Notes for Introductory Calculus (Math 120)

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[1] creativecommons.org/licenses/by-sa/4.0/

Introduction

These notes are an adjunct to the open source text for the course MATH 120 *Introductory Calculus*, Calculus, Volume 1[2] from OpenStax[3] They summarize the key ideas, examples and results from each section of that text that we will cover, along with additional examples and recommended homework exercises.

We start by restating some key ideal and objectives from the syllabus; see there for more about course organization, important dates, assessment, and so on.

Course Objectives and Student Learning Outcomes. The main goal of this course is for students to learn the basic concepts and skills of solving mathematical and scientific problems described by functions that vary "smoothly" (with no jumps, breaks or sharp corners in their graphs), and to solve problems whose solutions can at best only be approximated with algebra, geometry and trigonometry (like the areas of most regions), but can be solved exactly with the methods of calculus.

Applications include the description of motion in terms of velocity and acceleration, models of population growth, chemical reaction rates and growth of the value of an investment, and optimization problems such as minimizing the cost of a task or maximizing what can be achieved with a fixed amount of resources. This material is covered in the first five chapters of the text, with a few sections omitted or left until Calculus 2 (Math 220).

Students are expected to do not only the quizzes, assignments and class exercises, but also to review each section of the text after it has been covered in class and to attempt the exercises set for each section. This is because, more broadly, *a majority of the learning in this or any college course comes through students' efforts outside of class meetings.*

By the end of the course, students should be able to:

- Calculate a wide variety of limits, including derivatives using the limit definition and limits computed using l'Hospital's rule;
- Demonstrate understanding of the main theorems of one-variable calculus (including the Intermediate and Mean Value Theorems, and the Fundamental Theorem of Calculus) by using them to answer questions;
- Compute derivatives of functions with formulas involving elementary polynomial, rational, trigonometric, inverse trigonometric, exponential and logarithmic functions;
- Use information about the derivative(s) or antiderivative of a function (in graphical or symbolic form) to understand a function's behavior and sketch its graph;
- Construct models and use them to solve related rates and optimization problems;
- Recognize functions defined by integrals and find their derivatives;
- Approximate the values of integrals geometrically or by using Riemann sums; and

[2] openstax.org/details/books/calculus-volume-1/
[3] openstax.org

- Evaluate integrals by finding simple antiderivatives and by applying the method of substitution.

These outcomes will be assessed on the final exam.

General Education Student Learning Outcomes. This course can be used to satisfy some general education requirements, for which there are some standard goals. Students are expected to display a thorough understanding of the topics covered. In particular, upon completion of the course, students will be able to

1. model phenomena in mathematical terms,
2. solve problems using these models, and
3. demonstrate an understanding of the supporting theory behind the models apart from any particular application.

These outcomes will be assessed on the final exam.

Calculators. It might be useful to have a graphing calculator, and the standard recommendation is the Texas Instruments TI-84 Plus. However, many choices of free "calculating device" can work too, including websites, phone apps and computer software, and I will demonstrate some of them. One of my favorites is the Desmos Graphing Calculator[4]; in addition to that website interface, this is available as a free app for iPhones, iPads[5], and Android[6] devices.

There are also several other tools at the Desmos[7] website, such as a Scientific Calculator.

Such tools may be used for some homework and in-class exercises, but not on tests or the final exam.

[4] `www.desmos.com/calculator/`
[5] `apps.apple.com/us/app/desmos-graphing-calculator/id653517540`
[6] `play.google.com/store/apps/details?id=com.desmos.calculator`
[7] `www.desmos.com`

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Chapter 1

Functions and Graphs

This chapter is mostly a review of pre-calculus. In class we cover just some topics from the last two sections: exponentials, inverse functions, and logarithms. Also, within those two section we skip some topics for now:

- *Inverse trigonometric functions* will be reviewed when we encounter them in Section 3.7.
- *Hyperbolic functions and their inverses* are left till MATH 220, Calculus 2.

However, if you have not seen some of these omitted topics for a while, or are curious, you can review them; the three earlier sections of the chapter are:

- Section 1.1. Basic Classes of Functions[1]
- Section 1.2. Review of Functions[2]
- Section 1.3. Trigonometric Functions[3]

References.

- OpenStax Calculus Volume 1, Chapter 1[4], Sections 4 and 5.
- *Calculus, Early Transcendentals* by Stewart, Chapter 1, Sections 4 and 5.

1.1 Exponential Functions

References

- OpenStax Calculus Volume 1, Section 1.5 (the first part).[1]
- *Calculus, Early Transcendentals* by Stewart, Section 1.4.

A function like $f(x) = 2^x$ is called *exponential* because the argument x is the exponent in the formula. Exponential functions are the most basic and common *transcendental functions*, and are probably the most important functions in mathematics and science after polynomials.

We will see how exponential functions can be defined to have graphs that are continuous, unbroken curves with well defined slopes, rather being only a collection of separate points for integer values of x.

[1] `openstax.org/books/calculus-volume-1/pages/1-1-review-of-functions`
[2] `openstax.org/books/calculus-volume-1/pages/1-2-basic-classes-of-functions`
[3] `openstax.org/books/calculus-volume-1/pages/1-3-trigonometric-functions`
[4] `openstax.org/books/calculus-volume-1/pages/1-introduction`
[1] `openstax.org/books/calculus-volume-1/pages/1-5-exponential-and-logarithmic-functions`

Natural number powers of 2. With basic algebra, exponential functions are defined first for positive integer arguments, by formula

$$2^n = 2 \cdot 2 \cdot 2 \cdots 2, \text{ the product of } n \text{ copies of } 2.$$

Then to satisfy the rule $2^{n+m} = 2^n \cdot 2^m$ for the case $m = 0$ requires

$$2^0 = 1$$

so all non-negative integers n are covered.

Negative integer powers of 2. For a negative integer n, $|n| = -n$ is positive, and to satisfy the rule $2^n \cdot 2^m = 2^{n+m}$ we must have $2^n.2^{-n} = 2^{n+(-n)} = 2^0 = 1$, and so dividing by $2^{-n} = 2^{|n|}$,

$$2^n = \frac{1}{2^{-n}} = \frac{1}{2^{|n|}} \quad \text{for } n \text{ a negative integer.}$$

Rational powers of 2. Next we can make sense of exponentials for rational exponents. To get the exponential 2^r for any rational number r start with exponent $1/q$, q a positive integer. To satisfy the rule $(2^a)^b = 2^{ab}$ requires $(2^{1/q})^q = 2^{q/q} = 2^1 = 2$, so taking the q-th root of both sides of this equation,

$$2^{1/q} = \sqrt[q]{2} \text{ (the q-th root of 2) for q a positive integer.}$$

Finally, any rational number can be written as $r = p/q$ with p an integer, q a positive integer, and the same rule requires

$$2^{p/q} = ((2^p)^{1/q}) = \sqrt[q]{2^p}.$$

Irrational powers of 2 (so all power of 2). The graph of 2^x for all rational x looks like a dense collection of dots along a curve which increases to the right. Can we fill in the gaps at irrational values of x and get a smooth, uninterrupted curve? For example, can we make sense of an irrational power like $2^{\sqrt{3}}$?

A number like $\sqrt{3} = 1.73205\ldots$ is approximated by a succession of decimal fractions 1, 1.7, 1.73, 1.732, 1.7320, 1.73205 and so on: it is the **limit** of this **sequence** of rational numbers.

Raising 2 to each of these powers gives the following new sequence of numbers (everything rounded to five decimal places):

$$2^1 = 2 < 2^{1.7} = 3.24900 < 2^{1.73} = 3.31727 < 2^{1.732} = 3.32188 < 2^{1.73205} = 3.32200\ldots$$

All of these should be less that $2^{\sqrt{3}}$ since the values are increasing as the exponent increases and $\sqrt{3}$ is greater than each of these exponents.

On the other hand if we round up the decimal approximations of $\sqrt{3}$, the exponentials should all be greater than $2^{\sqrt{3}}$:

$$2^2 = 4 > 2^{1.8} = 3.48220 > 2^{1.74} = 3.34035 > 2^{1.733} = 3.32418 > 2^{1.7321} = 3.32211 > 2^{1.73206} = 3.32202\ldots$$

It appears that

$$2^{1.73205} = 3.32200 < 2^{\sqrt{3}} < 3.32202 = 2^{1.73206},$$

so that $2^{\sqrt{3}}$ rounded to four decimal places is 3.3220. We could continue with either sequence to compute a value for $2^{\sqrt{3}}$ to as many decimal places as we wish.

In this way, we can make sense of, and compute, any power of 2, rational or irrational, so we have made sense of the exponential function $f(x) = 2^x$ for all real arguments x.

Any power of any positive number. There is nothing special about the base 2 used above except that it is positive: we could do the same thing with any positive real number a, to compute the exponential function $f(x) = a^x$. The graphs for the different functions vary mostly in that they are

- increasing for $a > 1$, and increase faster for larger values of a,
- decreasing for $0 < a < 1$ and decrease faster for smaller values of a,
- and in the borderline case of $a = 1$, the graph is a constant: $1^x = 1$.

Rules for exponential functions. The familiar rules for exponentials still hold just as with with rational exponents: for a and b positive and any real numbers x and y,

- $a^{x+y} = a^x \cdot a^y$, and $a^{x-y} = a^x/a^y$
- $(a^x)^y = a^{x \cdot y}$
- $(a \cdot b)^x = a^x \cdot b^x$

Applications of exponential functions.

Example 1.1.1 Bacterial Growth. See Example 1.33 on Bacterial Growth in Section 1.5[2] of our text, *OpenStax Calculus Volume 1*. □

Example 1.1.2 Radioactive Decay. The **half-life** of strontium-90, ^{90}Sr, is 25 years. This means that half of any given quantity of ^{90}Sr will disintegrate in 25 years.

1. If a sample of ^{90}Sr initially has a mass of 24mg, find an expression for the mass $m(t)$ that remains after t years.
2. Find the mass remaining after 40 years, correct to the nearest milligram.
3. Use a graphing device to graph $m(t)$ and use the graph to estimate the time required for the mass to be reduced to 5 mg.

□

The number e. Of all possible choice of the base a of an exponential function a^x, one is most convenient for mathematics because it makes the slope of the graph simplest: the number called e, with value approximately $e \approx 2.71828$.

The graphs of all exponential functions pass through the point $P(0, 1)$ on the y-axis, but the bigger a is, the faster the function value grows as x increases, so the greater the slope is at this point. The slope is zero for $a = 1$, when the function is constant, and increases as a increases.

Experimenting with a graphing calculator suggests that the slope is less than 1 for 2^x, but greater than 1 for 3^x. So it seems that by increasing a to somewhere between 2 and 3, the slope will be 1 at $P(0, 1)$, with the slope greater than 1 for greater values of a, less than 1 for lesser values. That is, there is one special value for the base that gives slope 1: this is the value called e.

We have already seen that e lies between 2 and 3, and with ever more careful computation of slopes we could calculate the more accurate value given above.

We will soon see that any other exponential function can be written in terms of e^x, and this is very convenient in calculus, making this particular exponential function so important that it is often called simply "the exponential function".

There is another way to see the origins of this special number in terms of continuous growth, discussed in Section 1.5[3] of the text using the example of continuously compounded interest.

[2] `openstax.org/books/calculus-volume-1/pages/1-5-exponential-and-logarithmic-functions`
[3] `openstax.org/books/calculus-volume-1/pages/1-5-exponential-and-logarithmic-functions`

Exercises

Study Calculus Volume 1, Section 1.5[4], Exercises 233, 235, 239, 243, 278, 279, 299, 301, and 305(a) [we will get to part (b) soon].

1.2 Inverse Functions

References

- OpenStax Calculus Volume 1, Section 1.4.[1]
- *Calculus, Early Transcendentals* by Stewart, Section 1.5 (the first part).

Equation solving and inverse functions. In part (c) of Example A of the previous section, we knew that the value of the function $m(t)$ (mass of Strontium-90 remaining) was 5 mg, and wanted to know the corresponding value of its argument t (the time).

More generally it would be useful to have a formula giving time t as a function of mass m, $t = g(m)$. A function like this that takes values "backwards" compared to the function is the *inverse* of that function.

For example, with $y = f(x) = x^3$, we get back from a y value to the corresponding x value by treating y as known and solving $y = x^3$ for the unknown x: this gives $x = \sqrt[3]{y}$. Thus the cube root function is the inverse of the cube function, and we write $x = f^{-1}(y) = \sqrt[3]{y}$.

It is often convenient to go back to using the name x for the argument of this new function too, writing $f^{-1}(x) = \sqrt[3]{x}$.

The Horizontal Line Test [HLT] and one-to-one functions. Graphically, one gets the graph of the inverse function $x = f^{-1}(y)$ by flipping the graph of $y = f(x)$ along the diagonal line $y = x$. Since the role of the x and y values are swapped, the domain of the inverse is the range of the original function, and vice versa.

However, this flipping does not always give the graph of a function. The graph of any function must pass the *vertical line test* that no vertical line intersects it more than once, and for the flipped graph to pass, the original graph must have no *horizontal* line intersects it more than once.

This is the *Horizontal Line Test [HLT]*, and is exactly what is needed for a function to have an inverse. Algebraically, this means that no two different arguments x_1 and x_2 give the same value of the function:

Definition 1.2.1 Function f is **one-to-one** if for any $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$. ◊

Example 1.2.2 Is the function $f(x) = x^3$ one-to-one? This passes the HLT, since the function is increasing and so passes through any horizontal line just once and never returns. In fact, any function that is always **increasing** passes the HLT (same if it is always **decreasing**) □

Example 1.2.3 Is the function $g(x) = x^2$ one-to-one? No; this fails the HLT: the horizontal line $y = 1$ intersects for both $x = 1$ and $x = -1$. In fact every line $y = a$ for positive a intersects at two x values, $\sqrt{a}$ and $-\sqrt{a}$. □

Inverse functions. One-to-one functions are exactly the ones that have inverses:

Definition 1.2.4 (Inverse). If function f is *one-to-one*, if has an **inverse**, denoted f^{-1}, and defined by $x = f^{-1}(y)$ being given by the solution x of the equation $f(x) = y$. ◊

[4] openstax.org/books/calculus-volume-1/pages/1-5-exponential-and-logarithmic-functions
[1] openstax.org/books/calculus-volume-1/pages/1-4-inverse-functions

If a function is not one-to-one, this equation has several solutions for some values of y, so does not determine the value x; thus the inverse is not defined.

Domain and range of the inverse. The *domain* of f^{-1} is the *range* of f [the "y-values" in the above equation], and the *range* of f^{-1} is the *domain* of f [the "x-values"].

Caution: changing domain gives a different function. How do we reconcile $f(x) = x^2$ not being one-to-one with it having an inverse, the square root function?

Be careful: we are talking about two different functions here, even though they are described using the same formula $y = x^2$! When we use the formula with domain all the real numbers, it is not one-to-one, and has no inverse, but when we change the domain to non-negative real numbers, that is a different function, with inverse $\sqrt{x}$. The difference with this smaller domain is that the graph is only the right half of a parabola, which is increasing and so satisfies the HLT.

Algebraically, for any given value y, the equation $x^2 = y$ has only one non-negative solution x.

Example 1.2.5 If $f(1) = 5$, $f(3) = 7$ and $f(8) = -10$, find $f^{-1}(-10)$, $f^{-1}(5)$ and $f^{-1}(7)$. □

Notation warning: inverses are not reciprocals! Beware of a possible confusion:
$y = f^{-1}(x)$ [the *inverse* of function f applied to x]
is not the same as
$y = [f(x)]^{-1} = \frac{1}{f(x)}$ [the *reciprocal* of $f(x)$].

Example 1.2.6 Find the inverse function of $f(x) = x^3 + 2$. □

Exercises

Study Calculus Volume 1, Section 1.4[2], Exercises 183, 185, 189, 193, 195, 197, 201, 203, 217.

1.3 Logarithmic Functions

References

- OpenStax Calculus Volume 1, Section 1.5 (the second part).[1]
- *Calculus, Early Transcendentals* by Stewart, Section 1.5 (the second part).

Does an exponential function $y = f(x) = a^x$ have an inverse?

For $a > 1$, the value of a^x increases as x increases: the graph is increasing, which is enough to pass the HLT and ensure existence of an inverse.

For $0 < a < 1$, the graph is decreasing and so again passes the HLT, giving an inverse. (For $a = 1$, there is no inverse, and the function is boring: $f(x) = 1$.)

This inverse should be familiar: the number x for which $a^x = y$ is called the *logarithm of y base a*, written $\log_a y$, so the inverse of the exponential function $f(x) = a^x$ is the **logarithmic function base** a, $f^{-1}(x) = \log_a x$.

An exponential function for $a \neq 1$ is defined for all real numbers (its domain) and its values (range) are all positive numbers. Thus the logarithmic functions $\log_a$ have domain all the positive numbers, range all the reals: only positive numbers have logarithms.

Note that this simple domain and range for logarithms depends on exponential functions being defined for all real arguments, not just all rational arguments.

[2] `openstax.org/books/calculus-volume-1/pages/1-4-inverse-functions`
[1] `openstax.org/books/calculus-volume-1/pages/1-5-exponential-and-logarithmic-functions`

Rules for logarithms. Logarithms satisfy the following rules, all following from the rules for exponentials in Section 1.5; For any positive number a except 1, and any positive numbers x and y,

1. $\log_a(x \cdot y) = \log_a x + \log_a y$
2. $\log_a(x/y) = \log_a x - \log_a y$
3. $\log_a(x^p) = p \log_a x$ for any real power p.

Natural logarithms. Since e^x is the most commonly used exponential function, its inverse $\log_e$ is the most important logarithmic function: it is called the **natural logarithm**, and has the special name $\ln$ (from the initials of "logarithm" and "natural"):

$$\ln x = \log_e x.$$

Our first use of the natural logarithm is to put any exponential function a^x in terms of e^x. Using the properties of exponentials and the fact that $e^{\ln a} = a$,

$$a^x = (e^{\ln a})^x = e^{(\ln a)x}.$$

It can also be shown that

$$\log_a x = \frac{\ln x}{\ln a},$$

so we can also put all logarithmic functions in terms of natural logarithms.

Thus we mostly need just one exponential function, e^x, and just one logarithmic one: its inverse, the *natural logarithm*.

For now we omit the topic *Inverse trigonometric functions*; instead we will review those when we encounter them in Section 3.7.

Exercises

Study Calculus Volume 1, Section 1.5[2], Exercises 247, 251, 255, 261, 265, 271, 273, 277, 283, 285, 287, and 305(b).

[2] openstax.org/books/calculus-volume-1/pages/1-5-exponential-and-logarithmic-functions

Chapter 2

Limits

This chapter introduces the fundamental idea of calculating *limits* in order to calculate the *tangent slope* of a curve, the *instantaneous velocity* of a moving object, the instantaneous rate of growth of a population, and other *rates of change*.

The second key concept is *continuity* of a function: a nice property of many but not all common functions that makes limits easy to compute.

References.

- OpenStax Calculus Volume 1, Chapter 2.[1]
- *Calculus, Early Transcendentals* by Stewart, Chapter 2, Sections 1–6.

2.1 A Preview of Calculus

References.

- OpenStax Calculus Volume 1, Section 2.1[1]. (The last topic *The Area Problem and Integral Calculus* can be skipped for now; we return to it later in the semester.)
- *Calculus, Early Transcendentals* by Stewart, Section 2.1.

One of the beauties of mathematics is that often, several problems that seem to be quite different turn out to have very similar mathematical representations and solutions, so that there is a common way to solve them.

Two such problems are:

- Making sense of the slope at a point on a curve.
- Finding the velocity of a moving object from knowing its position as a function of time.

The Tangent Problem. We know how to compute the slope of a straight line, and how this is related to, say, the slope of an inclined plank when the graph describes height as a function of horizontal position. It is very useful to extend to this idea to calculating the slopes of curves. The slope can vary from point to point along a curve, so what we will calculate is the slope at each point of a curve. The geometrical idea is that near a point on a curve, the curve is very close to a certain straight line: the **tangent line** to that point.

[1] `openstax.org/books/calculus-volume-1/pages/2-introduction`
[1] `openstax.org/books/calculus-volume-1/pages/2-1-a-preview-of-calculus`

Example 2.1.1 A tangent line to a parabola. Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(2, 4)$. □

Tangent lines at each point of a curve. We often want the tangent slope or tangent line at multiple points on the curve, or at all of them, and then it is more efficient to proceed as follows:

Example 2.1.2 Tangent lines to a cubic curve. Find an equation of the tangent line to the cubic $y = x^3$ at the point $P(a, a^3)$ for any value a.

First we approximate the slope by the slope m_{PQ} of the secant line between this point P and a nearby point $Q(x, x^3)$ for x near a,

$$m_{PQ} = \frac{x^3 - a^3}{x - a}.$$

This should approach the tangent slope m as x approaches a [$x \to a$], and to see how m_{PQ} behaves then, it helps to *simplify first*.

The numerator vanishes for $x = a$, so has a factor $x - a$, and when we divide out this factor, $x^3 - a^3 = (x - a)(x^2 + x \cdot a + a^2)$. This gives

$$m_{PQ} = \frac{x^3 - a^3}{x - a} = \frac{(x - a)(x^2 + x \cdot a + a^2)}{x - a} = x^2 + x \cdot a + a^2, \text{ for } x \neq a.$$

For x near a, this has values close to what we get by *substituting* a for x: m_{PQ} gets close to $a^2 + a \cdot a + a^2 = 3a^2$. Thus, it seems that the tangent slope should be For x near a, this has values close to what we get by *substituting* a for x: m_{PQ} gets close to $a^2 + a \cdot a + a^2 = 3a^2$.

