,du = \frac{1}{n-1}\tan u\sec^{n-2}u + \frac{n-2}{n-1}\int \sec^{n-2}u\,du \tag{C.1.6}$$

$$\int \csc^n u\,du = \frac{-1}{n-1}\cot u\csc^{n-2}u + \frac{n-2}{n-1}\int \csc^{n-2}u\,du \tag{C.1.7}$$

$$\int u^n\sin u\,du = -u^n\cos u + n\int u^{n-1}\cos u\,du \tag{C.1.8}$$

$$\int u^n\cos u\,du = u^n\sin u - n\int u^{n-1}\sin u\,du \tag{C.1.9}$$

$$\int \sin^n u\cos^m u\,du = \int (1-\cos^2 u)^k\cos^n u\sin u\,du = -\int (1-v^2)^k v^n\,dv,\ m=2k+1,\ v=\cos u \tag{C.1.10}$$

$$\int \sin^n u\cos^m u\,du = \int (1-\sin^2 u)^k\sin^m u\cos u\,du = \int (1-v^2)^k v^m\,dv,\ n=2k+1,\ v=\sin u \tag{C.1.11}$$

$$\int \sin^n u\cos^m u\,du = \int\left(\frac{1-\cos 2u}{2}\right)^{k_1}\left(\frac{1+\cos 2u}{2}\right)^{k_2},\ n=2k_1,\ m=2k_2 \tag{C.1.12}$$

[1] openstax.org/books/calculus-volume-2/pages/a-table-of-integrals

$$\int \sin^n u \cos^m u \, du = -\frac{\sin^{n-1} u \cos^{m+1} u}{n+m} - \frac{n-1}{n+m} \int \sin^{n-2} u \cos^m u \, du, \; n \geq 2 \tag{C.1.13}$$

$$\int \sin^n u \cos^m u \, du = \frac{\sin^{n+1} u \cos^{m-1} u}{n+m} + \frac{m-1}{n+m} \int \sin^n u \cos^{m-2} u \, du, , \; m \geq 2 \tag{C.1.14}$$

C.2 Integrals Involving Inverse Trigonometric Functions

$$\int u^n \sin^{-1} u \, du = \frac{1}{n+1} \left[u^{n+1} \sin^{-1} u - \int \frac{u^{n+1} du}{\sqrt{1-u^2}} \right], \; n \neq -1 \tag{C.2.1}$$

$$\int u^n \cos^{-1} u \, du = \frac{1}{n+1} \left[u^{n+1} \cos^{-1} u + \int \frac{u^{n+1} du}{\sqrt{1-u^2}} \right], \; n \neq -1 \tag{C.2.2}$$

$$\int u^n \tan^{-1} u \, du = \frac{1}{n+1} \left[u^{n+1} \tan^{-1} u + \int \frac{u^{n+1} du}{1+u^2} \right], \; n \neq -1 \tag{C.2.3}$$

C.3 Integrals Involving $\sqrt{a+bu}$

$$\int u^n \sqrt{a+bu} \, du = \frac{2}{b(2n+3)} \left[u^n (a+bu)^{3/2} - na \int u^{n-1} \sqrt{a+bu} \, du \right] \tag{C.3.1}$$

$$\int \frac{u^n}{\sqrt{a+bu}} \, du = \frac{2u^n \sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)} \int \frac{u^{n-1}}{\sqrt{a+bu}} \, du \tag{C.3.2}$$

$$\int \frac{1}{u^n \sqrt{a+bu}} \, du = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)} \int \frac{1}{u^{n-1} \sqrt{a+bu}} \, du \tag{C.3.3}$$

Appendix D

Some Power Series

Reference. OpenStax Calculus Volume 2, Section 6.4[1]