Thus the tangent slope should be

$$m = \lim_{x \to a} m_{PQ} = \lim_{x \to a} \frac{x^3 - a^3}{x - a} = \lim_{x \to a} (x^2 + xa + a^2) = 3a^2.$$

The **point-slope formula** then gives the tangent line at $P(a, a^3)$:

$$y = a^3 + 3a^2(x - a).$$

Note that a is some constant, only x is variable, so this is a line, not a more complicated polynomial. For example, at the point $Q(2, 8)$ given by $a = 2$, the tangent line is $y = 8 + 12(x - 2)$. □

The Velocity Problem. One exercise is enough to reveal that the solution to this problem comes from the same calculations as seen above for computing tangent slopes:

Example 2.1.3 A falling object. Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450m above the ground.

Find the velocity of the ball after 5 seconds.

Then find the velocity at any given time after the ball is dropped. □

Recycling ideas and methods of calculation. This section on the velocity problem is very short because in fact *we have already solved the velocity problem by solving the tangent problem.*

The ability to solve a few core problems, like the tangent problem, and then "recycle" the ideas and computational methods discovered for them when solving various other problems, is one key to the efficiency and utility of calculus. The single most central idea discovered so far is finding **limits**: getting from various approximations to an exact answer, so we study that next.

Exercises

Study Calculus Volume 1, Section 2.1[2], Exercises 4, 5, 6, 16, and 17.

2.2 The Limit of a Function

References.

- OpenStax Calculus Volume 1, Section 2.2.[1]
- *Calculus, Early Transcendentals* by Stewart, Section 2.2.

In computing tangent slopes, velocities and areas, there is a common step, new with calculus; calculating a limit, like calculating that as x approaches 1, $\dfrac{x^2-1}{x-1}$ approaches 2:

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=2.$$

We will now make this idea of limits more clear, and learn how to calculate them. With this key new skill, most other calculations in this course can be handled with familiar algebra, geometry, trigonometry and such.

Definition 2.2.1 Limit, informal version. For a function f and numbers a and L, we say that the **limit** of $f(x)$ as x approaches a is L if we can force the value of $f(x)$ to be as close to L as we wish by considering only values of x sufficiently close to a, *but not equal toa*. This is written as

$$\lim_{x\to a} f(x)=L$$

Note that *the value of $f(a)$ is irrelevant*: f need not even be defined for $x=a$. ◊

Example 2.2.2 A limit where the formula gives 0/0. Guess the value that $f(x)=\dfrac{x-1}{x^2-1}$ approaches as x approaches 1; that is, guess the value of $\lim\limits_{x\to 1}\dfrac{x-1}{x^2-1}$.

Do this by trying x values that differ from 1 by 0.1, then 0.01 and so on.

Then try to corroborate your guess by simplifying the formula for $f(x)$. □

Example 2.2.3 Calculators can be fooled near 0/0. Investigate the behavior of $\dfrac{\sqrt{t^2+9}-3}{t^2}$ as t approaches 0.

First try x values $\pm 1, \pm 0.5, \pm 0.1, \pm 0.05, \pm 0.01$, and then the closer values $\pm 0.0005, \pm 0.0001, \pm 0.00005, \pm 0.00001$ etc.

Warning: Things are not as they first seem here!

The closer x values actually give a misleading result, due to the inability of a calculator to get sufficiently accurate results in this case. The limit is actually $1/6$, as suggested by the less close x values!

Using a graph on the calculator and zooming shows some probably unexpected and implausible behavior, revealing the accuracy limits of the calculator. □

Example 2.2.4 $(\sin x)/x$ for x near 0. How does the function $f(x)=\dfrac{\sin x}{x}$ behave as x approaches 0? (See Example 2.4 in OpenStax Calculus).[2]

Setting $x=0$ does not work as we get $0/0$ again, but experiments on a graphing calculator suggest

[2] openstax.org/books/calculus-volume-1/pages/2-1-a-preview-of-calculus
[1] openstax.org/books/calculus-volume-1/pages/2-2-the-limit-of-a-function

that the value approaches 1. (Note: remember to always use radian mode in math courses!)

This time, there is no simple algebraic way to simplify this formula and avoiding the "$0/0$ problem": we see in Chapter 3 how to compute this limit. □

Example 2.2.5 $\sin(\pi/x)$**.** Explore how the function $f(x) = \sin\dfrac{\pi}{x}$ behave as x approaches 0,

- first with $x = 1, 1/2, 1/3, 1/4, \ldots,$
- then with $x = 2, 2/3, 2/5, 2/7, 2/9, \ldots,$
- and finally with many more values, by using a graphing calculator and zooming in.

Again we see that looking at only some nearby x values can be misleading. This time it seems that there is no one value that $f(x)$ gets near to: no matter how close x is to 0, $f(x)$ can be anywhere from -1 to 1.

This function has no limit as $x \to 0$. □

Note well: Limits do not always exist! This example again shows that it is important to consider *all* values of x near a when studying a limit as $x \to a$, not just a selection.

Example 2.2.6 Measuring "closeness". Show that $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{2x^2-2}{x-1} = 4$.

We can simplify to $f(x) = 2x + 2$, valid for all $x \neq 1$.

Then we measure how close two numbers are by the absolute value of their difference. For example, if x is within 0.001 of 1, $|x - 1| < 0.001$, and so $|f(x) - 4| = |(2x + 2) - 4| = |2x - 2| = 2|x - 1|$ which is less than 0.002.

When we look only at x values ever closer to 1, in that $|x - 1|$ is ever smaller, $|f(x) - 4 = 2|x - 1|$ is ever smaller: $f(x)$ gets ever closer to 4. For example, the value $f(x)$ is sure to be within a tiny 10^{-100} of 4 when we look at x values within 0.5×10^{-100} of 1.

So the limit is 4. □

Example 2.2.7 A function with a jump. Consider the function $f(x)$ given by

$$f(x) = \begin{cases} \frac{2x^2-2}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$$

What is the limit of $f(x)$ as x approaches 1?

Since the limit as $x \to a$ is based on values of $f(x)$ for all x values *near to* a, but *not equal to* a, only the formula $(2x^2 - 2)/(x - 1)$ matters! And since it is equal to $2x + 2$ for all x values near 1, the value is near $2 \cdot 1 + 2 = 4$ there, and the limit is 4: $\lim_{x\to 1} f(x) = 4$, not 1. *Note well: The limit of $f(x)$ as x goes to a does not always equal the value $f(a)$, even when $f(a)$ makes sense!*The graph of this function has a **jump** at $x = 1$, but the limit calculation ignores this, and treats the function as as if it were "uninterrupted" or "continuous" there. □

Another type of jump: the Heaviside function. In the physical description of sudden changes, like turning on a power switch, the **Heaviside Function** is often useful:

$$H(t) = 0 \text{ for } t < 0; H(t) = 1 \text{ for } t \geq 0$$

For t near 0 and positive, $H(t)$ is 1, suggesting a limit of 1. But for t near 0 and negative, $H(t)$ is 0, suggesting a limit of 0.

The limit cannot be both zero and one, so again this function has *no limit as $t \to 0$*, due to this jump from one value to another, which breaks the graph at this point.

[2] openstax.org/books/calculus-volume-1/pages/2-2-the-limit-of-a-function

One-sided Limits. In the example above, we see that $H(t)$ has "no limit" as $t \to 0$, but it is useful also to describe what happens at times just before $t = 0$, and what happens at times just after $t = 0$: what happens to one side or the other of a point on the graph.

We want to note that "as t approaches 0 *from the right* ($t > 0$), $H(t)$ approaches 1." We use the notation $t \to 0^+$, with a plus sign superscript indicating that only t values to the right are considered: the relevant t values are "0 + something".

The value approached is the **right-hand limit**, or the **limit from the right**, with short-hand notation

$$\lim_{t \to 0^+} H(t) = 1.$$

One-sided limits: the left-hand limit. Similarly the behavior for t near 0 and less than zero is called the **left-hand limit** and we use a minus sign superscript, because the t value is "0 minus something:"

$$\lim_{t \to 0^-} H(t) = 0.$$

Note well: "$t \to 0^-$" is different from "$t \to -0$", which would be a funny way of writing a normal "two-sided" limit. And $t \to 1^-$ is very different than $t \to -1$; the former is about what happens for t just below 1; the latter is about what happens for t near -1.

Using one-sided limits to compute [two-sided] limits. Sometime it is easier to compute each one-sided limit at a and then use these to learn about the regular "two-sided" limit:

Theorem 2.2.8 *If both one sided limits at a exist and are equal*

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

then that common value is also the limit there:

$$\lim_{x \to a} f(x) = L$$

Otherwise, the latter limit does not exist.

Infinite limits. We have seen several ways that a function can fail to have a limit as $x \to a$, and decided that sometimes, there is still something useful to say about how the function behaves for x near a. Here is another case of that.

Example 2.2.9 Behavior of $1/x^2$ as $x \to 0$. Investigate the behavior of $f(x) = 1/x^2$ as x approaches 0. Does it have a limit?We see that there is no numerical value that $f(x)$ gets close to, but *there is a trend worth noting:The values of $f(x)$ get larger and larger, with no upper bound.*

Informally we could say that the value approaches infinity. □

To describe cases like this, we introduce the symbol ∞ for **infinity** and say that:

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

or in words, "as x approaches zero, the limit of $1/x^2$ is infinity."

One-sided infinite limits. Finally, it is natural to combine the ideas of infinite limits and one-sided limits.

Example 2.2.10 Describe how $f(x) = \frac{1}{x-2}$ behaves for x near 2, for the two cases $x > 2$ and $x < 2$

The values get large and positive on one side, large and negative on the other, so for x coming from the right, "the value approaches ∞", while from the left, "the value approaches $-\infty$".

Combining the above ideas and notation of one sided limits and infinite limits, we state this as

$$\lim_{x\to 2^-} \frac{1}{x-2} = -\infty, \quad \lim_{x\to 2^+} \frac{1}{x-2} = \infty.$$

But the limits from the two sides are different, so

$$\lim_{x\to 2} \frac{1}{x-2} \quad \text{Does Not Exist (DNE).}$$

□

Exercises

Study Calculus Volume 1, Section 2.2[3], Exercises 30, 31, 35, 36, 37, 46–49, 77 and 79.

2.3 The Limit Laws

References.

- OpenStax Calculus Volume 1, Section 2.3.[1]
- *Calculus, Early Transcendentals* by Stewart, Section 2.3.

We now have the fundamental idea of limits, and are ready to learn how to compute limits $\lim_{x\to a} f(x)$ as easily as possible, and to use this to compute the tangent slopes of curves, instantaneous velocities and such as easily as possible. Building this collection of calculational skills is the main goal of the rest of this chapter, with some related ideas and applications mixed in.

Limits for Two Very Basic Functions. We start with two very simple and intuitive cases, which are then surprisingly useful in handling other functions.

Firstly, the limit of a constant function $f(x) = c$ is easy:

for any value of x, $f(x)$ is extremely close to (in fact equal to!) c and so as $x \to a$ the limit is c. That is,

$$\lim_{x\to a} c = c.$$

Next, almost as easy, is to note that for $f(x) = x$, as x approaches a, $f(x) = x$ also approaches a since it is the same quantity. That is,

$$\lim_{x\to a} x = a.$$

Note: In each case, the limit is just the value of $f(a)$, which is true for many "nice" functions, but not for all functions: we have already seen some exceptions above. This is our first sighting of *continuity*, which we will explore more in Section 2.4, p. 15.

These two results are the building blocks that allow us to easily compute limits for any polynomial or rational function, once we know how to put together information about the limits of simple functions to get limits of more complicated ones.

The Limit of a Constant Multiple of a Function. The first combining rule comes from this intuitive idea: *when $f(x)$ is close to L, $C \cdot f(x)$ is close to $C \cdot L$.*

Thus if $f(x) \to L$ as $x \to a$, then $C \cdot f(x) \to C \cdot L$ as $x \to a$.

The first half of this says that $\lim_{x\to a} f(x) = L$, so the second half gives

[3] openstax.org/books/calculus-volume-1/pages/2-2-the-limit-of-a-function
[1] openstax.org/books/calculus-volume-1/pages/2-3-the-limit-laws

Fact 2.3.1 The Constant Factor Rule for Limits.

$$\lim_{x\to a}[C \cdot f(x)] = C \lim_{x\to a} f(x).$$

Example 2.3.2 Combining this with the previous result

$$\lim_{x\to 2} 7x = 7 \lim_{x\to 2} x = 7 \cdot 2 = 14.$$

□

The Limits of Sums and Differences of Functions. The next basic idea is that:
when $f(x)$ is close to L and $g(x)$ is close to M, their sum $f(x) + g(x)$ is close to $L + M$.
This leads to

Fact 2.3.3 The Sum Rule for Limits.

$$\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

Similarly

Fact 2.3.4 The Difference Rule for Limits.

$$\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x).$$

Other basic arithmetic works too: we have

Fact 2.3.5 The Product Rule for Limits.

$$\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

and with a little more caution,

Fact 2.3.6 The Quotient Rule for Limits.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \quad \textit{so long as } \lim_{x\to a} g(x) \neq 0.$$

The restriction here is just the requirement that the right-hand side makes sense.
Note well: when the right-hand side does not make sense (division by zero), the left-hand side still might! In fact, many of the most important limit calculations are like that.

The Power Rule, and Power Functions. Using the product rule repeatedly gives

Fact 2.3.7 The Power Rule for Limits.

$$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n \textit{ for } n \textit{ any natural number.}$$

The power rule applied to the simple function $f(x) = x$ gives the limits of power functions

$$\lim_{x\to a} x^n = a^n.$$

Limits of Polynomials and Rational Functions. The rule for constant multiples gives the limit for any **monomial** $f(x) = Cx^n$:

$$\lim_{x\to a} f(x) = \lim_{x\to a} C \cdot x^n = C \cdot \lim_{x\to a} x^n = C \cdot a^n = f(a).$$

Any polynomial $p(x) = c_0 + c_1x + c_2x^2 + \cdots$ is a sum of such monomials, so using the Addition Rule repeatedly gives

$$\lim_{x\to a} p(x) = \lim_{x\to a} c_0 + c_1x + c_2x^2 + \cdots = \lim_{x\to a} c_0 + \lim_{x\to a} c_1x + \lim_{x\to a} c_2x^2 + \cdots = c_0 + c_1a + c_2a^2 + \cdots = p(a).$$

So *we have every limit of every polynomial:* they are all given simply by evaluating at $x = a$.

Example 2.3.8 Find a simple strategy for calculating the limit of any rational function, $\lim_{x\to a} \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

Use the rules above first for quotients, then for polynomials. □

Theorem 2.3.9 The Direct Substitution Property. *For f a polynomial or rational function and any number a in its domain, the limit is given by simply substituting a for x:*

$$\lim_{x\to a} f(x) = f(a)$$

We will gradually expand the list of functions with this nice property, called **continuity**.

Checkpoint 2.3.10 Evaluate $\lim_{x\to 2} \frac{x-2}{x^2-4}$, using the limit laws and as little algebra as possible.

Theorem 2.3.11 The Root Law for Limits.

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a} \text{ for } n \text{ a positive integer } (a > 0 \text{ is needed if } n \text{ is even}).$$

More generally

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)},$$

this time with $\lim_{x\to a} f(x) > 0$ needed if n is even.

Proof. This can be shown by using the Power Rule. ■

Ignoring the Function Value at a. Remember that the value of $f(x)$ for $x = a$ is irrelevant to the limit as $x \to a$:

If $f(x) = g(x)$ for $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

This includes the possibility that neither limit exists: both are ``DNE''. Thus only $x < a$ and $x < a$ matter, and it sometimes helps to consider these two cases one at a time.

Using One-sided Limits to Compute Limits. As seen in the previous section, a limit exists exactly when both one-sided limits exist, and both of then have the same value, in which case the limit has that same value too. Moreover, all the rules seen above are also true for one sided limits.

Checkpoint 2.3.12 Evaluate $\lim_{x\to 2} f(x)$ where

$$f(x) = \begin{cases} x^3, & x < 2 \\ 3, & x = 2 \\ x^2 + 2x, & x > 2 \end{cases}$$

Hint: compute each one-side limit. Also, it might helps to sketch the graph first; it often does!

Limits Respect Inequalities. If $f(x)$ is no greater than $g(x)$, its limit at any point is not greater either. That is,

$$\text{If } f(x) \leq g(x), \text{ then } \lim_{x\to a} f(x) \leq \lim_{x\to a} g(x),$$

(so long as both limits exist.)

This inequality idea is particularly useful when a function $f(x)$ can be squeezed between two simpler functions $l(x)$ and $u(x)$ with both of them having the same limit at a point a; this situation forces the in-between function $f(x)$ to have that same limit:

Theorem 2.3.13 The Squeeze Theorem. *If* $l(x) \leq f(x) \leq u(x)$ *and* $\lim_{x\to a} l(x) = \lim_{x\to a} u(x) = L$, *then* $\lim_{x\to a} f(x) = L$ *also.*

Checkpoint 2.3.14 Sketch $y = f(x) = x^2 \sin\left(\frac{1}{x}\right)$ for x near 0, and then evaluate $\lim_{x\to 0} f(x)$.

In a while we will use squeezing to show that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, and use that to find the slope at any point on the graph of any trig. function.

Exercises

Study Calculus Volume 1, Section 2.3[2], Exercises 83, 85, 89, 91, 93, 97, 107, 111, 119, 121, 127, and 128.

2.4 Continuity

References.

- OpenStax Calculus Volume 1, Section 2.4.[1]
- *Calculus, Early Transcendentals* by Stewart, Section 2.5.

We have seen that many common functions f like polynomials have the nice property that the limit as x goes to a can be evaluated by simple evaluation of $f(a)$. This property is useful in many ways, so we now give it a name, explore its meaning in terms of graphs and other nice consequences, and expand our list of such functions.

Definition 2.4.1 Continuity at one point. The function f is **continuous at a** if $\lim_{x\to a} f(x) = f(a)$. (Note that this requires f to be *defined* at a!)

If not, we say that f is **discontinuous at a**. ◊

Definition 2.4.2 Continuity of a Function. If a function is continuous at every point a in its domain, we call it simply **continuous**. ◊

For example all polynomials are continuous. Indeed, *all rational functions are continuous*: continuity only fails at points where the denominator is zero, and those points are not in the domain!

Continuity at Some Places but not Others. Sometimes we have to be very careful with this definition:

Example 2.4.3 Consider the function

$$f(x) = \begin{cases} x^3, & x < 2 \\ x^2 + 2x, & x > 2 \end{cases}$$

The function f is not continuous at $x = 2$ due to being undefined there, even though the limit exists there: $\lim_{x\to 2} f(x) = 8$. On the other hand, this function is continuous at all x values *in its domain*, so it is continuous, despite this discontinuity outside its domain. □

[2] openstax.org/books/calculus-volume-1/pages/2-3-the-limit-laws
[1] openstax.org/books/calculus-volume-1/pages/2-4-continuity

Removable Discontinuities. There is also a remedy for the one discontinuity of f in the example above, which is to *extend* its definition to that one x value missing form its domain by setting $f(2) = 8$. This extended definition gives a function that is defined and continuous for all real x values, a bit more satisfying than having that gap in its domain.

If a function is discontinuous at some point $x = a$ because it is not defined there, but $\lim_{x \to a} f(x)$ exists, this discontinuity is called **removable**: the gap in the domain and the discontinuity can be removed by extending the domain of the function to include the value a, with $f(a)$ given the value of this limit.

An important case is the formula for the slope m(x) of the secant line from $P(a, f(a))$ to $Q(x, f(x))$, where the "missing value" at $x = a$ is the tangent slope at that point.

Jump Discontinuities. Another situation seen already is with the Heaviside function, where at some places, the two one-sided limits do not match up. Here is a more natural mathematical situation where this occurs:

Example 2.4.4 The **integer part function** $f(x) = \lfloor x \rfloor$ is defined as the largest integer less than or equal to x, so that $\lfloor 3.9 \rfloor = 3$, $\lfloor 5 \rfloor = 5$, $\lfloor -1.2 \rfloor = -2$ (not -1!). This is just the familiar process of "rounding down", like when you give your age in years.

- Sketch the graph of $y = \lfloor x \rfloor$.
- For which values a is this function *continuous* at a?
- For which values a is this function *discontinuous*?
- Is this function continuous?

□

The discontinuities seen here at each integer are called **jump discontinuities**: points where the function has one-sided limits from each side, but they are different.

Continuity From One Side Only. In the example above, the limits from the right do equal the function value, so that side seems "continuous". We talk about this situation as follows:

Definition 2.4.5 Right-continuity. The function f is **continuous from the right at a** or **right-continuous at a** if its right-limit there exists and equals its value there: $\lim_{x \to a^+} f(x) = f(a)$.

Similarly, **left-continuity at a** means $\lim_{x \to a^-} f(x) = f(a)$. ◊

For example, the integer part function is right-continuous at every x value, or simply **right-continuous**.

Loose Ends: Continuity on an Interval with Endpoints. With a function like $\sqrt{x(1-x)}$ whose domain $[0, 1]$ includes end-points, continuity at an end-point is take to mean one-sided continuity from the only side that makes sense: from inside the interval.

Definition 2.4.6 Continuity on an interval. A function f is **continuous on an interval** if it is continuous at each interior point (non-endpoint) and is also "continuous from the inside" at any endpoint that is in the interval: that is,

right-continuous at the left endpoint,

left-continuous at the right endpoint. ◊

Thus $f(x) = \sqrt{x(1-x)}$ can be shown to be continuous on interval $[0, 1]$ using the Root Law for limits and the simple limit behavior of polynomials:

$$\lim_{x \to 0^+} \sqrt{x(1-x)} = \sqrt{0(1-0)} = f(0),$$

$$\lim_{x \to 1^-} \sqrt{x(1-x)} = \sqrt{1(1-0)} = f(1),$$

and for $0 < a < 1$, $\lim_{x \to a} \sqrt{x(1-x)} = \sqrt{a(1-a)} = f(a)$.

New Continuous Functions from Old Ones with Arithmetic. The laws for limits of constant multiples, sums, difference products and quotients of functions also mean that if f and g are continuous at $x = a$, so are cf for any constant c, $f + g$, $f - g$, and fg.

f/g is also, so long as it even makes sense at a: $g(a) \neq 0$.

For example, continuity of the product at $x = a$ requires $\lim_{x \to a} f(x)g(x) = f(a)g(a)$, and in fact

$$\begin{aligned} \lim_{x \to a} f(x)g(x) &= \lim_{x \to a} f(x) \lim_{x \to a} g(x) \text{ (from the Product Rule for Limits)}, \\ &= f(a)g(a) \text{ (from continuity of each function)}, \end{aligned}$$

as needed.

As seen in Section 2.3 on limit rules, these facts allow us to show that all polynomials are continuous, and rational functions are continuous at each number in their domains (avoiding divisions by zero), so they are also continuous.

Next, let us consider two other important ways to produce new functions, and whether these preserve continuity.

Continuity of An Inverse Function at a Point. Inverses of continuous functions are also continuous. First:

Theorem 2.4.7 *If $f(x)$ is continuous at $x = a$, and has an inverse, then f^{-1} is continuous at the corresponding point $b = f(a)$.*

Intuitively, there is no break in the graph of f at point $(a, f(a)) = (a, b)$, so when the graph is flipped over the line $y = x$ to get the graph of f^{-1}, there is no break in its graph at point $(b, f^{-1}(b)) = (b, a)$.

Continuity of Inverse Functions. When the above applies at each point in the domain of f:

Theorem 2.4.8 *if a function is continuous at every number in its domain and is one-to-one, its inverse exists and is also continuous at every number in its domain.*

For example, since a root function $\sqrt[q]{x} = x^{1/q}$ is the inverse of the polynomial x^q, this shows that *all root functions are continuous.*

Continuity of Compositions. The Root Law for limits allows us to show that any root of a continuous function, $[f(x)]^{1/q}$, is continuous. This is one case of continuity of a composition.

To see this more generally, we need the last main law of limits, which could not be stated in Section 2.3 because it needs the idea of continuity:

Theorem 2.4.9 Limit Law for Compositions. *If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. That is*

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(b).$$

Loosely speaking, the limit operation can move from outside function f to inside it so long as f is continuous at the relevant point.

This is intuitive because for x near a, $g(x)$ is near b, so the argument $g(x)$ of f in $f(g(x))$ is near b, and continuity of f at b says that the value of $f(g(x))$ is close to $f(b)$. This shows that composition respects continuity, but we must be careful about the domain, since with

$$(f \circ g)(x) = f(g(x)),$$

f has a different domain than g:

Theorem 2.4.10 *If g is continuous* at number a *and f is continuous* at number $g(a)$, *then their composition $f \circ g$ is continuous* at a.
Proof. This is just the above result for the case where $b = g(a)$. ■

Putting this together over the whole domain of g:

Theorem 2.4.11 *If f and g are continuous and the range of g lies in the domain of f so that their composition $f \circ g$ is defined on the whole domain of g, then this composition is continuous.*

Continuity of Algebraic Functions. Combining all these rules gives the continuity of all functions built up from polynomials with arithmetic, powers, roots and inverses, giving all algebraic functions, and some more besides.

Theorem 2.4.12 *Algebraic functions are continuous at each point in their natural domains, and so are continuous.*

The next question is whether common transcendental (non-algebraic) functions are continuous.

Continuity of Trigonometric Functions. With trigonometric functions, radian angle θ is the distance along the unit circle anti-clockwise from point $P(1, 0)$ to point $Q(\cos\theta, \sin\theta)$. (Negative θ means going clock-wise by distance $|\theta|$.)

Thus the x-coordinate $\cos\theta$ of Q is not further than distance $|\theta|$ from the x-coordinate 1 of P and likewise for the y-coordinates, $\sin\theta$ and 0:

$$|\cos\theta - 1| < |\theta|, \ |\sin\theta - 0| = |\sin\theta| \leq |\theta|.$$

A simple use of the Squeeze Theorem shows that

$$\lim_{\theta\to 0} \cos\theta = 1 = \cos 0, \quad \lim_{\theta\to 0} \sin\theta = 0 = \sin 0,$$

so these functions are continuous at zero. Alternatively, the definition of limits in Section 2.4 can be applied, with $\delta = \epsilon$ in each case.

Then use of the trigonometric sum rules allows us to show that these functions are continuous everywhere.

With that, all other standard trigonometric functions like $\tan\theta$ are continuous everywhere that they are defined, since they all come from sines and cosines by quotients like $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$, etc.

(Again, $\tan$ is not continuous at $\theta = \pi/2, -\pi/2$ etc., but those points are not in its domain, so do not really matter!)

Continuity of Exponentials and Logarithms. We defined the exponential function a^x for all real x using limits of rational powers of a, and this naturally makes the limits in the definition of continuity match up:

Fact 2.4.13 *Exponential functions are continuous.*

Then since logarithms are inverses of exponentials, they are also continuous.

Continuity of all the Familiar "Elementary Functions". In summary, it seems that every familiar function given by a formulas in terms of polynomials, roots, powers, trigonometric, exponential and logarithmic functions, and inverses and compositions of these is continuous.

These functions are collectively known as **elementary functions**.

The only failures of continuity that we have seen so far are:

- at gaps in the domain due to division by zero, and

- at points where the "formula" for the functions changes, introducing a jump: for example, rounding in the nearest integer function, or "turning on the power" in the Heaviside function.

Solving Equations with Continuous Functions. One very important feature of continuous function is that there are no "gaps" in the graph, just as there are no "gaps" between the real numbers. (There are gaps between the rational numbers: the places where irrational numbers go.)

For example, if we deal only with rational numbers, the graph of $f(x) = x^2$ does not intersect the horizontal line $x = 2$: it comes very close above and below that line, but no rational number x gives $x^2 = 2$.

On the other hand, the graph of $f(x) = x^2$ for all real numbers x does intersect that line, at $x = \sqrt{2}$ and also $x = -\sqrt{2}$.

Defining Inverse Functions, Like Logarithms. This absence of gaps in the range of an exponential function was useful to define logarithmic functions, because with a function like $f(x) = a^x$, it is ensured that for every positive number y, there is a value x that solves $a^x = y$, and this solution is what we call $x = \log_a y$. This shows that $\log_a$ is defined for domain $(0, \infty)$.

The Intermediate Value Theorem. The sort of equation solving used to get the inverse of a function is always possible for a continuous function:

Theorem 2.4.14 The Intermediate Value Theorem. *If a function f is continuous on an interval $[a, b]$ and m is any value between $f(a)$ and $f(b)$, then f takes the value m somewhere in that interval: there is a number c, $a \leq c \leq b$, for which $f(c) = m$.*

A common case is that whenever a continuous function takes both positive and negative values on some interval, it has a root there too: $f(x) = 0$ has a solution.

Example 2.4.15 Show that the equation $\cos x = x$ has a solution.

1. First, any zero of the function $f(x) = x - \cos x$ will be a solution.
2. We know that $f(0) = -1$ and $f(2) = 2 - \cos 2 \geq 2 - 1 =$, so f changes sign on interval $[0, 2]$.
3. Also, the function f is continuous on this interval.
4. Thus for some number c between 0 and 2, $f(c) = c - \cos c = 0$, so $c = \cos c$.

□

Exercises

Study Calculus Volume 1, Section 2.4[2], Exercises 133, 137, 141, 147, 150, 151, 154, 157, 163, and 165.

2.5 The Precise Definition of a Limit

References.

- OpenStax Calculus Volume 1, Section 2.5.[1]
- *Calculus, Early Transcendentals* by Stewart, Section 2.4.

We have worked with limits so far using the intuitive idea that $\lim_{x \to a} f(x) = L$ means

As x gets "close" to a, $f(x)$ gets "close" to L.

To state this more precisely, we first put this in terms of guarantees of closeness to L:

[2] openstax.org/books/calculus-volume-1/pages/2-4-continuity
[1] openstax.org/books/calculus-volume-1/pages/2-5-the-precise-definition-of-a-limit

We can guarantee that the value of $f(x)$ is "close enough" to L by looking only at values of x that are "close enough" to a.

Next, we measure the closeness of two numbers by the absolute value of their difference being small:

We can guarantee that $f(x) - L$ is "as small as we want" by looking only at values of x with $x - a$ "small enough".

Finally we give a numerical meaning to "small":

- Guaranteeing $f(x) - L$ "as small as we want" means smaller than any chosen positive number, ϵ: $|f(x) - L| < \epsilon$;
- considering only $x - a$ "small enough" means only x values with $|x - a| < \delta$ for some positive value δ.

Putting this all together gives the precise definition of a limit:

Definition 2.5.1 Limit. The **limit** of $f(x)$ as x goes to a is L if for any given positive number ϵ, there is a positive number δ so that having $|x - a| < \delta$, $x \neq a$ ensures that $|f(x) - L| < \epsilon$.

When this is true, we write $\lim_{x\to a} f(x) = L$. ♢

Note that the value of f at $x = a$ is ignored, which in particular allows limits to exist even if f is not defined at $x = a$.

Example 2.5.2 For $f(x) = 2x + 3$, $a = 4$, verify that the limit is $L = 11$: $\lim\limits_{x\to 4}(2x + 3) = 11$.

This is confirmed by using $\delta = \epsilon/2$ (this δ is positive as required).

This is because for $|x-4| < \delta$, $|f(x)-11| = |(2x+3)-11| = |2x-8| = |2(x-4)| = 2|x-4| < 2\delta = \epsilon$. That is, $|f(x) - 11| < \epsilon$, as required.

For example, to get $|f(x) - 11| < 0.001$, so $10.999 < 2x + 3 < 11.001$,

it works to require $|x - 4| < 0.0005$, so that "x is close to 4" in that $3.9995 < x < 4.0005$. □

One-sided Limits. The other types of limits have similar precise definitions. Firstly,

Definition 2.5.3 Right-hand limit. The **right-hand limit** of $f(x)$ as x goes to a is L if for any given positive number ϵ, there is a positive number δ so that having $a < x < a + \delta$ ensures that $|f(x) - L| < \epsilon$. When this is true, we write $\lim_{x\to a^+} f(x) = L$. ♢

Checkpoint 2.5.4 $\sqrt{x}$ is only defined on one side of $x = 0$, so evaluate $\lim_{x\to 0^+} \sqrt{x}$.

Infinite Limits. We need a slightly different measure for $f(x)$ being "close to infinity", and what we use is $f(x) > M$ for large M; likewise having $f(x) < M$ for large negative M measures "closeness to $-\infty$".

Definition 2.5.5 Infinite Limit. The **limit of** $f(x)$ **as** x **goes to** a is **infinity** if for any given number M, there is a positive number δ so that

having $|x - a| < \delta$, $x \neq a$ ensures that $f(x) > M$.

When this is true, we write $\lim_{x\to a} f(x) = \infty$.

Similarly for f having a limit of $-\infty$, using $f(x) < M$ instead. ♢

Exercises

Study Calculus Volume 1, Section 2.5[2], Exercises 177, 184, 185, 187, and 191.

[2] openstax.org/books/calculus-volume-1/pages/2-5-the-precise-definition-of-a-limit

Chapter 3

Derivatives

This chapter introduces the concept of the derivative, and efficient rules for calculating the derivatives of functions.

References.

- OpenStax Calculus Volume 1, Chapter 3.[1]
- *Calculus, Early Transcendentals* by Stewart, Chapter 2, Sections 6 and 7, and Chapter 3, Sections 1–6.

3.1 Defining the Derivative

References.

- OpenStax Calculus Volume 1, Section 3.1.[1]
- *Calculus, Early Transcendentals* by Stewart, Section 2.7.

This section revisits ideas seen earlier in Section 2.1, now done more completely and efficiently using what we know about limits, and revisits examples from those sections. Thus we will not work all the examples in class, but I recommend that you read the whole section and study all the examples.

Tangents. In the Preview of Calculus we saw that the slope of the secant line on the curve $y = f(x)$ between a point $P(a, f(a))$ and another point $Q(x, f(x))$, $x \neq a$, is

$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}.$$

This "slope function" $\frac{f(x)-f(a)}{x-a}$ is undefined at $x = a$, but often it has a *removable discontinuity* there. We also saw there that it makes sense to define the slope of the curve at P as the limit of this secant slope as $x \to a$:

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Typically, the line of this slope m through P touches the curve but does not cross it, so we call it **the tangent line to** $y = f(x)$ **at point** $P(a, f(a))$, or **the tangent at** $x = a$.

[1] openstax.org/books/calculus-volume-1/pages/3-introduction
[1] openstax.org/books/calculus-volume-1/pages/3-1-defining-the-derivative

It is often convenient to let $h = x - a$, the horizontal increment, so that $x = a + h$ and the tangent slope is given by

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This makes it easier to identify the division by zero that we wish to eliminate in order to compute the limit. The formulation in terms of step size h is even more useful when the algebra gets more complicated.

Checkpoint 3.1.1 Find the slopes of the tangent line to $y = f(x) = \sqrt{x}$ at the points $(1, 1)$, $(4, 2)$ and $(9, 3)$, by computing it at a general point $P(a, \sqrt{a})$.

Use the "rationalizing factor"

$$1 = \frac{\sqrt{a+h} + \sqrt{a}}{\sqrt{a+h} + \sqrt{a}}.$$

Velocities. In the Preview we saw that **average velocity** is given by a formula like that for secant slope. For an object whose position at time t is $f(t)$, the average velocity over a time interval of duration h from time a to time $a + h$ is

$$v_{ave} = \frac{f(a+h) - f(a)}{h}.$$

The **instantaneous velocity at time a** is the limit of this as the length of the time interval h approaches zero:

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The quantity $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ has been seen to be important to computing slopes, velocities and other rates of change. It deserve a name, and a short-hand, $f'(a)$:

Definition 3.1.2 The **derivative** of function f at number a is the quantity

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if this limit exists. An alternative form is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \text{ where } x = a + h.$$

$\Diamond$

Interpretation of the Derivative as the Slope of a Tangent Line. The quantity now called $f'(a)$ has been seen as the slope m of the tangent line to a curve. We can now write that

Definition 3.1.3 The **tangent line** to curve $y = f(x)$ at point $P(a, f(a))$ is the line with equation

$$y = l(x) = f(a) + f'(a)(x - a)$$

$\Diamond$

Note well that only x is the variable in the function $l(x)$ here: a is a constant, and so $f(a)$ and $f'(a)$ are also constants.

Interpretation of the Derivative as a Rate of Change. The derivative of position as a function of time is velocity, or the *(time) rate of change of position*. Likewise the derivative of a function is the *rate of change of the value of the function value with respect to change in the value of its argument.*

For any quantity y related to another quantity x by $y = f(x)$, changing the value of x from x_1 to x_2 causes a change in y from $y_1 = f(x_1)$ to $y_2 = f(x_2)$, so that the change by $\Delta x = x_2 - x_1$ in x causes a change of by $\Delta y = y_2 - y_1$ in y. The **difference quotient**, defined by

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

gives *the average rate of change of y respect to x over the interval $[x_1, x_2]$.*

As we adjust x_2 to approach x_1, so that Δx approaches 0, this average rate of change approaches *the instantaneous rate of change of y with respect to x,*

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}, = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

With some name changes (x_1 becoming a; x_2 becoming $a + h$, so that Δx becomes h) this is the same as the definition of the derivative:

The derivative $f'(a)$ of function f at number a is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

Exercises

Study Calculus Volume 1, Section 3.1[2]; in particular Examples 1,2, 3, 5, 6 and 9, Checkpoint items 1, 3 and 4, and Exercises 1, 7, 11, 13, 15, 25, 37, 39, 41 and 51.

3.2 The Derivative as a Function

References.

- OpenStax Calculus Volume 1, Section 3.2.[1]
- *Calculus, Early Transcendentals* by Stewart, Section 2.8.

The formula for the tangent slope at one point $x = a$ on curve $y = f(x)$ can also be seen as a function, with argument x and value the slope at the corresponding point:

Definition 3.2.1 Derivative. The function f' given by

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

is the **derivative** of function f. The domain of f' is the x values for which this limit exists; this can be smaller than the domain of f. ◊

(The name comes from the ideas that this new function is *derived* from the original one.)

One basic physical example is that when $f(t)$ is position as a function of time, $f'(t)$ is velocity as a function of time. We will study the geometry of this new function, and some useful things that it can tell us about the original function.

This helps with questions like getting information about position from measurements of velocity.

Example 3.2.2

[2] openstax.org/books/calculus-volume-1/pages/3-1-defining-the-derivative
[1] openstax.org/books/calculus-volume-1/pages/3-2-the-derivative-as-a-function

- Draw some smooth graph of some function f: no formula needed.
- Try to sketch the graph of its derivative f'.

□

Example 3.2.3

- If $f(x) = x^3 - x$, find a formula for $f'(x)$.
- Illustrate by comparing the graphs of f and f'.

□

Example 3.2.4

- For $f(x) = \sqrt{x}$, find the derivative f'.
- What is the domain of f'?

□

Example 3.2.5 Calculate the derivative of $f(x) = (1 - x)/(2 + x)$. □

Other Notations. Many different notations are used for the derivative of $y = f(x)$. Besides f' as above (introduced by Joseph-Louis Lagrange) we often use $\frac{dy}{dx}$ or $\frac{df}{dx}$, introduced by Gottfried Leibniz. Another notation is Df, due to Leonhard Euler: it will be used less here, but can be convenient at times; when it is important to identify the independent variable, the variant $D_x f$ is used. So here is a collection of synonyms:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f = Df(x) = D_x f(x).$$

Motivation for the Leibniz Notation. The Leibniz notation is suggested by another way to write the formula for the derivative:

write Δx for h, the change in argument x, and let $\Delta y = f(x+h) - f(x)$ be the corresponding change in y. Then

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

and the change in notation from a capital Δ to a small d suggests the very small changes involved. In fact, Leibniz and other pioneers of calculus thought in terms of the derivative as the ratio of "infinitely small" ("infinitesimally small") increments in both the x and y values: the precise use of the limit idea came later. This connection back to the division used to define the derivative is useful in some calculations later.

Differentiability: at Some Values, and Everywhere. Limits do not always exist, so the derivative does not always exist. Thus, similar to continuity we have:

Definition 3.2.6 A function f is **differentiable at a** if $f'(a)$ exists.

It is **differentiable on an open interval** (a, b), $(-\infty, a)$, (a, ∞) or $(-\infty, \infty)$ if it is differentiable at every number in that interval.

If a function is differentiable at every number in its domain, we simply call it **differentiable**. ◊

Example 3.2.7

- Where is the function $f(x) = |x|$ differentiable?
- Give its derivative.
- Where is the function f continuous?

□

Differentiability and Continuity. The above example shows that sometimes, a function can be continuous at a, but not differentiable at a. The opposite is not true however:

Theorem 3.2.8 *If a function is differentiable at a, it is also continuous at a.*

Proof. Differentiability says that the limit $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists. Using the product law for limits,

$$\begin{aligned}
\lim_{h\to 0} f(a+h) - f(a) &= \lim_{h\to 0}\left(h \cdot \frac{f(a+h)-f(a)}{h}\right) \\
&= \left(\lim_{h\to 0} h\right)\left(\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}\right) \\
&= 0 \cdot f'(a) = 0.
\end{aligned}$$

Then the addition rule for limits gives

$$\begin{aligned}
\lim_{h\to 0} f(a+h) &= \lim_{h\to 0}[(f(a+h)-f(a)) + f(a)] \\
&= \lim_{h\to 0}[f(a+h)-f(a)] + \lim_{h\to 0} f(a) = 0 + f(a) = f(a).
\end{aligned}$$

With $x = a+h$, as $x \to a$, $h = x - a \to 0$, so

$$\lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h) = f(a): \text{ continuity at } a.$$

■

How Can a Function Fail to be Differentiable? The above theorem tells us one way that a function can fail to be differentiable at a: if f is not continuous at a, it is not differentiable there either. So a function is non-differentiable at jump discontinuities, removable discontinuities, places where it has vertical asymptotes, and places where wilder behavior occurs, as with $\sin(1/x)$ at 0. But Example 5 shows another situation, where a function can fail to be differentiable at a point even though it is continuous there. The problem there is a "corner" at the origin, where the graph does not have a well defined tangent line. In fact any line through the origin of slope between -1 and 1 is "tangent" to $y = |x|$ at the origin, in that it touches the curve but does not cross it. One other situation where the derivative does not exist is when a graph effectively has a vertical tangent line, or an infinite slope at a point.

Example 3.2.9 No derivative at a point due to a vertical tangent. Consider $y = f(x) = x^{1/3}$. It is continuous everywhere, but if we try to compute the derivative at $x = 0$, we get

$$\lim_{h\to 0} \frac{h^{1/3} - 0^{1/3}}{h} = \lim_{h\to 0} \frac{h^{1/3}}{h} = \lim_{h\to 0} \frac{1}{h^{2/3}}, \text{ DNE.}$$

Actually there is an infinite limit, so in a sense the graph has an infinite tangent slope at the origin: a *vertical tangent line*. □

Continuity vs Differentiability. Intuitively,

- the graph of a continuous function has no *breaks*, whereas
- the graph of a differentiable function has no *breaks*, *sharp corners*, or *vertical tangents*.