Function	Maclaurin Series	Interval and Radius of Convergence
$f(x) = \dfrac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$, $R = 1$
$f(x) = e^x$	$\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$	$-\infty < x < \infty$, $R = \infty$
$f(x) = \ln(1+x)$	$\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n} x^n$	$-1 < x \leq 1$, $R = 1$
$f(x) = \sin(x)$	$\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!} x^{2n+1}$	$-\infty < x < \infty$, $R = \infty$
$f(x) = \cos(x)$	$\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n)!} x^{2n}$	$-\infty < x < \infty$, $R = \infty$
$f(x) = \arctan(x)$	$\sum_{n=0}^{\infty} \dfrac{(-1)^n}{2n+1} x^{2n+1}$	$-1 \leq x \leq 1$, $R = 1$
$f(x) = (1+x)^r$	$\sum_{n=1}^{\infty} \binom{r}{n} x^n$	$-1 < x < 1$, $R = 1$ (see note)
	where $\binom{r}{n} = \dfrac{r(r-1)\cdots(r-n+1)}{n!}$	

Note: the Binomial Series for $(1+x)^r$ also converges for all x if r is a natural number (where the series is just a polynomial), and also can converge at the endpoints $x = \pm 1$ for some other values of r.

And all of these come from

Theorem D.0.1 Taylor's Theorem. *For a Taylor polynomial $T_n(x)$ for f with center a, and any number x in the domain of f, there is a number c between x and a such that*

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1} \tag{D.0.1}$$

and

Theorem D.0.2 Taylor's Inequality, and convergence to $f(x)$, *sometimes*. *If for some numbers M and $d > 0$,*

[1] openstax.org/books/calculus-volume-2/pages/6-4-working-with-taylor-series

$|x-a| \leq d$ *ensures that* $|f^{(n)}(x)| \leq M$, *for all* n, *then the remainder of the Taylor series for* f *with center* a *satisfies*

$$|R_N(x)| \leq \frac{M}{(N+1)!}|x-a|^{N+1} \quad \textit{for } |x-a| \leq d. \tag{D.0.2}$$

If this is true, then for $|x-a| \leq d$, $|R_N(x)| \to 0$ *as* $N \to \infty$, *so* $T_N(x) \to f(x)$, *so that the value of the Taylor Series is* $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \cdots \tag{D.0.3}$$

Appendix E

Some Trigonometry

$$
\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
1 + \tan^2 x &= \sec^2 x \\
\cos^2(ax) &= \frac{1}{2}[1 + \cos(2ax)] \\
\sin^2(ax) &= \frac{1}{2}[1 - \cos(2ax)] \\
\sin(ax)\cos(ax) &= \frac{1}{2}\sin(2ax) \\
\sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y \\
\cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \\
\cos x \cos y &= \frac{1}{2}\left[\cos(x - y) + \cos(x + y)\right] \\
\sin x \sin y &= \frac{1}{2}\left[\cos(x - y) - \cos(x + y)\right] \\
\sin x \cos y &= \frac{1}{2}\left[\sin(x - y) + \sin(x + y)\right]
\end{aligned}
$$

Values at key angles. In the first two quadrants, the main values with simple forms are

θ	$0(\rightarrow)$	$\frac{\pi}{6}$	$\frac{\pi}{4}\,(\nearrow)$	$\frac{\pi}{3}$	$\frac{\pi}{2}\,(\uparrow)$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}(\nwarrow)$	$\frac{5\pi}{6}$	$\pi(\leftarrow)$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	DNE	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

When sketching curves, it can help to know some numerical values:

$$\sqrt{2} \approx 1.4142 \quad \text{and} \quad \sqrt{3} \approx 1.7321$$

leading to

$$\frac{1}{\sqrt{2}} \approx 0.7071, \quad \frac{1}{\sqrt{3}} \approx 0.5774, \quad \text{and} \quad \frac{\sqrt{3}}{2} \approx 0.8660.$$

Appendix F

Strategy for Evaluating Integrals

F.1 A few general tactics

In trying to evaluate an integral with the collection of methods we have seen in Chapters 1 and 3, some general ideas are worth keeping in mind:

Find antiderivatives first It is often best to start by seeking any one antiderivative $F(x)$ for the function involved and deal later with the constant of integration in an indefinite integral and the limits of a definite integral.
However when using *substitutions on a definite integral*, you can avoid substituting back to the original variable by instead changing the limits of integration to the corresponding values for the new variable.

Start with the easiest possibilities Several techniques might apply to one integral, so start try the easiest first (recognizing a previously known integral, from memory or tables) and work through to the more sophisticated options (like inverse trigonometric substitutions and integration by parts.)

Repeat as necessary All methods except recognizing a known integral give one or several new simpler integrals, so the process needs to be applied repeatedly until every part of the answer is found by reducing to a previously known integral.

F.2 A detailed strategy

Here is one detailed approach to evaluating integrals. It is based on the discussion in Section 3.5 and covers all the methods we have seen in class. For practice and test review, I strongly recommend selecting exercises from Section 3.5, so that you must decide what methods to use as well as then applying the methods correctly: that is the way integration is in real life.

F.2.1 Use tables of integrals and known integrals

The easiest approach is to recognize an integral as one you have already worked out how to evaluate. The corollary of this is that you should memorize the most common integrals, and collect the ones that you encounter often but have not memorized (yet) on a *formula sheet*. The formulas should be as flexible as possible with adjustable constants to avoid routine substitutions: for example, not $\int \cos x \; dx = \sin x + C$ but instead

$$\int \cos ax \; dx = \frac{1}{a} \sin ax + C$$

F.2.2 Do basic simplifications

Simplify first is a good strategy in many mathematical situations: try to simplify the function involved before starting on the calculus itself.

The most general basic simplifications are breaking up *sums and differences* into separate integrals, taking *constant factors* out in front of each integral (including division by constant factors), and *rewriting roots as fractional powers*.

It is also often useful to eliminate divisions by rewriting powers in the denominator as negative powers in the numerator, and using trig identities like converting a factor $\cos x$ in the denominator into a factor $\sec x$ in the numerator.

For example, $\int \frac{x^2}{7} - \frac{1}{\sin 2x} + \frac{5x}{\sqrt[3]{1+x^2}}\ dx$ could be rewritten as

$$\frac{1}{7}\int x^2\ dx - \int \csc 2x\ dx + 5\int x(1+x^2)^{-1/3}\ dx$$

One important special type of simplification is used with integrals of *products of powers of trigonometric functions*, which will be discussed below.

F.2.3 Substitution

If the above steps do not give the solution, the easiest of the two most powerful general tools is *substitution*, especially with some compositions and products. That is, finding a function $u(x)$ so that the integrand is in the form $f(u)u'(x)$, or $f(u)\frac{du}{dx}$. Then you can use the ``cancellation of differentials'' idea

$$\int f(u)\frac{du}{dx}\ dx = \int f(u)\ du$$

to get a new, simpler integration problem.

That puts you back at the beginning with a new hopefully simpler integral. If the new integral is *not* easier to evaluate, the substitution was not useful, so try something else: another substitution or a different method.

F.2.4 Choosing a substitution function $u(x)$

As u will often appear inside a composition, one common choice is *the function inside a power or other composition.* In the example above of $5x(1+x^2)^{-1/3}$, you could try $u = 1+x^2$.

You must then check if the rest of the term is the derivative of u, up to a constant factor. In the current example $du/dx = 2x$ and the remainder of the term apart from the power of u is $5x$, which does match the derivative up to a harmless constant factor of $5/2$, so this substitution will work. (Exercise: complete this integral.)

In general, you must look to see if the term to be integrated consists of just the derivative of u times some expression that can be put in terms of u only (with no stray x terms.) For $\int \sin(x)\cos(x)\ dx$, the substitution $u = \sin(x)$ gives

$$\int u\frac{du}{dx}\ dx = \int u\ du = \frac{u^2}{2} + C = \frac{1}{2}\sin^2(x) + C$$

and also substitution $u = \cos(x)$ gives

$$\int -u\frac{du}{dx}\ dx = -\int u\ du = -\frac{u^2}{2} + C = -\frac{1}{2}\cos^2(x) + C$$

which despite initial appearances ("$u^2/2$" vs "$-u^2/2$") agree, due to $\cos^2(x) + \sin^2(x) = 1$.

This shows that there can be more than one useful substitution.