Second Derivatives. If function f is differentiable, its derivative f' is also a function, and so can have a derivative, called **the second derivative of f.** The Lagrange notation adds another prime: $(f')'$ or more commonly f''; The Leibnitz notations adds another "d/dx"; the Euler notation adds

another "factor" of D:

$$f' = \frac{d}{dx}f = \frac{df}{dx} = Df,\ (f')' = f'' = \frac{d}{dx}\frac{d}{dx}f = \frac{d}{dx}\frac{df}{dx} = \frac{d^2f}{dx^2} = D^2f.$$

The more compact version of Leibnitz notation treats each "d" on the top and each "dx" on the bottom as if they were factors in a fraction.

Example 3.2.10 For $f(x) = x^3 - x$:

- Calculate $f''(x)$.
- Give a geometrical interpretation of f''.

□

Third and Higher Derivatives. The above can be repeated, leading to the third derivative of f, denoted f''' or $\frac{d^3f}{dx^3}$ or D^3f, and so on to the n-th derivative for any natural number n. The primes can get messy when we do this too often, so the n-th derivative of f (it it exists) is also denoted $f^{(n)}$, or $\frac{d^nf}{dx^n}$. The Euler notation is perhaps the most elegant: D^nf. Note the parentheses in the Lagrange form: $f^{(n)}(x)$ is different from $f^n(x)$, which is $[f(x)]^n$.

Example 3.2.11 For $f(x) = x^3 - x$ as above:

- Calculate $f^{(3)}$.
- Calculate $f^{(4)}$.
- Calculate $f^{(17)}$.

□

Exercises

Study Calculus Volume 1, Section 3.2[2]; in particular all Examples and Checkpoint items are worth reviewing, along with Exercises 55, 57, 65, 67, 79, 80 and 96.

3.3 Differentiation Rules

References.

- OpenStax Calculus Volume 1, Section 3.3[1], and Section 3.9[2] for exponential functions.
- *Calculus, Early Transcendentals* by Stewart, Sections 3.1 and 3.2.

Derivatives of Linear Functions. Since the slope of a straight line $y = mx + c$ is the constant m, it is easy to check that the derivative of $f(x) = mx + c$ is m, for any constants m and c. It is often convenient to write calculations directly with formulas, without naming the functions, so to illustrate several notations:

Theorem 3.3.1 *The derivative of the linear function $f(x) = mx + c$ is*

$$f'(x) = (mx + c)' = \frac{d}{dx}(mx + c) = m.$$

The two most basic special cases are when the function is a constant c or just x:

$$(c)' = \frac{d}{dx}(c) = 0, \quad (x)' = \frac{d}{dx}(x) = 1.$$

[2] openstax.org/books/calculus-volume-1/pages/3-2-the-derivative-as-a-function
[1] openstax.org/books/calculus-volume-1/pages/3-3-differentiation-rules
[2] openstax.org/books/calculus-volume-1/pages/3-9-derivatives-of-exponential-and-logarithmic-functio

Derivatives of Power Functions. We have already computed the derivatives of a few powers functions like $(x^2)' = 2x$ and $(x^3)' = 3x^2$, and these fit a more general pattern:

Theorem 3.3.2 The Power Rule (for Derivatives). *For any non-negative integer n,*

$$(x^n)' = \frac{d}{dx}(x^n) = n \cdot x^{n-1}.$$

This also agrees with the results seen above for $f(x) = x^1 = x$ and $f(x) = x^0 = 1$.

Note that there is also a "Power Rule for Limits": from now on, when we simply say "power rule", we mean this one for derivatives.

To see the pattern that helps us to get the general rule, let us look at $n = 4$:

Example 3.3.3 Calculate the derivative of $f(x) = x^4$.

Use the first formula for the derivative $f'(a)$:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \lim_{x \to a} \frac{x^4 - a^4}{x - a}.$$

The numerator vanishes for $x = a$, so it has a factor $x - a$, and in fact the factorization is $x^4 - a^4 = (x - a)(x^3 + x^2 \cdot a + x \cdot a^2 + a^3)$. (Check by expanding!)

This gives

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{(x - a)(x^3 + x^2 \cdot a + x \cdot a^2 + a^3)}{x - a} \\ &= \lim_{x \to a} (x^3 + x^2 \cdot a + x \cdot a^2 + a^3) \\ &= 4a^3. \end{aligned}$$

That is, $f'(x) = (x^4)' = 4x^3$, as the Power Rule above says. □

Let us try the power rule, using various different notations for derivatives:

Example 3.3.4 For $f(x) = x^6$, find $f'(x)$. $f'(x) = 6x^5$ □

Example 3.3.5 For $y = x^{1000}$, find y'. $y' = 1000x^{999}$ □

Example 3.3.6 For $y = t^4$, find $\frac{dy}{dt}$. $\frac{dy}{dt} = 4t^3$ □

Example 3.3.7 Find $\frac{d}{dr}(r^3)$. $\frac{d}{dr}(r^3) = 3r^2$ □

Proof of the Power Rule. The key step is the factorization

$$x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1})$$

This can be checked by expanding the right hand side, distributing the left hand factor:

$$\begin{aligned} &(x - a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}) \\ &= x(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}) - a(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}) \\ &= x^n + x^{n-1}a + x^{n-2}a^2 + \cdots + x^2a^{n-2} + xa^{n-1} - x^{n-1}a - x^{n-2}a^2 - x^{n-3}a^3 - \cdots - xa^{n-1} - a^n \\ &= x^n - a^n \end{aligned}$$

because all the terms in between pair off and cancel out.

Much as with x^4, the definition of the derivative gives the derivative of $f(x) = x^n$ at $x = a$ as

$$f'(a) = \lim_{x \to a} \frac{x^n - a^n}{x - a}$$

$$
\begin{aligned}
&= \lim_{x\to a} \frac{(x-a)(x^{n-1}+x^{n-2}\cdot a + x^{n-3}\cdot a^2 + \cdots + a^{n-1})}{x-a} \\
&= \lim_{x\to a} (x^{n-1}+x^{n-2}\cdot a + x^{n-3}\cdot a^2 + \cdots + a^{n-1}) \\
&= a^{n-1}+a^{n-2}\cdot a + a^{n-3}\cdot a^2 + \cdots + a^{n-1} \\
&= a^{n-1}+a^{n-1}+a^{n-1}+\cdots+a^{n-1} \text{ (n copies)} \\
&= na^{n-1},
\end{aligned}
$$

so $f'(x) = (x^n)' = nx^{n-1}$, as claimed.

Constant Multiples, Sums and Differences. As with limits, we can build up polynomials from these power functions using constant multiples, sums and differences. The derivatives of these three basic combinations are as simple as with limits:

Theorem 3.3.8 The *Constant* Multiple Rule. *If a differentiable function f is multiplied by a constant c, this product is also differentiable, with derivative:*

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)], \text{ or } (cf)'(x) = cf'(x).$$

Theorem 3.3.9 The Sum Rule. *The sum of two differentiable functions f and g is differentiable, with the sum's derivative the sum of their derivatives:*

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x), \text{ or } (f+g)'(x) = f'(x)+g'(x).$$

Theorem 3.3.10 The Difference Rule. *The difference of two differentiable functions f and g is differentiable, with its derivative the difference of their derivatives:*

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x), \text{ or } (f-g)'(x) = f'(x)-g'(x).$$

Warning: The rules seen so far are the *only* ones that are as simple and "guessable" as for limits!

Checkpoint 3.3.11 Compute the derivative of $y = x^8 + 12x^5 - 4x^2 + 10x^3 - 6x + 5$. Note that this derivative is also a polynomial.

The same approach works for differentiating any polynomial:

All polynomials are differentiable, and their derivatives are polynomials, so the second and higher derivatives also exist.

Checkpoint 3.3.12 Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent is horizontal.

Derivatives of Other Power Functions. Example 3 in Section 2.8 shows that $\sqrt{x}$ has derivative $1/(2\sqrt{x})$. That is, $\frac{d}{dx}x^{1/2} = (1/2)x^{1/2-1}$. This fits the power rule, but for power $1/2$, not a positive integer. In fact, this works for all real powers:

Theorem 3.3.13 The Power Rule, Generalized Version. *For any real number a,*

$$\frac{d}{dx}x^a = ax^{a-1}, \text{ or } (x^a)' = ax^{a-1}. \tag{3.3.1}$$

This is most easily shown later when we know how to differentiate exponential functions and compositions of functions.

Example 3.3.14 If $f(x) = 1/x^2$, find $f'(x)$. $f'(x) = -2/x^3$ □

Example 3.3.15 If $y = \sqrt[3]{x^2}$, find y'. $y' = 3/(2\sqrt[3]{x})$ □

Checkpoint 3.3.16 Differentiate $3/x$.

To simplify differentiation of more functions, we would like to be able to deal with products, quotients, compositions and inverses, much as we did with limits and continuity.

Warning: none of these derivatives are given by rules quite as simple as for limits. Only *sums*, *differences* and *constant multiples* work that simply.

Example 3.3.17 A: a cautionary one. Compare

- the derivative of the product $x \cdot x^2 = x^3$
- the product of the derivatives of x and x^2.

The Power Rules tells us that the derivative of this product is $3x^2$; the product of the derivatives of x and x^2 is $1 \cdot 2x = 2x$: *not the same!* □

In this section we will see what the rules really are for products and quotients; compositions will be handled in Section 3.6.

The Derivative of a Product of Functions. The Leibniz notation is nice here. Let $u = f(x)$, $v = g(x)$, and compute the derivative of the product uv using the formula

$$\frac{d(uv)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta(uv)}{\Delta x} \tag{3.3.2}$$

What is $\Delta(uv)$?

First look at what Δu and Δv are: $\Delta u = f(x + \Delta x) - f(x)$, so $f(x + \Delta x) = u + \Delta u$, and likewise $g(x + \Delta x) = v + \Delta v$.

Next, $\Delta(uv)$ is the change in the value of the product $f(x)g(x)$ as the argument changes from x to $x + \Delta x$:

$$\Delta(uv) = f(x + \Delta x)g(x + \Delta x) - f(x)g(x) = (u + \Delta u)(v + \Delta v) - uv = u\Delta v + v\Delta u + \Delta u \Delta v$$

That is,

$$\Delta(uv) = u\Delta v + v\Delta u + \Delta u \Delta v$$

The difference quotient in (3.3.2) is thus

$$\frac{\Delta(uv)}{\Delta x} = \frac{u\Delta v + v\Delta u + \Delta u \Delta v}{\Delta x} = \frac{u\Delta v}{\Delta x} + \frac{v\Delta u}{\Delta x} + \frac{\Delta u \Delta v}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x}\frac{\Delta v}{\Delta x}\Delta x \tag{3.3.3}$$

The two difference quotients here have limits as $\Delta x \to 0$, as do the factors multiplying them: $\frac{\Delta u}{\Delta x} \to \frac{du}{dx}$, $\frac{\Delta v}{\Delta x} \to \frac{dv}{dx}$, $\Delta x \to 0$, and u and v do not vary as $\Delta x \to 0$. Thus we can compute the limit as $\Delta x \to 0$ in Equation (3.3.3):

Theorem 3.3.18 1. The Product Rule for Derivatives.

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}, \text{ or } (fg)' = f'g + fg'.$$

Note that change in each factor in the products adds to the total change in the product, so you *add* a term for the derivative of each factor to get the derivative of the product.

Example 3.3.19

- If $f(x) = xe^x$, calculate its derivative $f'(x)$.
- Compute the second and third derivatives f'' and f''' of this function.
- Compute all the derivatives $f^{(n)}$ of this function.

□

Example 3.3.20 Differentiate (compute the derivative of) the function $f(t) = \sqrt{t}(a + bt)$. Do this two ways, with and without the Product Rule. Hint: as usual, it can help to rewrite roots as powers, and division by powers and roots as negative powers. □

Sometimes, you only know the values of a function and its derivative at one point, like having measurements of the position and velocity of an object at one time. This can be enough to compute the derivative of another function got from the first with a product or such:

Example 3.3.21 If $f(x) = \sqrt{x}g(x)$, calculate $f'(x)$ in terms of x and $g'(x)$, and use this to find $f'(4)$ given that $g(4) = 2$ and $g'(4) = 3$. □

The Derivative of the Reciprocal of a Function. To deal with division, start with the simplest case, $f(x) = 1/g(x)$, and use a common strategy:

Rephrase a new problem in terms of a problem we have already solved.

In this case, we can restate the situation in terms of a product, and use the Product Rule. First, clear the denominator, getting $f(x)g(x) = 1$. Then use the product rule to get

$$f'(x)g(x) + f(x)g'(x) = 0.$$

Solve for f' and substitute in $f(x) = 1/g(x)$:

$$f'(x) = -g'(x)f(x)/g(x) = -g'(x)/[g(x)]^2,$$

or

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}.$$

The minus sign goes with the fact that an *increase* in g will cause a *decrease* in $1/g$.

The Quotient Rule for Derivatives. Now it is east to get a rule for the derivative of any quotient, by combining the product and reciprocal rules:

$$\left(\frac{f}{g}\right)' = \left(f \cdot \frac{1}{g}\right)' = f' \cdot \frac{1}{g} + f \cdot \left(\frac{1}{g}\right)' = \frac{f'}{g} - f \cdot \frac{g'}{g^2} = \frac{f'g - fg'}{g^2}.$$

Theorem 3.3.22 The Quotient Rule for Derivatives.

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}, \quad \textit{or} \quad \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)] \cdot g(x) - f(x) \cdot \frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Note well where the minus sign goes! *the derivative of the bottom factor gets a minus sign.* We saw that with products, the derivative of each factor *adds* to the total derivative of the product; now *the derivative of the top factors add*}, while *the derivative of the bottom factor subtracts*.

Example 3.3.23

1. Differentiate $y = \dfrac{x^2 + x - 2}{x^3 + 6}$.
2. Find the equation of the tangent line to this curve at point $P(-1, -2/5)$.

□

Remark 3.3.24 The derivatives of exponential functions are not covered in the OpenStax text till Section 3.9, but these functions are so important that I like to introduce these facts as soon as possible.

Derivative of the Natural Exponential Function. One way to define the number e is so that the slope of $y = e^x$ at point $(0, 1)$ is 1. That is,

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

This choice makes the derivative of e^x simple:

$$\begin{aligned}
\frac{d}{dx}e^x &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} \\
&= e^x \lim_{h\to 0} \frac{e^h - 1}{h} \\
&\text{(as } e^x \text{ is a constant as far as this limit is concerned: } h \text{ is the variable!)} \\
&= e^x \cdot 1 \\
&= e^x.
\end{aligned}$$

So the natural exponential function is equal to its own derivative:

Theorem 3.3.25 Derivatives of Exponential Functions.

$$\frac{d}{dx}e^x = e^x.$$

Further, as will be shown in Section 3.4,

$$\frac{d}{dx}a^x = (\ln a)a^x \quad \textit{for any constant } a > 0.$$

Thus all exponential functions have a rate of change proportional to their current value.

This fits for example with the simple exponential model of a population whose growth rate is proportional to its current size because the rates of births and deaths are both proportional to current population.

Geometrical Explanation of the Derivative of $\mathbf{a^x}$. The result for the derivative of a^x can be seen graphically by writing $a = e^{\ln a}$ so that

$$a^x = (e^{\ln a})^x = e^{(\ln a)x}.$$

The effect of changing from $f(x)$ to $g(x) = f(kx)$ is to compress the graph horizontally by a factor of k, increasing the slope at corresponding points by a factor k: in terms of derivatives,

$$\frac{d}{dx}f(kx) = kf'(kx)$$

Thus, the graph of a^x is a compression of the graph of e^x by factor $\ln a$ and

$$\frac{d}{dx}a^x = (\ln a)e^{(\ln a)x} = (\ln a)a^x..$$

In Section 3.4 we will see another way to compute this derivative, using a derivative rule for composition of functions.

Example 3.3.26 If $f(x) = e^x - x$, find $f'(x)$ and $f''(x)$, and then compare the graphs of f and f''.

1. $f'(x) = e^x - 1$

2. $f''(x) = e^x$ □

Checkpoint 3.3.27 At what point on the curve $y = e^x$ is the tangent parallel to the line $y = 2x$?

Differentiation Facts So Far.

- $\frac{d}{dx}(c) = 0$
- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(a^x) = (\ln a)a^x$
- $\frac{d}{dx}(cf) = c\frac{df}{dx}$
- $\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$
- $\frac{d}{dx}(f-g) = \frac{df}{dx} - \frac{dg}{dx}$
- $\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$
- $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{df}{dx}g - f\frac{dg}{dx}}{g^2}$

What is missing so far? Mostly, derivatives of

- trigonometric functions,
- inverses of functions, including logarithms, and
- compositions of functions.

These gaps will be filled in the next few sections.

Exercises

Study Calculus Volume 1, Section 3.3[3]; in particular all Examples are worth reviewing, along with Checkpoint items 12 to 19 and Exercises 107, 109, 111, 119, 122, 127, 129, 130, 131, 133, 142, 143 and 147.

3.4 Derivatives as Rates of Change

Reference. OpenStax Calculus Volume 1, Section 3.4[1]

Now that we know how to evaluate the derivatives of some common functions efficiently, let us look at some of the places that this is useful in scientific problems. We will look at velocity and acceleration, population growth, and marginal cost.

Rate of Change of One Quantity Relative to Another. The basic idea of a rate of change is that there is a relationship between changes in one quantity x and another, y, due to there being a function connecting the two quantities, $y = f(x)$. (Note: you might not have a formula for this function!) The pairs might be

- time and position,
- pressure and volume in a quantity of gas,
- horizontal and vertical position of a person traveling over hilly terrain,

or many other combinations.

Average Rate of Change of One Quantity Relative to Another. Suppose that the first quantity changes in value from x_1 to x_2, the second changes from y_1 to y_2; in terms of the function, from $f(x_1)$ to $f(x_2)$.

The change in the first quantity is $\Delta x = x_2 - x_1$ (spoken "delta x"), the corresponding change in the second quantity is $\Delta y = y_2 - y_1 = f(x_2) - f(x_1)$ ("delta y").

[3] openstax.org/books/calculus-volume-1/pages/3-3-differentiation-rules
[1] openstax.org/books/calculus-volume-1/pages/3-4-derivatives-as-rates-of-change

The **average rate of change of y with respect to x** over the interval from x_1 to x_2 is given by the **difference quotient**

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

This is the slope of the secant line between the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the graph of y vs. x.

Instantaneous Rate of Change of One Quantity Relative to Another. In the limit of $x_2 \to x_1$, with x_1 unchanging, so $\Delta x \to 0$, we get the *instantaneous rate of change of y with respect to x* at x_1,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}(x_1) = f'(x_1).$$

Thus, whenever a derivative arises like this in a physical context, it has an interpretation as a *rate of change.*

Physics: Velocity and Acceleration. We have already seen one example: when position s (along a line) is a function of time t, $s = f(t)$ the the rate of change of position with respect to time is the **velocity**,

$$v(t) = s'(t) = f'(t) = \frac{ds}{dt}.$$

Many basic laws of physics describe the way that a force causes the velocity of an object to change at a certain rate, and the rate of change of velocity is **acceleration**,

$$a(t) = v'(t) = \frac{dv}{dt} = s''(t) = f''(t) = \frac{d^2 s}{dt^2}.$$

Checkpoint 3.4.1 An object moving back and forth along a line is at position $s = f(t) = t^3 - 6t^2 + 9t$ meters to the right of its starting point at time t seconds (a negative s value means that it is to the left.)

- Find its velocity at time t.
- What is the velocity after 2 seconds? After 4 seconds?
- When is the object at rest?
- When is it moving to the right?
- Draw a diagram to represent the motion of the object.
- Find the total distance traveled by the object during the first five seconds.

Bacterial Population Growth. A population of bacteria often grows by cells dividing at a roughly fixed time period, T. If the initial number of bacteria is P_0, then at a time nT the number of bacteria is $P = f(nT) = P_0 2^n$. To write this as a function of t, use $t = nT$ so $n = t/T$, so

$$P = f(t) = P_0 2^{t/T}.$$

If this exponential patterns hold at all times, not just multiples of T (the bacteria are probably not synchronized to divide at the same time), we can try to compute the rate of change of the population,

$$\frac{dP}{dt} = \frac{d(P_0 2^{t/T})}{dt} = P_0 \ln 2 \cdot 2^{t/T} \frac{1}{T} = kP, \quad k = \frac{\ln 2}{T}.$$

The exponential function fits the expected pattern of population growth rate being proportional to current population size. As always, the exponential can be rewritten using the natural exponential:

$$P(t) = P_0 e^{\ln 2 \cdot t/T} = P_0 e^{kt}, \quad k = \frac{\ln 2}{T}.$$

Example 3.4.2 If the initial size of a bacterial population is $P_0 = 100$ and it doubles every five hours:

1. How fast is it growing initially?
2. How fast is it growing after two days?
3. Compare the second answer to the average rate of change between days 1 and 3.

1. The population as a function of time is $100 \cdot 2^{t/5}$, measuring time in hours. Thus the growth rate is $100k2^{t/5}$, $k = (\ln 2)/5$, approximately $13.862^{t/5}$. This gives the initial growth rate as $100k \approx 13.86$ bacteria per hour.

2. The growth rate two days [48 hours] later as $100k2^{48/5} \approx 10,758$ bacteria per hour.

3. The average growth rate "around" day two, time $t = 48$, from $t = 24$ to $t = 72$, is $[f(72) - f(24)]/48 = 44,981$. The average is distinctly higher, because the growth rate increases so much between days 2 and 3. □

Economics: Cost Functions and Marginal Cost. If the cost C of producing x units of a product depends only on the number produced, it is given by some function $C(x)$. Increasing production level from x_1 units to x_2 incurs a change (increase) in cost of $\Delta C = C(x_2) - C(x_1)$, for a change (increase) in production of $\Delta x = x_2 - x_1$. The average rate of increase of cost with respect to increase in production is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}$$

Even though the production level is an integer, it can be extremely large, so that x might be in units of thousands or millions, and then increments Δx in production level can be small, so the average rate of change of cost is well approximated by the derivative at x_1:

$$\frac{\Delta C}{\Delta x} \approx \frac{dC}{dx}(x_1) = C'(x_1).$$

This is called the **marginal cost**, also well approximated by $\Delta C/\Delta x$ for an increase in production by a single unit (which might be Δx far smaller than 1, depending on the units used.) A decision to increase production might be made by comparing this marginal cost to the price at which extra units could be sold.