F.2.5 Integration by Parts

Our second and last general tool is Integration by Parts, as summarized by

$$\int u\,dv = uv - \int v\,du.$$

Here u and v are both functions of the actual integration variable x, so that in more detail the rule is

$$\int u(x)\frac{dv}{dx}\,dx = u(x)v(x) - \int v(x)\frac{du}{dx}\,dx$$

Note that if one does a definite integral directly, the formula becomes

$$\int_{x=a}^{x=b} u\,dv = \Big[uv\Big]_{x=a}^{x=b} - \int_{x=a}^{x=b} v\,du$$

The word "parts" refers to the fact that only one part of the integrand gets integrated, at least initially: dv/dx is integrated to find v, while u gets differentiated. The main hint that this method might be useful is that the function is a product of several functions and we know how to integrate at least one of them: particularly, powers of x, trigonometric functions, exponentials, logarithms, and inverses of familiar functions. There are always many different ways to choose which factor is to be the factor u to be differentiated (with all the rest of the integrand going into the factor dv/dx to be integrated; or in other words into the differential dv). Some guidelines are

- Try to integrate the most complicated part you can.
- It it necessary that you can integrate the function that goes into the differential dv.
- It is desirable for the function u to have a simple derivative, and three very common choices are positive integer powers of x, logarithms, and any inverses of familiar functions.

F.2.6 Inverse Substitution, especially with trigonometric functions

All substitutions can be done in inverse form, where one specifies $x = g(u)$, $dx = g'(u)du$ instead of $u = f(x)$, $du = f'(x)dx$: g is the inverse of f. This has the great advantage that the formulas for x and dx automatically put everything in terms of the new variable u: there are never any stray x terms. Of course, the inverse g might be a messier function to work with, so this method is at its best when f is itself the inverse of a familiar function, and the most common examples are when the forward substitution function f is an inverse trigonometric function, so that g is a basic trigonometric function: therefore, the new variable is traditionally called θ instead of u. The three main cases are

- For an integral containing integer powers of $\sqrt{a^2 - x^2}$ and of x, use $x = a\sin\theta$, so $dx = a\cos\theta\,d\theta$ and $\sqrt{a^2 - x^2} = a\cos\theta$.
- For an integral containing integer powers of $\sqrt{a^2 + x^2}$ and of x, use\\ $x = a\tan\theta$, so $dx = a\sec^2\theta\,d\theta$ and $\sqrt{a^2 + x^2} = a\sec\theta$.
- For an integral containing integer powers of $\sqrt{x^2 - a^2}$ and of x, use\\ $x = a\sec\theta$, so $dx = a\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2 - a^2} = a\tan\theta$.

In each case, an appropriate right-triangle summarizes all the formulas needed. The resulting integrals involve products of powers of trigonometric functions, and often the methods of section~F below are needed to evaluate them. In that case you can get solutions involving sine and coinse on multiples of a new variable θ. These can be put in erms of just $\sin\theta$ and $\cos\theta$ using the double angle formulas

$$\sin 2\theta = 2\sin\theta\cos\theta, \qquad \cos 2\theta = 2\cos^2\theta - 1, \quad = 1 - 2\sin^2\theta$$

F.2.7 Special simplifications and substitutions for products of trigonometric functions

It often helps to convert into an expression with just *sines and cosines* using $\tan x = \sin x / \cos x$ and so on. The example above of $\int \sin x \cos x dx$ illustrates some important special trigonometric substitutions: With *products of integer powers of sine and cosine*

- if there is an *odd power of cosine* one can gather a term like $\cos x dx$ at the end and use the substitution $\sin x = u$, $\cos x\, dx = du$, leaving over an even power of $\cos x$ which can be written as an integer power of $\cos^2 x$ which in turn gets converted with

  $$\cos^2 x = 1 - \sin^2 x = 1 - u^2$$

  Then all the remaining x terms are in terms of $\sin x$, which becomes u, so you get an integral of a polynomial (easy), or rational function if there were some negative powers (see below).