Checkpoint 3.4.3 Consider the cost function $C(x) = 10,000 + 5x + 0.01x^2$

1. Find the marginal cost where the production level is 500 units.
2. Compare this to the actual added cost of making one more unit (501 instead of 500).
3. At what production level does the marginal cost reach \$20 per item? (Important to know if that is the selling price!)

Exercises

Study Calculus Volume 1, Section 3.4[2]; in particular Examples 34 to 36, Checkpoint item 22, and Exercises 151, 159 and 165.

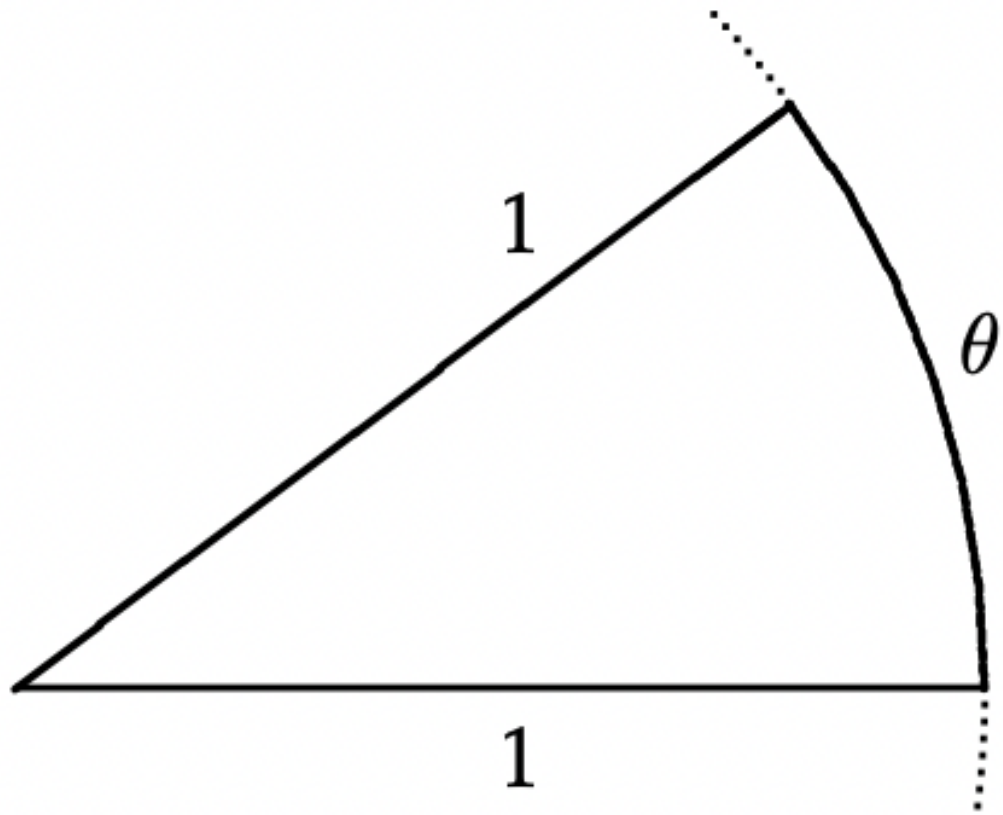
3.5 Derivatives of Trigonometric Functions

References.

- OpenStax Calculus Volume 1, Section 3.5[1]
- *Calculus, Early Transcendentals* by Stewart, Section 3.3.

[2] openstax.org/books/calculus-volume-1/pages/3-4-derivatives-as-rates-of-change
[1] openstax.org/books/calculus-volume-1/pages/3-5-derivatives-of-trigonometric-functions

To find the derivatives of trigonometric functions, note first that we always use radian measure which means that the angle θ is the length along the arc of the unit circle bounding a sector of angle θ

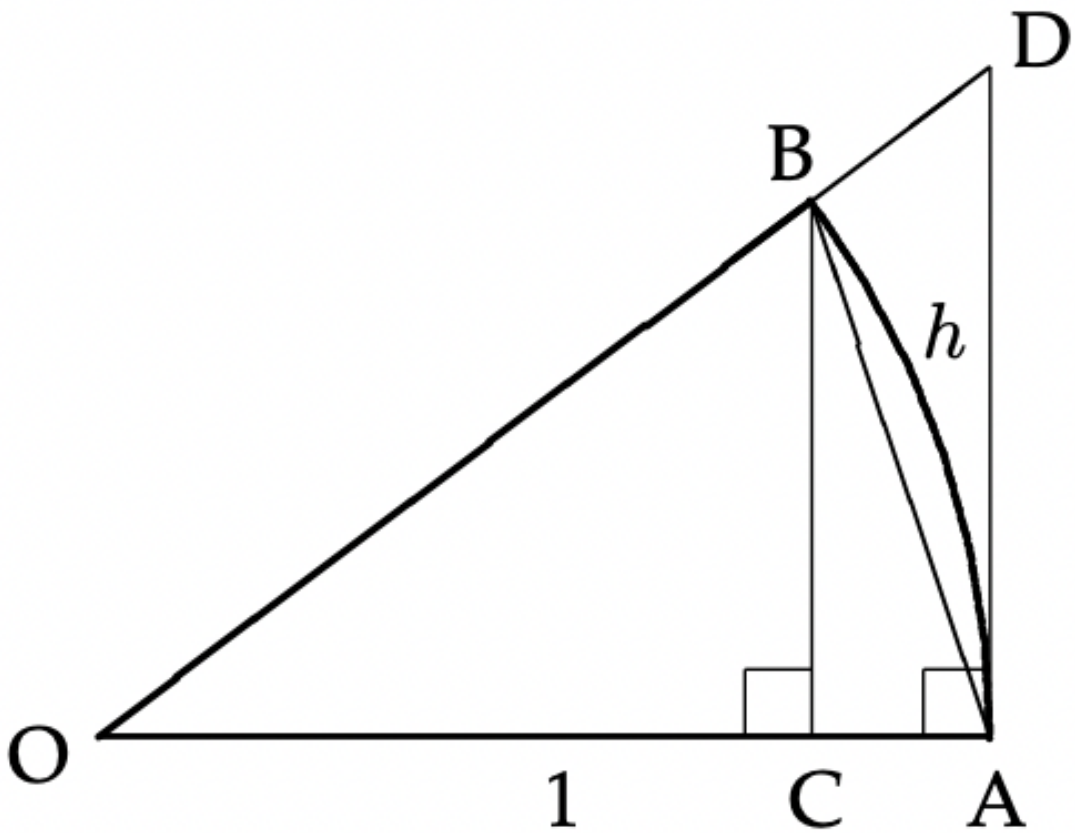


Two calculations are the basis of all others: the derivatives of sine and cosine at the origin:

$$\sin'(0) = \lim_{h\to 0} \frac{\sin(0+h) - \sin 0}{h} = \lim_{h\to 0} \frac{\sin h}{h} = 1, \tag{3.5.1}$$

$$\cos'(0) = \lim_{h\to 0} \frac{\cos(0+h) - \cos 0}{h} = \lim_{h\to 0} \frac{\cos h - 1}{h} = 0. \tag{3.5.2}$$

These will be calculated with the Squeeze Theorem from Section 2.3, p. 12 and the trigonometry in this picture:



The line segment CB has length $\sin h$, so triangle $\triangle OAB$ has base 1, height $\sin h$, area $\frac{1}{2}\sin h$.
The sector OAB with arc of length h and sides of length 1 has area $\frac{1}{2}h$.
The line segment AD has length $\tan h$, so triangle $\triangle OAD$ has base 1, height $\tan h$, area $\frac{1}{2}\tan h$.
Comparing these areas,

$$\sin h \leq h \leq \tan h \quad \text{for angle in the first quadrant.}$$

(For negative angles in the fourth quadrant, the order is reversed.)
Using $\tan h = \dfrac{\sin h}{\cos h}$, and dividing through by h,

$$\frac{\sin h}{h} \leq 1 \leq \frac{\sin h}{h}\frac{1}{\cos h}, \text{ true also for negative } h.$$

Multiplying the second inequality by $\cos h$ gives $\cos h \leq \dfrac{\sin h}{h}$; since we already have $\dfrac{\sin h}{h} \leq 1$, we now have $\dfrac{\sin h}{h}$ squeezed:

$$\cos h \leq \frac{\sin h}{h} \leq 1$$

Thus, as $h \to 0$, we have $\cos h \to \cos 0 = 1$ and $1 \to 1$, which combine to force $\dfrac{\sin h}{h} \to 1$: this confirms Equation (3.5.1).

We can also use this to calculate the derivative of cosine at the origin. The difference quotient used to get the derivative is:

$$\frac{\cos h - 1}{h} = \frac{\cos h - 1}{h} \cdot \frac{1 + \cos h}{1 + \cos h} = \frac{\cos^2 h - 1}{h(1 + \cos h)} = \frac{-\sin^2 h}{h(1 + \cos h)} = -\frac{\sin h}{h} \cdot \frac{\sin h}{1 + \cos h}.$$

Thus,

$$\begin{aligned}\lim_{h \to 0} \frac{\cos h - 1}{h} &= \lim_{h \to 0} \left(-\frac{\sin h}{h} \cdot \frac{\sin h}{1 + \cos h} \right) \\ &= -\lim_{h \to 0} \left(\frac{\sin h}{h} \right) \lim_{h \to 0} \left(\frac{\sin h}{1 + \cos h} \right) = -1 \cdot \frac{0}{1 + 1} = 0,\end{aligned}$$

confirming Equation (3.5.2).

With this we can compute the derivative of $\sin x$ at any point, using the addition formulas

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \tag{3.5.3}$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y \tag{3.5.4}$$

$$\begin{aligned}\frac{d}{dx} \sin x &= \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h} \\ &= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}, \text{ using the above sin-of-sum formula} \\ &= \lim_{h \to 0} \left[\sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h} \right] \\ &= (\sin x) \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + (\cos x) \cdot \lim_{h \to 0} \frac{\sin h}{h} \\ &= (\sin x) \cdot 0 + (\cos x) \cdot 1, \text{ using the above results for the derivatives at zero} \\ &= \cos x\end{aligned}$$

that is,

$$\frac{d}{dx} \sin x = \cos x. \tag{3.5.5}$$

With a similar calculation, we get:

$$\frac{d}{dx} \cos x = -\sin x. \tag{3.5.6}$$

Note well where the minus sign is!

Checkpoint 3.5.1 Sketch the graphs of $\sin x$, $\cos x$ and their derivatives, and use it to explain why the minus sign above makes sense.

Checkpoint 3.5.2 Find the derivative of $\dfrac{x^2 \sin x}{1 + \cos x}$.

Checkpoint 3.5.3 Complete this list of the derivatives of the other four standard trigonometric functions:

- $\dfrac{d}{dx} \tan x =$
- $\dfrac{d}{dx} \cot x =$
- $\dfrac{d}{dx} \sec x =$
- $\dfrac{d}{dx} \csc x =$

Note that the derivative of each complementary "co-" function has a minus sign and swapping of "co-" and "non-co" functions.

Checkpoint 3.5.4

- Differentiate $f(x) = \dfrac{\sec x}{1 + \tan x}$
- For what values of x does $f(x)$ have a horizontal tangent?

Exercises

Study Calculus Volume 1, Section 3.5[2]; in particular Examples 39 to 44, Checkpoint items 25 to 30, and Exercises 175, 178, 181, 182, 191, 197 and 206.

3.6 The Chain Rule

References.

- OpenStax Calculus Volume 1, Section 3.6[1] (and Section 3.9[2] for exponentials and logarithms)
- *Calculus, Early Transcendentals* by Stewart, Section 3.4.

The Chain Rule shows how to differentiate compositions of functions, which also allows us to differentiate inverses. These are the last main methods for building new functions from old, and so complete the tools we need to compute the derivatives of all the *elementary* functions. Along the way we will verify that the power rule works for all real powers, as stated in Section 3.3, p. 26, and see the easy way to compute the derivative of cosine.

Example 3.6.1 Unit Conversions as Rates of Change. Suppose that m is length in miles, y is length in yards, and i is length in inches. The rate of change of y relative to m is 1760: each one mile increase is a 1760 yard increase. Likewise, rate of change of i relative to y is 36. So what is the rate of change of i relative to m?Clearly $1760 \cdot 36$, the number of inches in a mile: the rates of change *multiply*. In formulas, $y = 1760m$ and $i = 36y$, and the above results are

$$\frac{dy}{dm} = 1760, \quad \frac{di}{dy} = 36, \quad \frac{di}{dm} = \frac{di}{dy}\frac{dy}{dm} = 36 \cdot 1760.$$

In terms of functions, $i = f(y) = 36y$, $y = g(m) = 1760m$ and inches as a function of miles is $i = f(g(m))$. Composition takes miles m, first applies the "inside" function g to "input" m to get yards y, and then applies the "outside" function f to its "input" y (which is the "output" of g) to get the final "output" value for inches i. □

The simple pattern above of multiplying derivatives works for all compositions:

Theorem 3.6.2 1. The Chain Rule for the Derivative of a Composition. *If two differentiable functions f and g are composed giving $F = f \circ g$, its derivative is the product of their derivatives, each derivative evaluated at the same argument as the corresponding original function:*

$$\begin{aligned} F'(x) &= f'(g(x)) \cdot g'(x), \quad \textit{or} \\ (f \circ g)'(x) &= f'(u) \cdot g'(x), \ \textit{with } u = g(x). \end{aligned} \tag{3.6.1}$$

Note well that the argument of f' is $g(x)$, just as the argument of f is $g(x)$ in the composition.

In Leibniz notation, with $y = f(u)$ and $u = g(x)$ so $y = f(g(x))$,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}. \tag{3.6.2}$$

[2] openstax.org/books/calculus-volume-1/pages/3-5-derivatives-of-trigonometric-functions
[1] openstax.org/books/calculus-volume-1/pages/3-6-the-chain-rule
[2] openstax.org/books/calculus-volume-1/pages/3-9-derivatives-of-exponential-and-logarithmic-functio

Mnemonic: Treat Derivatives as Quotients of dx, dy, du, etc.. Note the resemblance to simplification of a product of fractions in the last form. The Leibniz notation also emphasizes that the argument u of the outside function $y = f(u)$ is different from the argument x of the inside function $g(x)$.

Checkpoint 3.6.3 The derivative of $(x^3+1)^2$ and higher powers of x^3+1.

1. Compute the derivative of $F(x) = (x^3+1)^2$ two ways: with and without the Chain Rule.
2. Repeat for $F(x) = (x^3+1)^9$ using the Chain Rule.
3. What would be involved in differentiating $F(x) = (x^3+1)^9$ without using the Chain Rule?

Checkpoint 3.6.4 The derivative of $(\sqrt[5]{x})^5$, with a check.

1. Compute the derivative of $f(x) = (\sqrt[5]{x})^5$, using the Power Rule and the Chain Rule;
2. then check by noting that this function is just $f(x) = x$!

Checkpoint 3.6.5 Verifying that $(\cos x)' = -\sin x$. Use the facts that $(\sin x)' = \cos x$ and $\cos(x) = \sin(x + \pi/2)$ to verify that

$$(\cos x)' = -\sin x.$$

Partial Verification of the Chain Rule. Write u for $g(x)$ and let $\Delta u = g(x+\Delta x) - g(x)$, the change in the value of g caused by changing x by an amount Δx, so that $g(x+\Delta x) = g(x) + \Delta u$. The change Δx changes the composition by

$$\Delta y = f(g(x+\Delta x)) - f(g(x)) = f(u+\Delta u) - f(u)$$

so the derivative of $y = F(x)$ is the limit as $\Delta x \to 0$ of

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}, \tag{3.6.3}$$

so long as Δu is never zero.

Ignoring that possibility for now, as $\Delta x \to 0$, $\Delta u \to 0$ because g is continuous, and so

$$\frac{\Delta y}{\Delta u} \to \frac{dy}{du} = f'(u) = f'(g(x)), \frac{\Delta u}{\Delta x} \to \frac{du}{dx} = g'(x), \frac{\Delta y}{\Delta x} \to \frac{dy}{dx} = F'(x).$$

Equation (3.6.3) says that the last of these limits is the product of the first two, which is the Chain Rule in Equations (3.6.1) and (3.6.2) above.

The examples so far test the Chain Rule against previous results; let us now use it where there is no other way to get the answer:

Checkpoint 3.6.6 Differentiate $\sqrt{x^2+1}$. Do this two ways, first using the Chain Rule in the Lagrange form (3.6.1) and then using the Leibniz form (3.6.2).

1. It is useful at first to introduce a name like u for the intermediate quantity, the value of the first or "inside" function in the composition.

However with practice, it becomes quicker and more convenient to work directly with formulas all in terms of one variable, the argument of the first function.

2. As is often the case with calculus, it can help to rewrite roots in terms of fractional powers.

Checkpoint 3.6.7 Some notation to be careful with, and the order of composition matters! Differentiate

- $\sin x^2$ $(= \sin(x^2))$
- $\sin^2 x$ $(= (\sin x)^2)$

What is the intermediate quantity "u" in each case?

Suggestion: write each function with an abundance of parentheses to avoid ambiguity about the order of operations.

Note from this example the importance of which function comes first in a composition! Also note the convention for squaring the value of a trigonometric function by writing as if one is squaring the name of the function [$\sin^2$] as opposed to squaring its argument [x^2].

Derivatives of Exponentials. We can now confirm the result for the derivative of any exponential function $y = f(x) = a^x$. Note that the text leaves this till Section 3.9[3] but I like to introduce exponential functions as soon as possible.

Write a as $e^{(\ln a)}$, so that $y = a^x = \left[e^{(\ln a)}\right]^x = e^{(x \ln a)}$, a composition with

- inside function $u = (\ln a)x$, with derivative the *constant* $\ln a$
- outside function e^u, derivative e^u.

The derivative is thus $\dfrac{d}{dx}a^x = \dfrac{d}{du}e^u \cdot \dfrac{d}{dx}[(\ln a)x] = e^u(\ln a) = (\ln a)a^x$, or

$$\frac{d}{dx}a^x = (\ln a)a^x. \tag{3.6.4}$$

Checkpoint 3.6.8 Differentiate $y = e^{\sin x}$.

The Generalized Power Rule. A common and convenient case of the Chain Rule is when the outside function is a power: $y = u^r$, $u = g(x)$. Then

$$\frac{d}{dx}(u^r) = ru^{r-1}\frac{du}{dx} \tag{3.6.5}$$

or

$$[(g(x))^r]' = r[g(x)]^{r-1} \cdot g'(x).$$

Checkpoint 3.6.9 Simplifying with roots and reciprocals. Differentiate $f(x) = \dfrac{1}{\sqrt[3]{x^2+x+1}}$.Rewrite roots as powers, and reciprocals of powers as negative powers: this is often useful in calculus.

In fact, the rule for the derivative of a reciprocal can be got just using the chain rule; writing $1/g(x))$ as $F(x) = [g(x)]^{-1} = u^{-1}$ so $u = g(x)$,

$$\frac{d}{dx}\frac{1}{g(x)} = \frac{d(u^{-1})}{du}\frac{du}{dx} = (-1)u^{-2}\frac{dg}{dx} = -\frac{dg/dx}{g^2(x)}$$

When one or both of the inner and outer functions is itself complicated, it can be worth first computing their derivatives separately, and then combining them with the Chain Rule:

Checkpoint 3.6.10 Combining compositions with quotients. Differentiate $g(t) = \left(\dfrac{t-2}{2t+1}\right)^9$.

Checkpoint 3.6.11 Differentiate $y = (2x+1)^5(x^3-x+1)^4$. Note that this is not overall a composition, but *contains* two compositions, and we need to use the Product Rule first.

The above example involves using the Chain Rule several times, and next we look at another way that this can happen.

Nested Compositions. Functions can be produced with multiple nested compositions and then the Chain Rule must be applied repeatedly. My usual guideline applies: look at the order in which the steps in evaluation of the function must be done, and apply differentiation rules from last to first; "from the outside inwards." This means that the Chain Rule is applied first to the "outer" composition, as this is evaluated last.

[3] `openstax.org/books/calculus-volume-1/pages/3-9-derivatives-of-exponential-and-logarithmic-functio`

Example 3.6.12 Differentiate $f(x) = \sin(\tan(x^2))$.

1. The outside function is $\sin$, with derivative $\cos$, which is evaluated at $\tan(x^2)$ because $\sin$ was: this gives a factor $\cos(\tan(x^2))$.
2. Next in is the function $\tan$, which has derivative $\sec^2$, and this is evaluated at x^2: a factor $\sec^2(x^2)$.
3. Finally, the innermost function is x^2, with derivative $2x$.

Altogether,

$$\begin{aligned}\frac{d}{dx}\sin(\tan(x^2)) &= \sin'(\tan(x^2)) \cdot \tan'(x^2) \cdot (x^2)' \\ &= \cos(\tan(x^2)) \cdot \sec^2(x^2) \cdot 2x \\ &= 2x\cos(\tan(x^2))\sec^2(x^2).\end{aligned}$$

□

Note: put parentheses around the arguments of trig. functions whenever the argument is more than a single letter like "x": that avoids any possibly ambiguity about what the argument is, such as in products like this.

Checkpoint 3.6.13 Differentiate $y = e^{\sec 3\theta}$. That innocent looking "3θ" means there are two compositions here.