- if there is an *odd power of sine*, one can gather a term like $\sin x dx$ and use the substitution {\boldmath $\cos x = u$, $\sin x\, dx = -du$}, and deal with the remaining even power of $\sin x$ using

  $$\sin^2 x = 1 - \cos^2 x = 1 - u^2.$$

- if there are *products of even powers of both sine and cosine* (including the case where only one of these functions is present, like $\sin^4 x$), one can reduce the powers using the half-angle formulas

  $$\cos^2 x = \frac{1}{2}(1 + \cos 2x), \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

  Then expand, simplify, and if necessary apply further trigonometric simplifications to some parts. Eventually you will get an odd power of sine or cosine in each term, plus a constant. For example

  $$\begin{aligned} \sin^2 x \cos^2 x &= \frac{1}{2}(1 - \cos 2x)\frac{1}{2}(1 + \cos 2x) \\ &= \frac{1}{4}(1 - \cos^2 2x) \\ &= \frac{1}{4}\sin^2 2x \\ &= \frac{1}{4} \cdot \frac{1}{2}(1 - \cos 4x) \\ &= \frac{1}{8} - \frac{1}{8}\cos 4x \end{aligned}$$

F.2.8 Integration of rational functions (ratios of polynomials)

It is possible to integrate any rational function $f(x) = \dfrac{P(x)}{Q(x)}$ if you can factorize the polynomial $Q(x)$ in the denominator into linear factors $(x - r)$ from roots plus irreducible quadratic factors $x^2 + bx + c$. *Irreducible* means that the quadratic has no real roots, which by the discriminant test means that $b^2 < 4c$.

It is easy and convenient to divide $P(x)$ and $Q(x)$ by a constant so that the lead coefficient in $Q(x)$ is one; this form is assumed from now on.

Also, if the numerator has the same or higher degree as the denominator, the function should first be simplified to the sum of a polynomial plus a *proper rational function*, one with numerator of lower degree than the denominator. This is done by *synthetic division of polynomials*.

The integration is done by rewriting the rational function as the sum of a polynomial plus terms that can be integrated with a few basic rules. The main integration rules needed are

$$\int \frac{du}{u - r} = \ln|u - r| + C$$

$$\int \frac{du}{(u-r)^n} = \frac{-1}{n-1}\frac{1}{(u-r)^{n-1}} + C, \quad n > 1$$
$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$$
$$\int \frac{u}{a^2+u^2}du = \frac{1}{2}\ln(a^2+u^2) + C$$

plus occasionally *completing the square* to get into the latter two forms.

1. The first step is to *convert to the case of a proper rational function*, one with the polynomial $P(x)$ in the numerator of degree less than $Q(x)$, if this is not already true.

   Example F.2.1
   $$\frac{x^3+3x+1}{x^2-4} = \frac{x(x^2-4)+7x+1}{x^2-4} = x + \frac{7x+1}{x^2-4}.$$
   □

   The polynomial part is easy to integrate, so I will deal from now on only with proper rational functions.

2. Next, *factorize the denominator* into linear and irreducible quadratic factors.

   Example F.2.2 The denominator in Example 1 has roots 2 and -2, and so
   $$\frac{7x+1}{x^2-4} = \frac{7x+1}{(x-2)(x+2)}.$$
   □

   Sometimes the factorization will have to involve quadratic factors that have no real roots and so cannot be written in terms of two linear factors. This is true if the discriminant $b^2-4c<0$.

   For an irreducible quadratic factor x^2+bx+c, complete the square to get the form $x^2+bx+c = (x+e)^2+a^2$, where $e=b/2$, $a^2=c-b^2/4$.

   Example F.2.3 $\frac{13}{x^3-4x^2+13x}$. The denominator here has only one real root 0:
   $$\frac{13}{x^3-4x^2+13x} = \frac{13}{x(x^2-4x+13)} = \frac{13}{x[(x-2)^2+3^2]}.$$
   □

3. Next the rational function can be *written as a sum of constant multiples of simple functions*, for which we know the integrals. The simple functions needed are as follows

   - For a factor $(x-r)$ in the denominator, a term like
   $$\frac{A}{x-r}$$
   - For a factor $(x-r)^n$ in the denominator, n terms
   $$\frac{A_1}{x-r}, \quad \frac{A_2}{(x-r)^2}, \text{ and up to } \frac{A_n}{(x-r)^n} \text{ (the power in the original denominator)}$$
   - For an *irreducible* quadratic factor $(x+e)^2+a^2$ in the denominator, two terms, which can be put together as
   $$\frac{A+Bx}{a^2+(x+b)^2}$$

- For the most complicated case of a repeated irreducible quadratic factor like $[a^2 + (x + b)^2]^n$, put together the previous two ideas: a succession of terms each with a linear factor like $A + Bx$ on top, and powers of $a^2 + (x + b)^2$ on the bottom, ranging up to the same n-th power as in the original denominator.

Example F.2.4 For Examples 2 and 3,

$$\frac{7x+1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}.$$

and

$$\frac{13}{x[(x-2)^2+3^2]} = \frac{A}{x} + \frac{B+Cx}{(x-2)^2+3^2}$$

□

4. The constants in the expansion can be determined, and the first step is to *clear the denominator:* multiply through by the denominator $Q(x)$ of the original rational function to get an equation between two polynomials.

Example F.2.5 For the two examples above we get

$$7x + 1 = A(x + 2) + B(x - 2)$$

and

$$13 = A[(x - 2)^2 + 3^2] + (B + Cx)x.$$

□

5. To *find the numerical values of the constants in this equation*, one method (not my favorite) is to expand out the right hand side into multiples of powers of x, including a constant, set the coefficients of each power of x equal to those at left, and solve the resulting simultaneous equations for the constants A, B, etc.

However this equation solving can be made easier or avoided entirely by the strategy of **strategic substitution**: first looking at the easier equations you get by *substituting each root of the denominator $Q(x)$ in for x.*

When there are repeated factors or irreducible quadratic factors, this substitution method will only give some of the constants. To get the others, one can seek other easy equations by substituting in a few other "nice" integer values such as $x = 0, 1, -1, \ldots$.

Example F.2.6 For the first example function above, $x = -2$ and $x = 2$ give

$$-13 = B(-4), \quad 15 = A(4)$$

which immediately give $B = 13/4$, $A = 15/4$:

$$\frac{7x+1}{(x^2-4)} = \frac{7x+1}{(x-2)(x+2)} = \frac{15/4}{x-2} + \frac{13/4}{x+2}.$$

□

Example F.2.7 For the second example, $x = 0$ is the only root to substitute in, giving $13 = A[4 + 9]$ so $A = 1$. Using this A value and substituting also $x = 1$ and $x = -1$ gives

$$13 = [1 + 9] + (B + C), 13 = [9 + 9] - (B - C)$$

or $B + C = 3$, $B - C = 5$, with the solution $B = 4$, $C = -1$:

$$\frac{13}{x^3-4x^2+13x} = \frac{13}{x[(x-2)^2+3^2]} = \frac{1}{x} + \frac{4-x}{(x-2)^2+3^2}.$$

The integral of this can be found using the four basic forms above plus a substitution $u = x - 2$. (You get $\ln|x| + \arctan((x-2)^2 + 3^2) - \frac{1}{2}\ln((x-2)^2 + 3^2) + C$.) □

6. Finally, you can evaluate the resulting integrals, using the four basic integrals above:

$$\int \frac{7x+1}{(x^2-4)}\,dx = \int \frac{15/4}{x-2} + \frac{13/4}{x+2}\,dx = \frac{15}{4}\ln|x-2| + \frac{13}{4}\ln|x+2| + C$$

and

$$\int \frac{13}{x^3 - 4x^2 + 13x}\,dx = \int \frac{1}{dx} + \frac{4-x}{(x-2)^2+3^2}\,dx = \ln|x| + \arctan((x-2)^2+3^2) - \frac{1}{2}\ln((x-2)^2+3^2) + C$$

where the substitution $u = x - 2$ is needed for the last term.

F.2.9 Final steps: make sure that you answer the original question

Remember a few things that must be done before you are finished:

- Substitute if necessary to get the solution *in terms of the original variable*.
- For an *indefinite integral*, *add the constant of integration*: if $F(x)$ is an antiderivative of $f(x)$,

$$\int f(x)\,dx = F(x) + C.$$

- For a *definite integral*, *evaluate* at the limits of integration:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

- For a *definite integral using substitution* $u = g(x)$, you might instead do the evaluation in the new u variable to avoid converting back to the original variable:

$$\int_{x=a}^{x=b} f(g(x))g'(x)\,dx = \int_{u=c}^{u=d} f(u)\,du,\ c = g(a),\ d = g(b).$$

- Similarly, for a *definite integral using inverse substitution* $x = h(t)$, you might do the evaluation in the new variable using.

$$\int_{x=a}^{x=b} f(x)\,dx = \int_{t=c}^{t=d} f(h(t))h'(t)\,dt,$$

But you have to solve equations to get the new limits, c from a and d from b: $h(c) = a$, $h(d) = b$.