Exercises

Study Calculus Volume 1, Section 3.6[4]; in particular Examples 48, 48, 50, 52 and 53, all Checkpoint items, and Exercises 215, 217, 219, 221, 224, 229, 233, 235, 245, 251 and 257.

3.7 Derivatives of Inverse Functions

Reference. OpenStax Calculus Volume 1, Section 3.7[1]

The Chain Rule for compositions, Equation (3.6.2) in Section 3.6, p. 38, gives us a rule for the derivative of the inverse a function, because a function and its inverse are connected by composition: For $y = g(x) = f^{-1}(x)$ the inverse of $x = f(y)$, their composition brings you back where you started:

$$f(g(x)) = x$$

The Chain Rule then connects their derivatives in a way that looks quite simple in Leibniz notation:

- the derivative of $y = g(x)$ that we seek is $g'(x) = dy/dx$,
- the derivative of its inverse $x = f(y)$ is $f'(y) = dx/dy$, so
- the derivative of their composition $x = f(g(x))$ is

$$1 = \frac{dx}{dx} = \frac{dx}{dy}\frac{dy}{dx}:$$

- thus in words, *the derivative of the inverse is the reciprocal of the derivative*:

$$\frac{dy}{dx} = \frac{1}{dx/dy}. \tag{3.7.1}$$

[4] openstax.org/books/calculus-volume-1/pages/3-6-the-chain-rule
[1] openstax.org/books/calculus-volume-1/pages/3-7-derivatives-of-inverse-functions

This is just as you might guess from thinking of derivative notations like dy/dx as being arithmetic with small quantities dx and dy!

However, we need to be careful with the fact that the function and its inverse have different input arguments; x and $y = f(x)$ respectively. The Lagrange "f'" notation makes the issue clearer:

$$g'(x) = \frac{1}{f'(y)}$$

showing that this gives the derivative — a function of x — in terms of y instead. To get a formula in terms of x, eliminate y by using $y = g(x)$:

$$g'(x) = \frac{1}{f'(g(x))}$$

Finally, to put it all in terms of the original function f whose derivative we already know,

$$[f^{-1}]'(x) = \frac{1}{f'(f^{-1}(x))}. \tag{3.7.2}$$

Example 3.7.1 Derivative of the square root. Let's start with an example where we can check the answer: the derivative of the square root function $y = g(x) = \sqrt{x}$. As will often be the case, the first step is to rewrite in terms of a function that we already know how to differentiate, by solving for x: $x = f(x) = y^2$.

Using Leibniz notation first, $dx/dy = d(y^2)/dy = 2y$ so $y = \sqrt{x}$ has derivative

$$\frac{d\sqrt{x}}{dx} = \frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{2y}.$$

Then to get this in terms of x, use $y = \sqrt{x}$ to get

$$\frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{x}},$$

as we have seen before. □

3.7.1 A procedure using just the Chain Rule

Though these formulas can be useful, it is in many cases easier and safer to use a strategy of

1. "inverting" the equation $y = f^{-1}(x)$ to the equation $x = f(y)$ which involves only a function we already know how to differentiate, then
2. differentiate both sides of that equation using the Chain Rule, and finally
3. solve a simple equation by division.

This procedure will be used in most examples from now on, and is the basis of an important strategy introduced in Section 3.8, p. 45: *Implicit Differentiation*. (In fact, I often prefer avoiding the memorizing of yet another formula by instead having a "procedure" or "algorithm" that break the calculation into steps each of which uses facts and methods that I already know.)

3.7.2 The Power Rule $(x^r)' = rx^{r-1}$ for any rational number r

The result above for the square root can be extended to compute the derivative of any root function $f(x) = \sqrt[q]{x} = x^{1/q}$ for q a natural number, and that is the main step in verifying the power rule for all rational powers. This will be done using the new strategy described above.

Firstly, since the q-th root is the inverse of the q-th power, $y = x^{1/q}$ means that $x = y^q$ (so the inverse function "$\sqrt[q]{}$" disappears). Then differentiate both sides of this equation, noting well that *the variable is x, not y*. The Leibniz notation is safest here: On one side, $\dfrac{dx}{dx} = 1$; on the other side,

$$\frac{d}{dx}y^q = \frac{d(y^q)}{dy}\frac{dy}{dx} = qy^{q-1}\frac{dy}{dx}$$

Equating these two, and inserting the formula for y,

$$1 = q(x^{1/q})^{q-1}\frac{dy}{dx} = qx^{1-1/q}\frac{dy}{dx}$$

Finally, solve for the desired derivative:

$$\frac{dy}{dx} = \frac{d}{dx}x^{1/q} = \frac{1}{q}x^{1/q-1},$$

confirming the Power Rule for this case of exponent $1/q$.

Now we can get the result for x^r where $r = p/q$ is any rational number, by using the Chain Rule again. First, $x^r = (x^{1/q})^p, = u^p$ with $u = x^{1/q}$. Then

$$\frac{d}{dx}(x^r) = \frac{du^p}{dx} = \frac{du^p}{du}\frac{du}{dx} = \frac{du^p}{du}\frac{d(x^{1/q})}{dx}$$

and we already have the power rule for each of these factors:

$$\frac{d}{dx}(x^r) = pu^{p-1}\frac{1}{q}x^{1/q-1}$$

Finally, get it all in terms of the variable x using $u = x^{1/q}$ and $p/q = r$:

$$\frac{d}{dx}(x^r) = \frac{p}{q}(x^{1/q})^{p-1}x^{1/q-1} = \frac{p}{q}x^{p/q-1/q+1/q-1} = \frac{p}{q}x^{p/q-1} = rx^{r-1},$$

as advertised.

In fact we will soon be able to verify the power rule for any real power, x^a, so the above was not essential, but gives some useful examples of using the Chain Rule and of this strategy for differentiating inverses.

To complete the story of the Power Rule, we first need the derivative of the natural logarithm.

3.7.3 Derivative of the Natural Logarithm

The Chain Rule can also be used to compute the last of the derivative of the last basic elementary functions, the logarithm. The text leaves this till Section 3.9[2] but again, I like to introduce all the elementary functions as soon as possible.

For this we again use the above strategy of solving the equation so that the inverse function temporarily "disappears" and we only have to deal with functions and operations (like composition) that we already know how to handle; this will also be very useful in the next few sections.

Let $u = \ln x$ and solve for x, giving

$$x = e^u = e^{\ln x}$$

Differentiating both sides *with respect to x* and using the Chain Rule gives

$$1 = \frac{de^u}{dx} = \frac{de^u}{du}\frac{du}{dx} = e^u\frac{du}{dx}$$

Inserting $e^u = x$ and $u = \ln x$, this says that $x\frac{d}{dx}\ln x = 1$, so dividing by x gives

$$\frac{d}{dx}\ln x = \frac{1}{x}, = x^{-1}. \tag{3.7.3}$$

[2] openstax.org/books/calculus-volume-1/pages/3-9-derivatives-of-exponential-and-logarithmic-functio

Example 3.7.2 The derivative of any logarithmic function: $(\log_a x)' = \dfrac{1}{\ln a}\dfrac{1}{x}$**.** Though it is rarely needed, the same method gets the derivative of any logarithmic function $y = \log_a x$; do this by differentiating the "inverse form" $x = a^y$. □

3.7.4 Verification of the Power Rule for all Real Powers

To differentiate $f(x) = x^a$ for $x > 0$ and any real a, write $x = e^{(\ln x)}$ so that $x^a = [e^{(\ln x)}]^a = e^{(a \ln x)}$. Now use the Chain Rule, with

- inside function $u = a \ln x$, with derivative $a \cdot \dfrac{1}{x}$;
- outside function e^u, with derivative $e^u, = e^{a \ln x} = x^a$.

$$\frac{d(x^a)}{dx} = \frac{d(e^u)}{dx} = \frac{d(e^u)}{du}\frac{du}{dx} = e^u\left(a \cdot \frac{1}{x}\right) = \frac{x^a \cdot a}{x} = ax^{a-1}.$$

3.7.5 Derivatives of the Inverse Trigonometric Functions

We can compute the derivative of an inverse trigonometric function like $y = \arcsin x = \sin^{-1} x$ by again using the strategy described in Subsection 3.7.1, p. 42 above of first "solving for x" to hide the inverse function: writing an equation involving the original "non-inverse" function whose derivative we know and then using implicit differentiation. Here we can use

$$x = \sin y$$

One thing we must be careful about is that *no trigonometric function is invertible on its entire natural domain*, so we limit the domain to make it satisfy the Horizontal Line Test. Here, we restrict the domain of $\sin$ to $[-\pi/2, \pi/2]$ where it is increasing, cuting off the domain at the points $x = \pm\pi/2$ beyond which it flips to being decresing. Thus $\sin^{-1} x$ has range $[-\pi/2, \pi/2]$, and domain $[-1, 1]$.

Implicit differentiation of $x = \sin y$ gives $1 = \cos y \cdot \dfrac{dy}{dx}$, so

$$\frac{dy}{dx} = \frac{1}{\cos y}, \quad \cos y \neq 0.$$

But we want this as a function of x, not y!

Using the identity $\sin^2 y + \cos^2 y = 1$ along with $x = \sin y$, we get $\cos^2 y + x^2 = 1$, so $\cos y = \pm\sqrt{1 - x^2}$. The range of y values $[-\pi/2, \pi/2]$ ensures that $\cos y$ is not negative, so $\cos y = \sqrt{1 - x^2}$ and

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1 - x^2}}, \quad -1 < x < 1.$$

The restriction on the x values is to avoid division by zero, and shows that $\arcsin x$ is not differentiable at the endpoints of its domain, due to "vertical tangents" (as with $\sqrt{x}$ at $(0, 0)$). This happens because sine has horizontal tangents at the endpoints of its domain.

This is rather typical with inverse functions, because restricting the domain of a function to make it satisfy the Horizontal Line Test often requires ending the domain at a point where the tangent is horizontal. For example, the same happens when we cut off the domain of $y = x^2$ at $x = 0$ to get its inverse $\sqrt{x}$. It also happens with the inverses of $\cos$, $\sec$ and $\csc$.

The Derivatives of Other Inverse Trig. Functions. Similar calculations give

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\csc^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$$

Checkpoint 3.7.3 Reciprocals vs inverses. Differentiate $y = \dfrac{1}{\sin^{-1}x}$, being careful with the two uses of superscript -1!

Checkpoint 3.7.4 Compositions vs products. Differentiate $f(x) = x\arctan\sqrt{x}$, being careful to distinguish products from compositions.

Exercises

Study Calculus Volume 1, Section 3.7[3]; in particular Examples 61 to 67, Checkpoint items 43 to 46, and Exercises 265, 267, 269, 271, 279, and 291.

Hint for Exercise 279. One approach is to use the "equation solving" strategy of making the inverse function disappear: solve for $\sin(y) = x^2$ and then differentiate each side of that equation.

3.8 Implicit Differentiation

References.

- OpenStax Calculus Volume 1, Section 3.8[1]
- *Calculus, Early Transcendentals* by Stewart, Section 3.5.

In Section 3.7, p. 41, *Derivatives of Inverse Functions*, we computed the derivatives of $y = \ln x$ and $y = \arcsin x$ functions by using the fact that they are the inverses of the natural exponential and $\sin$ functions respectively: then we got equations for their values y in terms of those more familiar functions, $e^y = x$ and $\sin y = x$. That allowed us to use the Chain Rule to get a formula for dy/dx in terms of the derivative of those original functions.

These are examples of the strategy of **implicit differentiation**, where the function to be differentiated is given *implicitly* as the solution of an equation, rather than by an explicit formula.

Here we will see other uses for this strategy, like computing the slope at a point on a curve when the curve is given by an equation, not as the graph of a known function.

Looking forward, this strategy will be useful in Chapter 4, p. 50; in particular, Section 4.1, p. 50

Checkpoint 3.8.1 Find the tangent line to point $P(3,-4)$ on circle $x^2+y^2=25$. Do this two ways:

1. By finding an explicit equation $y = f(x)$ for the curve.
2. by differentiating the equation
$$x^2 + [f(x)]^2 = 25$$
without using a formula for $f(x)$.

Checkpoint 3.8.2 Tangent to a curve that cannot be expressed as $y = f(x)$. Find the tangent line to the point $P(1,1)$ on the curve $2x^7 + y^7 = 3xy$.

[3] openstax.org/books/calculus-volume-1/pages/3-7-derivatives-of-inverse-functions
[1] openstax.org/books/calculus-volume-1/pages/3-8-implicit-differentiation

Procedure for Implicit Differentiation, with example $x^2 + xy^2 + e^y = 8$. To find a formula for the derivative of a function $y = f(x)$ given implicitly by an equation involving x and y:

1. Differentiate each side of the equation.

   Note that every time y appears, you must use the Chain Rule.

$$2x + \left(y^2 + x2y \cdot \frac{dy}{dx}\right) + e^y \cdot \frac{dy}{dx} = \frac{d}{dx}8 = 0.$$

   (The parentheses are around the derivative of xy^2, which also requires the product rule.)

2. Add and subtract to get terms with factor $\frac{dy}{dx}$ at left, all others at right.

$$2xy \cdot \frac{dy}{dx} + e^y \cdot \frac{dy}{dx} = -2x - y^2.$$

3. Collect the common factor $\frac{dy}{dx}$ present in every term at left.

$$\frac{dy}{dx}(2xy + e^y) = -(2x + y^2).$$

4. Divide out to get a formula for $\frac{dy}{dx}$, which is the desired answer.

$$\frac{dy}{dx} = -\frac{2x + y^2}{2xy + e^y}.$$

Note that to use this formula, you need both coordinates of a point of the curve given by the original equation, $P(x_0, y_0)$, not just an x value.

Example 3.8.3 Find $\frac{dy}{dx}$ for $\sin(x + y) = y^2 \cos x$.

Hint. Note that the "hidden" Chain Rule comes up in two places, $\frac{d}{dx}(y^2)$ and $\frac{d}{dx}(x + y)$.

Solution.

1. The chain rule at left and product rule at right give:

$$\cos(x + y)\frac{d}{dx}(x + y) = \frac{d}{dx}(y^2)\cos x + y^2\frac{d}{dx}(\cos x),$$

   so

$$\cos(x + y)\left(1 + \frac{dy}{dx}\right) = 2y\frac{dy}{dx}\cos x - y^2 \sin x.$$

2. Moving terms with factor $\frac{dy}{dx}$ to the left, others to the right,

$$\frac{dy}{dx}\cos(x + y) - 2y\frac{dy}{dx}\cos x = -y^2 \sin x - \cos(x + y)$$

3. Gathering the common factor $\frac{dy}{dx}$ at left (and a factor -1 at right),

$$\frac{dy}{dx}[\cos(x + y) - 2y \cos x] = -(y^2 \sin x + \cos(x + y)).$$

4. Dividing out,

$$\frac{dy}{dx} = -\frac{y^2 \sin x + \cos(x + y)}{\cos(x + y) - 2y \cos x}.$$

□

Higher Derivatives: No Further Implicit Differentiation Needed! Once a derivative has been found by implicit differentiation, you can compute the second and higher derivatives with no further implicit differentiation:

Example 3.8.4 Find y'' if $x^4 + y^4 = 16$.

Solution. First, implicit differentiation gives $\dfrac{dy}{dx} = -\dfrac{x^3}{y^3}$.

Then the second derivative is $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\dfrac{dy}{dx} = -\dfrac{d}{dx}\dfrac{x^3}{y^3}$, and the quotient rule combined with the hidden chain rule result

$$\frac{d(y^3)}{dx} = \frac{d(y^3)}{dy}\frac{dy}{dx}$$

gives

$$\frac{d^2y}{dx^2} = -\frac{3x^2y^3 - x^3 3y^2 dy/dx}{(y^3)^2}.$$

Finally, inserting the above result for dy/dx gives

$$\frac{d^2y}{dx^2} = -\frac{3x^2y^3 - x^3 3y^2(-x^3/y^3)}{y^6}, = -\frac{3x^2y^4 + 3x^6}{y^7}.$$

□

Exercises

Study Calculus Volume 1, Section 3.8[2]; in particular Examples 68, 69, 71, 72, both Checkpoint items and Exercises 301, 303, 305, 307, 311, 316, 325, and 329.

3.9 Derivatives of Exponential and Logarithmic Functions (and *Logarithmic Differentiation*)

References.

- OpenStax Calculus Volume 1, Section 3.9[1]
- *Calculus, Early Transcendentals* by Stewart, Section 3.6.

In this course, we have already seen and worked with the derivatives of exponential functions (in Section 3.6, p. 38 and of logarithmic functions (in Section 3.7, p. 41). Thus for us, the first part of this section of the text is just for review and further worked examples and homework exercises, and I will just summarise briefly with some examples. I suggest looking at Theorems 14, 15 and 26 in the text to review the formulas, and its Examples 74, 75, 77, and 79.

The last part of this section introduces a useful new technique, *Logarithmic Differentiation*, which is a use of implicit differentiation to simplify differentiation of functions that involve a mix of products, quotients and powers.

Derivatives of exponential and logarithmic functions (recap). The natural exponential function has derivative

$$\frac{d}{dx}e^x = e^x. \tag{3.9.1}$$

[2] openstax.org/books/calculus-volume-1/pages/3-8-implicit-differentiation
[1] openstax.org/books/calculus-volume-1/pages/3-9-derivatives-of-exponential-and-logarithmic-functio

and more generally

$$\frac{d}{dx}a^x = (\ln a)a^x,\ a > 0 \tag{3.9.2}$$

The natural logarithm function has derivative

$$\frac{d}{dx}\ln x = \frac{1}{x},\ x > 0 \tag{3.9.3}$$

Further, for $x < 0$, $(\ln|x|)' = (\ln(-x))' = \frac{1}{-x}\cdot(-1) = \frac{1}{x}$, so

$$\frac{d}{dx}\ln|x| = \frac{1}{x},\ x \neq 0 \tag{3.9.4}$$

Checkpoint 3.9.1 Differentiate $y = \ln(x^3+1)$.

Checkpoint 3.9.2 Find $\frac{d}{dx}\ln(\sin x)$.

Always remember to look simplify first, before differentiating:

Checkpoint 3.9.3 Differentiate $f(x) = \sqrt{\ln x}$.

Here even more, simplify first!

Checkpoint 3.9.4 Find $\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}}$.

Two Useful Derivative Formulas. The Chain Rule gives

$$\frac{d}{dx}\ln f(x) = \frac{df/dx}{f(x)}, \quad \text{or} \quad (\ln u)' = \frac{u'}{u}. \tag{3.9.5}$$

One common special case is when function u is linear:

$$\frac{d}{dx}\ln(mx+a) = \frac{m}{mx+a}, \tag{3.9.6}$$

and even more specifically,

$$\frac{d}{dx}\ln(x+a) = \frac{1}{x+a}. \tag{3.9.7}$$

Warning: This is the* only *case where the derivative of* $\ln f(x)$ *is* $1/f(x)$*!
Having $f'(x) = 1$ is the key.

Example 3.9.5 Verify that the derivative of $\ln(|\sec x|)$ is $\tan x$. Note that $\sec x = \frac{1}{\cos x}$ and $\ln(1/u) = -\ln u$, so

$$\ln(|\sec x|) = \ln\left(\frac{1}{|\cos x|}\right) = -\ln(|\cos x|).$$

Then using the Chain Rule with $u = \cos x$,

$$\frac{d}{dx}\ln(|\sec x|) = -\frac{d}{du}(\ln|u|)\cdot\frac{du}{dx} = -\frac{1}{u}\cdot(-\sin x) = \frac{\sin x}{\cos x} = \tan x.$$

□

Logarithmic Differentiation. Logarithms have the nice property of converting products to sums, quotients to differences and exponentials to products. The leads to the method of **logarithmic differentiation**, which can simplify the differentiation of functions built of products, quotients and exponentials.

Example 3.9.6 Compute $\frac{dy}{dx}$ for $y = \frac{x^3}{(2x+3)^5}$. For $y = \frac{x^3}{(2x+3)^5}$, $\ln y = \ln \frac{x^3}{(2x+3)^5} = \ln(x^3) - \ln((2x+3)^5) = 3\ln x - 5\ln(2x+3)$.

The left side has derivative $\frac{d}{dx}\ln y = \frac{d}{dy}(\ln y)\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$.

Using Eq. (3.9.6), the right side has derivative

$$\frac{d}{dx}[3\ln(x) - 5\ln(2x+3)] = 3\frac{1}{x} - 5\frac{2}{2x+3} = \frac{3}{x} - \frac{10}{2x+3}.$$

Comparing the two sides, $\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} - \frac{10}{2x+3}$.

Finally, multiplying each side by y,

$$\frac{dy}{dx} = \left(\frac{3}{x} - \frac{10}{2x+3}\right)y = \left(\frac{3}{x} - \frac{10}{2x+3}\right)\frac{x^3}{(2x+3)^5}.$$

□

The working of this example reveals a strategy for usign logrutmns to simplify teh differentiation of a function $y = f(x)$ when the formuals for is butrok fomr products quotients, powers and expontiations.

1. First, take the derivative of both sides, getting $\ln(y) = \ln(f(x))$
2. Next, a critical step: *simplify* the right-hand side as much as possible, using the properties of logarithms; in particular:
   - $\ln(ab) = \ln(a) + \ln(b)$,
   - $\ln(a/b) = \ln(a) - \ln(b)$, and
   - $\ln(a^b) = b\ln(a)$.

   (Also, as usual, convert roots to fractional powers!) This will give something like

   $$\ln y = \ln(f_1(x)) + \cdots$$

3. Differentiate both sides of the equation, with much use of the rule $\frac{d}{dx}(\ln(u)) = \frac{1}{u}\frac{du}{dx}$; This gives an equation that starts

   $$\frac{1}{y}\frac{dy}{dx} = \cdots$$

4. Multiply both sides by the quantity $y = f(x)$, the original function being differentiated.

Also look at Examples 81 to 83 in Section 3.9 of the text[2].

Exercises

Study Calculus Volume 1, Section 3.9[3]; in particular Examples 74, 75, 77, 78, 81 and 82, Checkpoint 54, and Exercises 333, 339, 347, 351 and 353.

We in particular emphasize the last topic of *Logarithmic Differentiation*, using the strategy of simplifying functions of the form $\log(\dots)$ using the laws of logarithms like $\log(ab) = \log(a) + \log(b)$.

[2] openstax.org/books/calculus-volume-1/pages/3-9-derivatives-of-exponential-and-logarithmic-functio
[3] openstax.org/books/calculus-volume-1/pages/3-9-derivatives-of-exponential-and-logarithmic-functio

Chapter 4

Applications of Differentiation

This chapter introduces the concept of the derivative, and efficient rules for calculating the derivatives of functions.

Note: The topic *Newton's Method* will probably be omitted, so for now at least the notes for the corresponding Section 4.9 are just references and a brief study guide.

References.

- OpenStax Calculus Volume 1, Chapter 4.[1]
- *Calculus, Early Transcendentals* by Stewart, Chapter 3, Sections 9 and 10 and Chapter 4, Sections 1–5 and 7–9, and also Section 2.6 (for horizontal asymptotes)

4.1 Related Rates

References.

- OpenStax Calculus Volume 1, Section 4.1[1]
- *Calculus, Early Transcendentals* by Stewart, Section 3.9.

In many physical situations, several related quantities change with time, such as the pressure P and volume V of a fixed amount of a gas at a fixed temperature ($PV = c$.) When the changing quantities are related by a known formula, their rates of change are also related, so that a measurement of one rate of change can be used to determine the other. In the above example, if the applied pressure is increased at a given rate, one can predict how fast the volume will be decreasing.

Since the rates of change are with respect to the variable time, this leads to implicit differentiation of the formula relating the two quantities. For example, the above equation could be spelt out as $P(t) \cdot V(t) = c$, with c a known constant.

Strategy for Related Rates Problems. Firstly, what identifies a related rates problem is that you are asked to *find one rate of change* (a derivative, such as a velocity) using information about one or more *other rates of change* (other derivatives), and you often have that rate of change information *only at one time*, not as functions of time. Most often the independent variable is time, t, so I assume that here. In the following strategy, always keep track of your main goal: *to compute the derivative of a certain variable with respect to t.Keep your eye on that "key" variable!*

[1] openstax.org/books/calculus-volume-1/pages/4-introduction
[1] openstax.org/books/calculus-volume-1/pages/4-1-related-rates

Strategy for Related Rates Problems.

1. *Read* all the given information carefully, and identify the main goal: *What is the quantity whose rate of change you wish to know?*
2. Give *names to* all relevant *variables*; in particular *name variable quantities whose rate of change is either wanted or is known.* (Constants usually do not need names.)
3. *Draw a diagram* relating all the variables and other known values, if appropriate.
4. Describe *the rate of change sought* as the derivative of a certain variable.
5. *Seek an equation* (or several equations) relating just the relevant variables. That is, variables whose derivatives are either known or wanted. *Make sure that these equations are true at all times, not just at one moment.*
6. *Differentiate* the equation[s] *with respect to time. This is implicit differentiation, with "hidden" compositions.*
7. *Substitute known values* into the equation[s] got by differentiating. At this stage only, the values might only be valid at the one time of interest.
8. *Solve* the resulting equation[s] for the desired rate of change (derivative). It might be necessary to also substitute all known values into the original equations (from step 5), and use this larger collection of equations to solve for the desired rate of change.
9. *Answer the original question!* That is, relate your mathematical results back to the question asked, preferably as a verbal statement of the result. Put back in physical units, and interpret the *sign* of the derivative as saying whether the quantity is *increasing or decreasing*.

Example 4.1.1 Inflating a balloon. Air is being pumped into a spherical balloon at a rate of 100cm^3/s. How fast is the radius increasing when the diameter is 50cm?

Writing $V = V(t)$ for the volume, a function of time, and $r = r(t)$ for the radius:

- What we want to know is dr/dt at a certain moment.
- what we know is that $dV/dt = 100$cm^3/s and $V = \frac{4}{3}\pi r^3$ at all times, and at that moment, $r = 50/2$ cm (half the diameter.)

How do we relate these two derivatives? The equation relating the two relevant variables is

$$V(t) = \frac{4}{3}\pi[r(t)]^3, \quad \text{or just} \quad V = \frac{4}{3}\pi r^3.$$

This can be differentiated with respect to time, involving the Chain Rule, or implicit differentiation of r with respect to t:

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

At the moment of interest, $r = 25$ and $dV/dt = 100$, and substituting in these numbers: $100 = 4\pi(25)^2\frac{dr}{dt}$, so

$$\frac{dr}{dt} = \frac{100}{4\pi(25)^2} = \frac{1}{25\pi} \approx 0.0127\text{cm/s}.$$

We could also solve for the unknown derivative at any moment in terms of known quantities:

$$\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2}.$$

□

Exercises

Study Calculus Volume 1, Section 4.1[2]; in particular the *Problem Solving Strategy*, all Examples and Checkpoints, and Exercises 1, 3, 5, 7, 9, 17, and 25.

4.2 Linear Approximations and Differentials

References.

- OpenStax Calculus Volume 1, Section 4.2[1]
- *Calculus, Early Transcendentals* by Stewart, Section 3.10.

When we first introduced the tangent line to a curve at a point, one characteristic was that this appears to be the line that is closest to the curve itself when one looks near that point: zooming in towards that point, the curve looks more and more like that tangent line. This means that the tangent line can be useful as an approximation of the curve, and thus can be used to approximate the value of a function from its value at one point plus its derivative at that point.

The tangent line to $y = f(x)$ at $x = a$. At the point $P(a, f(a))$ on the curve $y = f(x)$, the tangent line has slope $m = f'(a)$, and is

$$y = f(a) + f'(a)(x - a) \tag{4.2.1}$$

Since this line is close to the curve at least when x is close to a, this gives

$$f(x) \approx f(a) + f'(a)(x - a) \tag{4.2.2}$$

as the **linear approximation** or **tangent line approximation** of f at a. The linear function occurring here,

$$L(x) = f(a) + f'(a)(x - a) \tag{4.2.3}$$

is called the **linearization** of f at a.

Note well:* $f(a)$ *and* $f'(a)$*, not* $f(x)$ *and* $f'(x)$*! Once the point $P(a, f(a))$ is chosen, these are just numbers, with the only variable in $L(x)$ being the x in $(x - a)$.

Example 4.2.1 At 10am, a car has traveled 200 miles since the start of the day and its speed is 55mph. Use the linearization of the function giving distance traveled in terms of time to estimate the total distance traveled by a slightly later time like 10:15am.

With $s = f(t)$ where t is time (in hours since midnight) and s is distance traveled in miles, we know that $f(10) = 200$ and $f'(10) = 55$. Thus the linearization at $t = 10$ is

$$L(t) = 200 + 55(t - 10).$$

For example, time 10:15 is $t = 10.25$, so the linear approximation of total distance traveled by then is

$$L(10.25) = 200 + 55(10.25 - 10) = 213.75 \text{ miles.}$$

□

Linearization: Assuming Constant Rate of Change. This procedure should be familiar: it is approximation by assuming that the speed stays constant at the given value of 55, or at least close to that speed. More generally, linear approximation is the assumption that the rate of change is constant, or at least does not change much when the independent variable is changed by only small amount. This is the same idea as used when a population's size in the near future is estimated using the population's current size and rate of growth.

[2] openstax.org/books/calculus-volume-1/pages/4-1-related-rates
[1] openstax.org/books/calculus-volume-1/pages/4-2-linear-approximations-and-differentials

Linearization to Evaluate Functions Approximately. Sometimes it is easy to find out about a function at one value of its argument, and we can use this information to approximate it at other arguments, where the exact calculation is harder:

Checkpoint 4.2.2 The linearization of $f(x) = \sqrt{x+3}$ at $a = 1$. Find the linearization of $f(x) = \sqrt{x+3}$ at $a = 1$, and use this to approximate $\sqrt{3.98}$ and $\sqrt{4.05}$.

Checkpoint 4.2.3 A calculator exercise based on the above. For which values of x is the above linear approximation accurate to within 0.1?

$\sin x \approx x$ **For Small Angles.** It is often useful to have an approximation of $\sin x$ for small angles. For example in optics, angles of only a few degrees are often involved, and then x is smaller than about 0.1 (in radians). Thus complicated optical formulas involving trig. functions and their inverses are accurately approximated by far simpler linear formulas. This approximation is done with the linearization of $f(x) = \sin x$ at $a = 0$, where $f'(0) = \cos 0 = 1$. Since $\sin 0 = 0$, we get the very simple approximation

$$\sin x \approx L(x) = x.$$

This is one reason why radian measure is convenient!

Differentials: dx, dy, etc. The Leibniz notation $\dfrac{dy}{dx}$ came from the intuition that the slope of curve $y = f(x)$ is given by the ratio of a very small difference dx in the value of argument x to the very small difference dy that this causes in the value of $y = f(x)$. These very small differences were called **differentials**, and the subject of derivatives is sometimes called **differential calculus**, meaning the subject of calculating with differentials. This intuitive, approximate idea can be made more precise using linearization: for a function $y = f(x)$,

- any change in the independent variable x is denoted dx, a **differential**. (Likewise dt if the independent variable is t, etc.)
- the resulting linear approximation of the change in the y value is called the differential dy, given by
$$dy = f'(x)\,dx.$$
- That is, $dy = L(x + dx) - L(x)$, the change in the value of the linearization of f at x.

Note that this also means that the ratio of the differentials is

$$\frac{dy}{dx} = f'(x),$$

so the Leibniz notation is now a genuine fraction!

Example 4.2.4 (Example 4.2.1 above revisited). In Example 1, the differential for the independent variable t is the change in time, $dt = 0.25$, and the resulting differential in the independent variable is

$$ds = s'(10)dt = 55 \cdot 0.25 = 13.75,$$

the estimated change in position. □

Example 4.2.5 (Checkpoint 4.2.2 revisited). In Example 2 with $f(x) = \sqrt{x+3}$ linearized at 1, the values 3.98 and 4.05 corresponding to differential in x of $dx = -0.02$ and $dx = 0.05$. The differential in the value of the function is

$$dy = \frac{dx}{2\sqrt{1+3}} = \frac{dx}{4},$$

giving $dy = -0.005$ and $dy = 0.0125$ respectively: the estimated changes in the value of the function from its value of 2 when $x = 3$. □

Differential vs Difference. For the independent variable, a differential dx is the same thing as the difference denoted Δx: the change in the value of x. However for a dependent variable $y = f(x)$, $\Delta y = f(x + \Delta x) - f(x)$ is the *exact change* in the value of y, while $dy = f'(x)dx$ is the *linear approximation of this change*.

Checkpoint 4.2.6 For $y = f(x) = x^3 + x^2 - 2x + 1$, compare the values of Δy and dy when x changes:

- from 2 to 2.05
- from 1 to 0.99

Estimating the Effects of Measurement Error. One important use of differentials and linearization in experimental science is estimating the effect of error in measuring one quantity on the error in some other quantity computed from the measured value.

Example 4.2.7 Accuracy in computing the volume of a sphere fom its diameter. The radius of a sphere has been measured and found to be 21 cm to the nearest mm, so the measurement error is at most 0.05 cm. Use differentials to estimate the maximum error in the value of the volume got by using this measured value for the radius.

The exact error is ΔV, where $V = \frac{4}{3}\pi r^3$; we approximate this with the differential $dV = 4\pi r^2 dr, = 4\pi(21)^2 dr$.

The actual radius can vary from 21 by up to 0.05 in either direction, so all we know is that $-0.05 \leq dr \leq 0.05$, or $|dr| \leq 0.05$. This tells us that $-4\pi(21)^2 0.05 \leq dV \leq 4\pi(21)^2 0.05$ or

$$|dV| = \left|\frac{dV}{dr}\right| \cdot |dr| = 4\pi(21)^2|dr| \leq 4\pi(21)^2 0.05 \approx 277 \text{ cm}^3.$$

Exact calculation gives $-276.4 \leq \Delta y \leq 277.7$, so this is a good error estimate. □

Exercises

Study Calculus Volume 1, Section 4.2[2] including all Examples and Checkpoints and a few Exercises from each of the ranges 50–55, 62–67, 68–71, 72–77, 78–83, 84–86; for example, Exercises 49, 51, 52, 57, 69, 73, 79 and 84.

4.3 Maxima and Minima

References.

- OpenStax Calculus Volume 1, Section 4.3[1]
- *Calculus, Early Transcendentals* by Stewart, Section 4.1.

Questions of optimization are one of the two most important applications of calculus that we will see this semester. (The other is finding a function from information about its rate of change, coming up in Section 4.10, p. 69 and Chapter 5, p. 72) For example, choosing the shape of a product that minimizes its weight or cost, or that maximizes strength for a given weight, or finding a route that minimizes travel time, or the price at maximizes profit.

We have already seen one key idea intuitively and graphically: the graph of function tends to have a low point or a high point where the derivative is zero, and not where the derivative is non-zero. We now investigate carefully questions like:

- When does a function have an overall **minimum** or **maximum** value?

[2] `openstax.org/books/calculus-volume-1/pages/4-2-linear-approximations-and-differentials`
[1] `openstax.org/books/calculus-volume-1/pages/4-3-maxima-and-minima`

- How can we find these **extreme values** of a function, and the arguments of the function that give them?

Checkpoint 4.3.1 Try to draw a graph where the minimum value occurs at a point $P(a, f(a))$, where it is not true that $f'(a) = 0$.

Checkpoint 4.3.2 Try to draw a graph where $f'(a) = 0$ for some a, but the value $f(a)$ is not a minimum or maximum, even compared to nearby points on the graph.

Definition 4.3.3 Global Extrema.

A function has an **absolute maximum**, or **global maximum**, at c if $f(c) \geq f(x)$ for all x in its domain D. The value $f(c)$ is the **maximum value** of f on domain D.

An **absolute (global) minimum** and **minimum value** are defined similarly. Collectively, global maxima and minima are **global extrema**, and the values of f there are **extreme values**. ◊

Definition 4.3.4 Local Extrema.

A function has a **local maximum**, or **relative maximum**, at c if $f(c) \geq f(x)$ for x near c. That is, on some open interval (a, b) containing c, $f(x)$ is never greater than $f(c)$.

The value of the function at a local maximum is a **local maximum value**.

Local (relative) minima, extrema and such are defined similarly. ◊

Checkpoint 4.3.5 Find the local [relative] and global [absolute] minima and maxima of the function $f(x) = \cos x$, and the points at which they occur.

Checkpoint 4.3.6 Try to find the locations of the global extrema and corresponding extreme values for $f(x) = x^2$.

Checkpoint 4.3.7 Do the same for $g(x) = x^3$.

Checkpoint 4.3.8 Find the global extrema of $f(x) = x^2$ with domain $D = [-1, 2]$.

Find all local extrema for this function.

Checkpoint 4.3.9

1. Using a graph, try to find the local extrema for $f(x) = 3x^4 - 16x^3 + 18x^2$ on domain $D = [-1, 4]$.
2. Then find the global extrema of this function.
3. Is the derivative of f zero at every local extremum?
4. Where are the local extrema with $f'(x) \neq 0$?

Theorem 4.3.10 The Extreme Value Theorem. *If function f is continuous on a closed interval $[a, b]$, then it attains a global maximum value $f(c)$ and a global minimum value $f(d)$ at some numbers c and d in this interval.*

Note that either extreme value can possibly occur at more than one number, so that c and d are not always unique.

Theorem 4.3.11 Fermat's Theorem. *If function f has a local extremum on an* open *interval at number c and if $f'(c)$ exists, then $f'(c) = 0$.*

Note well:

- $f'(c)$ might not exist at a local extremum.
- This refers to open intervals, so excluding endpoints: endpoints are also always candidate locations for extrema.

Checkpoint 4.3.12 Use $f(x) = x^3$ to show that having $f'(c) = 0$ does not always give a local extremum at $x = c$.

Checkpoint 4.3.13 Use $f(x) = |x|$ to show that not all local extrema on open intervals occur at points where $f'(c) = 0$.

Checkpoint 4.3.14 Use $f(x) = x$ on domain $D = [0, 1]$ to show another way that local extrema can occur.

The above three exercises show that $f'(x) = 0$ is only part of the puzzle!

Definition 4.3.15 Critical Points, Values, and Numbers.

A **critical number** of a function f is a value c in its domain such that either $f'(c) = 0$ or $f'(c)$ does not exist.

A **critical value** is the value $f(c)$ of a function at a critical number c.

A **critical point** is a point $P(c, f(c))$ on the graph of function f for c a critical number. ◊

Loosely, a critical point is *a point on the graph where the curve does not either rise or fall as it passes through the point, where rise or fall of the curve is indicated by a rising or falling tangent line.*

Checkpoint 4.3.16 Find the critical points of $f(x) = x^{3/5}(x - 4)$.

Fermat's Theorem can now be rephrased this way:

Theorem 4.3.17 (Where the extrema are, and are not). *Local extrema can occur only at critical points and end points.*

Thus, if f has a non-zero derivative at a point that is not an end point, that point does not give a local extremum.

Note: the "open interval" part of the original statement of Fermat's Theorem meant that it did not say anything about end points. Typically a function has only a finite number of critical points (and of end points), so once these are found, working out which of them give global minima or maxima is just a matter of computing and comparing the values at those points. This is a lot better than having to check at the infinite number of points in the domain of f!

Checking if a point is a *local* minimum or maximum sometimes requires a few more ideas, coming in the next few sections.

The Closed Interval Method for Finding Global Extrema. The results above can be turned into a procedure for finding and classifying the extrema of a continuous function f on a closed bounded interval $[a, b]$.

1. Compute the derivative of f, and find where it is zero or does not exist, plus the end points.
2. Find the value of f at each of these points.
3. Compare these values: the largest and smallest are the global maximum and minimum of f.

Exercises

Study Calculus Volume 1, Section 4.3[2]; in particular the *Problem Solving Strategy*, all Examples and Checkpoints, and a few Exercises from each of the ranges 91–98, 100–103, 104–107, 108–117, 118–128 and 129–134. (Some suggested selections are Exercises 91, 93, 97, 101, 107, 109, 119 and 129.)

4.4 The Mean Value Theorem

References.

- OpenStax Calculus Volume 1, Section 4.4[1]

[2] openstax.org/books/calculus-volume-1/pages/4-3-maxima-and-minima
[1] openstax.org/books/calculus-volume-1/pages/4-4-the-mean-value-theorem

- *Calculus, Early Transcendentals* by Stewart, Section 4.2.

The Mean Value Theorem is the intuitive fact that the slope between the endpoints of a curve is a mean (average) of the slopes at the various points along that curve, and so is between the extreme values of the tangent slopes, so that this secant slope equals the tangent slope at at least one point. A few details are needed to make this precise:

Theorem 4.4.1 The Mean Value Theorem, or MVT. *If a function f is continuous on a closed interval $[a, b]$ and is also differentiable there except possibly at the endpoints, then for at least one number c in (a, b), the average slope over the whole interval equals the tangent slope:*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

That is, the change $\Delta x = b - a$ over the interval produces a change

$$f(b) - f(a) = f'(c)(b - a), \text{ or } \Delta y = f'(c)\Delta x.$$

We will verify the MVT for all cases soon, but for now:

Checkpoint 4.4.2 Verify the Mean Value Theorem for $f(x) = x^3 - x$, $a = 1$, $b = 3$.

The Case of Zero Mean Slope: Rolle's Theorem. The MVT is even more intuitive in the special case when the value at each endpoint is the same, so the secant line is horizontal: the MVT then says that $f'(c) = 0$ somewhere in between.

Theorem 4.4.3 Rolle's Theorem. *If a function f is continuous on a closed interval $[a, b]$ and differentiable there except possibly at the endpoints, and if $f(a) = f(b)$,* then *$f'(c) = 0$ for at least one number c in the open interval (a, b).*

This has to be true basically because f must have a global maximum or minimum at some point c between a and b, and then Fermat's Theorem in Section 4.1, p. 50 says that $f'(c) = 0$. The only way that Fermat's Theorem might fail to give $f'(c) = 0$ is that the maximum and minimum both occur at the endpoints. But then the common endpoint value is both the global maximum and the minimum so f is constant, making $f'(c) = 0$ for any c!

Example 4.4.4 If an object's position s is a differentiable function of time t, $s = f(t)$, and the object is at the same position at two different times a and b, then Rolle's Theorem shows that the velocity $v = s'$ is zero at some intermediate time: *to return to its starting point, an object moving on a line must be stationary at some intermediate time.*

Note that this confirms an intuition already used in Section 3.7, p. 41. □

Checkpoint 4.4.5 Prove that equation $x^3 + x - 1 = 0$ has exactly one solution. Use the Intermediate Value Theorem to show that there is at least one solution, and then Rolle's Theorem to show that there is not more than one.

See Example 4 in Section 4.4 of the text,[2] which introduces the following intuitive physical version of the MVT:

Average Velocity. An intuitive application is that the average velocity over an interval of time must equal the instantaneous velocity at at least one moment: your speed cannot always be above average or always below average, and to swap between above and below, it must at some moment be exactly average. This comes from letting distance traveled by time t be $s = f(t)$ so the average velocity between times a and b is $v_{ave} = \dfrac{f(b) - f(a)}{b - a}$, and at some intermediate time $t = c$, the instantaneous velocity is $v_{inst} = f'(c) = v_{ave}$.

[2] `openstax.org/books/calculus-volume-1/4-4-the-mean-value-theorem`

Checkpoint 4.4.6 Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all x. How large can $f(2)$ possibly be?