Appendix G

Some Formulas Worth Knowing

It is worth knowing all the Differentiation Rules from the Reference Pages at the back of the text, and all the Basic Forms for integrals, items 1 to 20 in the Table of Integrals in those Reference Pages. Appendix A is a guide to checking your knowledge of these.

Beyond that, here are some important formulas, mostly integrals.

There are some useful formulas not given here but better reviewed in the notes where their usage is explained. In particular see Section 7.4 on integrating rational functions and 7.8 on approximate (numerical) integration.

$$\begin{aligned}
\cosh x &= \frac{e^x + e^{-x}}{2}, \ \sinh x = \frac{e^x - e^{-x}}{2}. \\
\tanh x &= \frac{\sinh x}{\cosh x}, \text{ etc.} \\
\cos^2(ax) &= \frac{1}{2}[1 + \cos(2ax)] \\
\sin^2(ax) &= \frac{1}{2}[1 - \cos(2ax)] \\
\sin(ax)\cos(ax) &= \frac{1}{2}\sin(2ax) \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 \cdots, |x| <; 1
\end{aligned}$$

The area between curves $y = g(x) \leq f(x)$, $a \leq x \leq b$. This has infinitesimal *area differential*

$$dA = (f(x) - g(x))dx$$

giving area integral

$$A = \int dA = \int_{x=a}^{b} (f(x) - g(x))dx$$

The solid produced by rotating curve $y = f(x)$ about the horizontal axis. This solid can be sliced perpendicular to the x-axis into disks with volume differential

$$dV = (\text{disk area})dx = \pi r^2 dx = \pi y^2 dx$$

giving volume integral

$$V = \int dV = \int_{x=a}^{b} \pi r^2 dx = \int_{x=a}^{b} \pi y^2 dx = \int_{x=a}^{b} \pi [f(x)]^2 dx$$

As always I recommend using the versions in terms of variables like x and y or geometric quantities like the radius r, without using function names like $f(x)$: you will know the formula for y, or be able to work out the relevant formula for the radius r.

The solid produced by rotating the region $g(x) \leq y \leq f(x)$, $a \leq x \leq b$ about the horizontal axis. This can be sliced perpendicular to the x-axis into annuli with volume differential

$$dV = (\text{annulus area})dx = \pi R^2 - \pi r^2 dx,\ R = f(x) > r = g(x)$$

giving volume integral

$$V = \int dV = \pi \int_{x=a}^{b} R^2 - r^2 dx$$

The solid produced by rotating the region $g(x) \leq y \leq f(x)$, $0 \leq a \leq x \leq b$ about the vertical axis. This can be sliced into cylindrical shells of volume differential

$$dV = (\text{shell circumference})(\text{shell height})dx = 2\pi r[f(x) - g(x)]dx = 2\pi x[f(x) - g(x)]dx$$

(since the radius of each shell is $r = x \geq 0$) giving volume integral

$$V = \int dV = \int_{x=a}^{b} 2\pi x[f(x) - g(x)]dx$$

The average value of the function $f(x)$ on interval $a \leq x \leq b$.

$$\bar{f} = f_{ave} = \frac{1}{b-a}\int_{x=a}^{b} f(x)\,dx$$

Substitution in a Definite Integral with $u = g(x)$.

$$\int_{x=a}^{b} f(u(x))\frac{du}{dx}dx = \int_{u=c}^{d} f(u)du,\ c = g(a),\ d = g(b)$$

Note that the limits of integration change, from x values to u values.

Integration by Parts.

$$\int u\,dv = uv - \int v\,du \text{ or } \int u(x)\frac{dv}{dx} = u(x)v(x) - \int v(x)\frac{du}{dx}\,dx$$

With definite integrals, this becomes

$$\int_{x=a}^{b} u\,dv = [uv]_{x=a}^{b} - \int_{x=a}^{b} v\,du$$

Note that the limits of integration are still x values, as du and dv are shorthands $du = \frac{du}{dx}dx$ and $dv = \frac{dv}{dx}dx$: the dummy variable in the integration is still x.

For Integrals Involving the Square Root of a Quadratic.

- With $\sqrt{a^2 - x^2}$, try $x = a\sin\theta$ so that $\sqrt{a^2 - x^2} = a\cos\theta$, $dx = a\cos\theta\,d\theta$.
- With $\sqrt{a^2 + x^2}$, try $x = a\tan\theta$ so that $\sqrt{a^2 + x^2} = a\sec\theta$, $dx = a\sec^2\theta\,d\theta$.
- With $\sqrt{x^2 - a^2}$, try $x = a\sec\theta$ so that $\sqrt{x^2 - a^2} = a\tan\theta$, $dx = a\tan\theta\sec\theta\,d\theta$.

For more general quadratics, first extract the coefficient of x^2 as a common factor and then complete the square

$$x^2 + bx + c = [x + b/2]^2 + c - (b/2)^2$$

Then the substitution $u = x + b/2$ is useful, and $a^2 = c - (b/2)^2$ or $(b/2)^2 - c$, whichever is positive.

For a Separable Differential Equation $\frac{dy}{dx} = f(x)g(y)$**.** Write in differential form

$$\frac{1}{g(y)}dy = f(x)dx$$

and integrate, but beware of division by zero: check the case $g(y) = 0$.

The Arc Length of the Curve $y = f(x)$**,** $a \le x \le b$**.** The *arc length differential* is

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

so the total arc length is

$$L = \int ds = \int_{x=a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

The parametric curve $x = f(t)$**,** $y = f(t)$**.** This has slope at point $P(x(t), y(t))$ given by

$$m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

but beware of division by zero, whihc happens at points where $\frac{dx}{dt} = 0$. There, the above formula needs to be simplified before using it.

The arc length of the above parametric curve.. For $\alpha \le t \le \beta$ the **arc length differential** is

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

so the total arc length is

$$L = \int ds = \int_{t=\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Polar coordinates. To convert between polar coordinates r, θ and cartesian (rectangular) coordinates x, y, use

$$x = r\cos\theta,\ y = r\sin\theta,$$
$$r^2 = x^2 + y^2,\ \tan\theta = \frac{y}{x}$$

with special handling of points on the y-axis: the ones with $x = 0$.

The polar curve $r = f(\theta)$**.** This is a case of parametric curve, with angle θ as the parameter and

$$x = r\cos\theta = f(\theta)\cos\theta,\ y = r\sin\theta = g(\theta)\sin\theta$$

The slope of polar curve $r = f(\theta)$**.** This is given by combining formulas above, but simplifies to

$$m = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

The area of the region inside polar curve $r = f(\theta)$, $a \le \theta \le b$. This has area differential

$$dA = \frac{1}{2}r^2 d\theta = \frac{1}{2}(f(\theta))^2 d\theta$$

so the area is

$$A = \int dA = \int_{\theta=a}^{b} \frac{1}{2}r^2 d\theta = \frac{1}{2}\int_{\theta=a}^{b} (f(\theta))^2\, d\theta.$$

The arc length of the above polar curve. This can be derived using formulas above for parametric curves but simplifies nicely: the *arc length differential* is

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

so the total arc length is

$$L = \int ds = \int_{\theta=a}^{b} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Integral error bound for the series $S = \sum a_n$ with $a_n = f(n)$, f positive and decreasing. The *error* R_N for the partial sum $S_N = \sum_{n=1}^{N} a_n$ as an approximation of $S = \sum a_n$ has a size limit

$$0 \le R_N = S - S_N \le \int_N^\infty f(n)\, dn$$

The Taylor Series.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 \cdots$$

and the N-the degree Taylor polynomial

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \cdots \frac{f^{(N)}(a)}{N!}(x-a)^N,$$

the *remainder*

$$R_N(x) = f(x) - T_N(x)$$

has an upper size limit

$$|R_N(x)| \le \frac{M}{(N+1)!}|x-a|^{N+1}$$

if you have a number M such that $|f^{(n)}(x)| \le M$. If the later is only true for some x, $|x-a| \le d$, then the limit on $R_N(x)$ is also only for these x values.