Verifying the MVT in all cases. It can now be seen that the MVT is true in all cases by simply "twisting" the graph to get a horizontal secant line. For f satisfying the conditions of the MVT and with $m = \dfrac{f(b) - f(a)}{b - a}$ its mean slope, the new function

$$g(x) = f(x) - m(x - a)$$

has the same value $f(a)$ at each end point, and Rolle's Theorem gives $g'(c) = 0$ somewhere in between. Since $f(x) = g(x) + m(x - a)$, we get the claimed result that

$$f'(c) = g'(c) + m = 0 + m = \frac{f(b) - f(a)}{b - a}.$$

Zero Derivative Means Constant on an Interval. The MVT also helps us to find a function from information about its rate of change. For now, we deal with just the simplest case of zero rate of change:

Theorem 4.4.7 *If a function has derivative equal to zero at every point of an open interval (a, b), it is constant on that interval. This also applies to infinite intervals like $(-\infty, \infty)$. That is,* on intervals, *"zero derivative everywhere" means "constant everywhere".*

Proof. This is true because if the function were not constant, there would be a pair of numbers $\alpha < \beta$ in the interval with $f(\alpha) \neq f(\beta)$, and then the MVT applied to this smaller closed interval $[\alpha, \beta]$ would give a number c with derivative $f'(c) = (f(\beta) - f(\alpha)/(\beta - \alpha) \neq 0$, which would contradict what we know about f'. ■

Getting f from f', part I. The above says that the obvious functions with derivative $f' = 0$ are the only functions with this derivative, at least with domain being an interval. This has another important consequence:

Theorem 4.4.8 *If two functions have equal derivatives at each number in an interval, they differ by a constant on that interval.*

That is, if $f'(x) = g'(x)$ for all x in (a, b), then for some constant C, $f(x) = g(x) + C$.

Checkpoint 4.4.9

(a) Find every function whose derivative is $\cos x$.

(b) Find every function whose derivative is $1/x$. Be careful!

Checkpoint 4.4.10 Verify the identity $\arcsin x + \arccos x = \pi/2$. Do this two ways:

1. With a diagram and trigonometry.
2. Using Theorem 4.4.8, p. 58 above.

Exercises

Study Calculus Volume 1, Section 4.4[3]. Pay particular attention the *Corollaries of the Mean Value Theorem* in the second half: Theorems 6, 7 and 8: these will be extremely useful for applications later in this chapter.

Study Examples 14 and 15, Checkpoint 14, and a selection from Exercises 148–150, 152–156, 161–166, 167–169, 182–184, and 190–193.

Here I group the exercises in ranges by "question type", so start by trying one or two from each of the six ranges. For example, some suggested selections are Exercises 149, 153, 161, 169, 182 and 192.

[3] `openstax.org/books/calculus-volume-1/pages/4-4-the-mean-value-theorem`

4.5 Derivatives and the Shape of a Graph

References.

- OpenStax Calculus Volume 1, Section 4.5[1]
- *Calculus, Early Transcendentals* by Stewart, Sections 4.3 and 4.5.

We have seen a number of connections between derivatives and the shape of a graph, and these can be useful in both directions: using derivatives to understand how the graph of a function will look, and using the graph to summarize and identify useful information about a function, such as the locations of its extrema.

For example, we have seen that positive derivatives are associated with function values that increase as the argument increases, zeros of the derivative are associated with minimum and maximum values, and the second derivative is associated with whether the curve is bending up or down.

f′ Tells Where a Curve Increases and Where it Decreases. First let's make precise some intuitive concepts:

Definition 4.5.1 Increasing/Decreasing. Function f is **increasing** on an interval if for any numbers $c < d$ in that interval, $f(c) < f(d)$.

Likewise f is **decreasing** on an interval if $f(d) < f(c)$ for all such $c < d$. ◊

Theorem 4.5.2 Increasing/Decreasing Test.

1. *If $f'(x) > 0$ throughout an interval, then f is increasing on that interval.*
2. *If $f'(x) < 0$ throughout an interval, then f is decreasing on that interval.*
3. *Only at critical points can a function change between increasing and decreasing ...*
4. *... but not every critical point gives a change between increasing and decreasing.*

Proof. The proofs of (a)-(c) are based on $f(b) - f(a) = f'(c)(b - a)$, as given by the Mean Value Theorem.

Item (d) is shown by examples like $f(x) = x^3$. ■

Checkpoint 4.5.3 Make sketches illustrating each of the above four statements.

Checkpoint 4.5.4 Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing, and where it is decreasing.

Which Critical Points Are Local Minima? Local Maxima? Increasing/decreasing behavior can only change at critical points, and how it changes (if at all) determines whether a critical point is a local minimum or local maximum (or neither):

Theorem 4.5.5 The First Derivative Test for Local Extrema. *Consider c a critical number of a continuous function f (not an endpoint):*

1. *If f' changes from positive to negative at $x = c$ (as x increases), then f has a local maximum at c.*
2. *If f' change from negative to positive at $x = c$, then f has a local minimum at c.*
3. *If f' does not change sign at c (i.e. it has the same sign for x near c to either side), then there is no local extremum at c.*

All of these results are intuitive when one sketches the situations described, and are proved by the increasing/decreasing properties given by the signs of the derivatives.

Checkpoint 4.5.6 Make sketches illustrating all three cases above.

[1] `openstax.org/books/calculus-volume-1/pages/4-5-derivatives-and-the-shape-of-a-graph`

Checkpoint 4.5.7 Find the local minimum and maximum points of

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5.$$

Checkpoint 4.5.8 Find all local minima and maxima of

$$g(x) = x + 2\sin x, \text{ domain } 0 \le x \le 2\pi.$$

Using f′ at Endpoints. At endpoints we only need to look to one side:

Theorem 4.5.9 The First Derivative Test at Endpoints.

1. *If c is a left endpoint of an interval in the domain and $f'(x) > 0$ nearby, then f has a local minimum at c.*
2. *If c is a left endpoint and $f'(x) < 0$ nearby, then f has a local maximum at c.*
3. *If c is a right endpoint, the above are reversed: $f' > 0$ gives a local maximum, $f' < 0$ gives a local minimum.*

Checkpoint 4.5.10 Make sketches illustrating all four cases.

Checkpoint 4.5.11 If the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ has domain $[-2, 3]$, check the endpoints for local minima and maxima, and then determine the global extrema.

f″ Tells Which way a Curve is Bending: Concavity. Though we can usually classify critical points using just the first derivative, the second derivative gives an alternative that is sometime more convenient, and can improve the visual accuracy of a sketch graph.

Definition 4.5.12 Concavity. If the graph of f lies above all its tangents on an interval I it is called **concave upward** on that interval.

If the graph of f lies below all its tangents on an interval I it is called **concave downward** on that interval. ◊

Theorem 4.5.13 Concavity Test. *If $f''(x) > 0$ for all x in interval I, f is concave upward on I.*

If $f''(x) < 0$ for all x in interval I, f is concave downward on I.

Checkpoint 4.5.14 Illustrate each of the above with sketches of simple functions like $f(x) = x^2$ and $g(x) = \sin(x)$.

Concavity can be used in place of increasing/decreasing behavior to check a critical number c for a local minimum and maximum. This is sometimes easier, as you need only think about function values at one argument c, not at all nearby ones.

However, it only works where $f'(c) = 0$, not where $f'(c)$ does not exist.

Theorem 4.5.15 The Second Derivative Test.

1. *If $f'(c) = 0$ and $f''(c) > 0$, f has a local minimum at c.*
2. *If $f'(c) = 0$ and $f''(c) < 0$, f has a local maximum at c.*

Proof. Part (a) is true because when $f'(c) = 0$, the tangent there is the horizontal line $y = f(c)$, and $f''(c) > 0$ makes f concave up, so that the graph lies above this horizontal tangent line: nearby values of $f(x)$ are greater than $f(c)$, which is a local minimum.

Part (b) is the same but upside down. ■

Checkpoint 4.5.16 Illustrate each part of the above theorem.

Where Concavity Changes: Inflections. We have seen unusual cases like $f(x) = x^3$ where a critical point is not a local extremum. This is related to concavity flipping at such points:

Definition 4.5.17 A point P on a curve is called an **inflection point** if the curve changes from being concave up to being concave down (with continuity at that point.) $\Diamond$

This is akin to a local extremum being a point where the curve changes from increasing to decreasing. Inflections can occur at points where $f'' = 0$, and also at points where f'' does not exist. Locating inflections along with local extrema can help get the overall shape in a sketch of a function.

A Curve Sketching Strategy. A key to sketching a curve is to find "interesting" points (domain end-points, axis intercepts, vertical asymptotes, critical points, inflection points) and then find out about behavior in each interval between these, such as whether the curve in increasing or decreasing, concave up or concave down.

Note that within each interval between two consecutive "interesting" points, there is no change between increasing and decreasing, and no change in concavity.

It helps to summarize this information in a table, with the top row listing in increasing order the "interesting" values of the argument (x-values), a column in between each for the intervals in between, and all other useful information gathered in rows below.

As a variant, you can draw the number line for the domain, mark the interesting values on there, and summarize other information below it.

To sketch $y = f(x)$,

1. Compute the first and second derivatives, $y' = f'(x)$ and $y'' = f''(x)$.
2. Note any endpoints of the domain and points where the function is undefined.
3. If feasible, find the x-intercepts, points where $y = 0$.
4. Find the critical points, where y' is zero or does not exist (possible local extrema).
5. Find the points where y'' is zero or DNE (possible inflections).
6. Evaluate the function at all these x values, and summarize on a table with a column for each of these x values (a, b, c etc.) and a column in between each:

x	2		4		7		$\cdots$
y	5		6		4		$\cdots$
y'							
y''							

7. Determine the sign of y' and of y'', meaning positive, negative or zero, and add this information in the next rows of the table as $+$, $-$ or 0. (One way to do this is to evaluate y' and y'' at one x value in each interval between "interesting" values.)

x	2		4		7		$\cdots$
y	5		6		4		$\cdots$
y'	$+$	$+$	0	$-$	$-$	$-$	
y''	$-$	$-$	$-$	$-$	0	$+$	

8. At the bottom of the table, you might want to draw a little fragment of curve in each column with the correct increasing/decreasing behavior and correct concavity, with a dot on the curve where each "interesting value" occurs. Alternatively do this directly as you sketch the graph. (In this table, $x = 2$ is the left end-point, $x = 4$ a critical number and $x = 7$ gives an inflection.)
9. Sketch the graph using the points in the top two lines and (if drawn) the shapes drawn at the bottom of the table.

Exercises

Study Calculus Volume 1, Section 4.5[2]; in particular the *Problem Solving Strategy*, the *First Derivative Test*, the *Second Derivative Test* all Examples and Checkpoints, and a selection from Exercises 194–200, 201–205, 206–210, 211–215, 216–220, 221–223 and 224–230.

Some suggested selections are Exercises 199, 201, 203, 213, 215, 217, 223, 225, 229.

4.6 Limits at Infinity and Asymptotes

References.

- OpenStax Calculus Volume 1, Section 4.6[1]
- *Calculus, Early Transcendentals* by Stewart, Section 2.6

So far we have used the idea of limits to describe how a function behaves as its argument approaches a real value a: as $x \to a$. We also introduced the idea that if the value $f(x)$ gets larger and larger without bound, we say that "$f(x)$ is approaching infinity", and use shorthand $f(x) \to \infty$.

We now consider what happens as the argument x gets larger and larger without bound, using the similar wording that "x is approaching infinity", shorthand $x \to \infty$.

For x negative and increasing in magnitude, we talk of "x approaching negative infinty", shorthand $x \to -\infty$.

The idea of limits at infinity is useful to describe what happens to the extreme right and left on the graph of a function like $f(x) = \frac{x^2-1}{x^2+1}$, whose value is very close to 1 for x values of large magnitude. In the new shorthand, the value approaching 1 for ever larger positive x is "$f(x) \to 1$ as $x \to \infty$", or

$$\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = 1$$

The same happens going to the left, with x negative and of ever larger magnitude:

$$\lim_{x\to-\infty} f(x) = 1.$$

Graphically, the curve gets very close to the horizontal line $y = 1$ both to the right and left: this line is called a **horizontal asymptote.**

Limits at Infinity. We measure x being close to ∞ or $-\infty$ by $x > M$ for large M and $x < M$ for large negative M, as we measured $f(x)$ being close to infinity in the precise definition of infinite limits in Section 2.4. Thus, similar to that definition we have

Definition 4.6.1 Limits at Infinity. For function f defined on infinite interval (a, ∞) [i.e., for $x > a$], we say that the limit of $f(x)$ as x goes to ∞ is L if:

For any given positive number ϵ, there is a number M so that

having $x > M$ ensures that $|f(x) - L| < \epsilon$.

When this is true, we write

$$\lim_{x\to\infty} f(x) = L.$$

Similarly for the limit at $-\infty$, using $x < M$ instead. ◊

Limit Laws for Limits at Infinity. All the familiar limit laws apply for this new type of limits, so sums, products, compositions and such are all easily handled.

[2] openstax.org/books/calculus-volume-1/pages/4-5-derivatives-and-the-shape-of-a-graph
[1] openstax.org/books/calculus-volume-1/pages/4-6-limits-at-infinity-and-asymptotes

Example 4.6.2 The asymptotes of $\tan$ and $\tan^{-1}$. The tangent function on interval $(-\pi/2, \pi/2)$ has one-sided limits

$$\lim_{x\to-\pi/2^-} = -\infty, \qquad \lim_{x\to-\pi/2^+} = \infty$$

and is one-to-one with range (∞, ∞). Thus it has an inverse, $\tan^{-1}$, with domain (∞, ∞), range $(-\pi/2, \pi/2)$, and the vertical asymptotes flip over to give horizontal asymptotes $y = -\pi/2$ to the left, and $y = \pi/2$ to the right for this inverse.

These correspond to limits at infinity:

$$\lim_{x\to-\infty} = -\pi/2, \quad \lim_{x\to\infty} = \pi/2.$$

□

$\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$ are both zero; these facts that "As $x \to \pm\infty$, $1/x \to 0$" are useful building blocks in computing the horizontal asymptotes of other rational functions.

For example, these zero limits also occur for any negative power of x:

$$\text{For any } r > 0, \lim_{x\to\infty} \frac{1}{x^r} = \lim_{x\to-\infty} \frac{1}{x^r} = 0.$$

which follows from the above by using the power rule for limits.

Checkpoint 4.6.3 Calculate $\lim_{x\to\infty} \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$. As so often, the key idea is finding a useful simplification, so that limit laws and results above then give the answer easily. Here this means getting rid of division by infinity, by dividing top and bottom by x^2. Thus the limit is

$$\lim_{x\to\infty} \frac{3 - 1/x - 2/x^2}{5 + 4/x + 1/x^2} = \lim_{x\to\infty} \frac{3 - 1/x - 2(1/x)^2}{5 + 4(1/x) + (1/x)^2} = \frac{3 - 0 - 2(0)^2}{5 + 4(0) + (0)^2} = \frac{3}{5}.$$

Exponential functions like e^x have a horizontal asymptote $y = 0$ to the left, so

$$\lim_{x\to-\infty} e^x = 0, \text{ and in fact } \lim_{x\to-\infty} a^x = 0 \text{ for any } a > 1.$$

Checkpoint 4.6.4 Calculate $\lim_{x\to0^-} e^{1/x}$. Convert this to a limit at infinity with the change of variable $t = 1/x$.

Infinite limits at infinity. Many familiar functions have values that grow without bound ("$f(x) \to \infty$") as their argument grows without bound ("$x \to \infty$"). The simplest example is $f(x) = x$, and $g(x) = e^x$ is another. This situation combines function values going to infinity (as seen with infinite limits in Section 2.2) with the argument x going to infinity (as just seen with limits at infinity).

Combining these ideas and the notation for them, we say that $\lim_{x\to\infty} x = \infty$, $\lim_{x\to\infty} e^x = \infty$. In general in this situation, we write that

$$\lim_{x\to\infty} f(x) = \infty.$$

and likewise with the various $-\infty$ options.

Example 4.6.5 Calculate $\lim_{x\to\infty} x^3$, $\lim_{x\to-\infty} x^3$, $\lim_{x\to\infty} x^2$, and $\lim_{x\to-\infty} x^2$. □

Note: all the limits are infinite, but note how the signs differ for even and odd powers of x.

Example 4.6.6 Calculate $\lim_{x\to\infty} x^2 - x$. Factorize. □

Checkpoint 4.6.7 Sketching $y = f(x) = \dfrac{x^2 - 1}{x^2 + 1}$. When a function has horizontal asymptotes or infinite limits at infinity, we can enhance the sketching procedure from Section 4.5, p. 59 with the

help of fictitious "infinite endpoints" at $x = \pm\infty$ added to the table described there.

First, we need the derivatives, to seek critical and inflection points:

$$y' = \frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) = \frac{2x(x^2+1)-(x^2-1)2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

so the only critical point is at $x = 0$: the point $(0, -1)$. Next

$$y'' = \frac{d}{dx}\left(\frac{4x}{(x^2+1)^2}\right) = \frac{4(x^2+1)^2 - 4x2(x^2+1)2x}{(x^2+1)^4} = \frac{4x^2+4-16x^2}{(x^2+1)^3} = \frac{4(1-3x^2)}{(x^2+1)^3}$$

so the second derivative is zero at $x = \pm 1/\sqrt{3}$, giving the points $(\pm 1/\sqrt{3}, -1/2)$.

In this case, we can also easily get the x-axis intercepts, where $y = 0$: they are at $x = \pm 1$ Thus there are five "interesting" x-values, or seven counting the infinities; arranged in order left-to-right, $-\infty, -1, -1/\sqrt{3}, 0, 1/\sqrt{3}, 1, \infty$.

x	$-\infty$		-1		$-1/\sqrt{3}$		0		$1/\sqrt{3}$		1		∞
y	1		0		$-1/2$		-1		$-1/2$		0		1
y'							0						
y''					0				0				

It is easy to check that $y' < 0$ for $x < 0$ and $y' > 0$ for $x > 0 y''$

x	$-\infty$		-1		$-1/\sqrt{3}$		0		$1/\sqrt{3}$		1	
y	1		0		$-1/2$		-1		$-1/2$		0	
y'		$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$
y''		$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$

Sketches

With this information we can sketch the function; do this in three ways:

- Using just the axis intercepts and the increasing/decreasing information given by the signs of the first derivative, y'.
- Using just the axis intercepts and the concavity information given by the signs of the second derivative, y''.
- Using all the above information.

Exercises

Study Calculus Volume 1, Section 4.6[2]; particularly Examples 21 to 26, 28, 29 and 31 and Checkpoints 20, 23–25, 27, 28 and 30 (We omit *oblique asymptotes*, so skip Example 30 and Checkpoint 29), and a selection from Exercises 251–255, 256–260, 261–270, 271–281, 285–288 and 294–298.

Here the exercises are grouped in ranges by "question type", so start by trying one or two from each of the seven ranges; some suggested selections are Exercises 251, 256, 257, 259, 261, 263, 265, 267, 271, 279, 281, 285, 306 and 307.

4.7 Applied Optimization Problems

References.

- OpenStax Calculus Volume 1, Section 4.7[1]

[2] openstax.org/books/calculus-volume-1/pages/4-6-limits-at-infinity-and-asymptotes
[1] openstax.org/books/calculus-volume-1/pages/4-7-applied-optimization-problems

- *Calculus, Early Transcendentals* by Stewart, Section 4.7.

Many scientific and engineering questions can be phrased in terms of finding the global minimum or maximum of a function, such as minimizing cost or weight or maximizing the use of limited resources. The solution to such questions often breaks into several steps:

1. rephrasing a verbal question in terms of a quantity to be minimized, given in terms of other quantities whose values one can adjust in seeking the optimal outcome;
2. expressing the quantity to optimize as a function of a single variable;
3. finding the local extrema of this function; and finally
4. identifying the optimum (global minimum) from amongst these local extrema.

The First and Second Derivative Tests for Global Extrema. There is bad news and good news in using the ideas of previous section to find a global extremum.

The bad news is that the domain is often not a closed interval for which the Extreme Value Theorem applies. Instead it is often an open interval, such as "any positive value".

The good news is that there is often a unique critical point, and then the only question is whether it is a global maximum, a global minimum, or neither. Derivatives again answer this question:

Theorem 4.7.1 (The first derivative test for global extrema). *Suppose that function f has a unique critical number c on an interval.*

- *If $f'(x) > 0$ to the left of c ($x < c$) and $f'(x) < 0$ to the right of c ($x > c$), then $f(c)$ is the* maximum *value of f on that interval.*
- *If $f'(x) < 0$ to the left of c and $f'(x) > 0$ to the right of c, then $f(c)$ is the* minimum *value of f on that interval.*
- *If $f'(x)$ has the same sign on either side of c then f* has no global extremum *on that interval, except possibly at an endpoint.*

Checkpoint 4.7.2 Make sketches illustrating each case.

Theorem 4.7.3 (The second derivative test for global extrema). *Suppose that function f has a unique critical number c on an interval.*

- *If $f''(c) < 0$ then $f(c)$ is the* maximum *value of f on that interval.*
- *If $f''(c) > 0$ then $f(c)$ is the* minimum *value of f on that interval.*

Note: if $f''(c) = 0$ or DNE, other tests must be used, like the one above.

Checkpoint 4.7.4 Make sketches illustrating each case.

A Strategy for Optimization Problems. The approach blends some elements from related rates problems with ideas from this chapter.

- *Read the question carefully.* (Familiar?) Note all the relevant information.
- If appropriate, *draw a diagram* to summarize this information.
- *Name all relevant quantities*, in particular one to be optimized (let us call it Q) and others whose values can be adjusted (say x, y, etc.)
- Find *a formula for the quantity to be optimized* in terms of the other quantities: say $Q(x, y, \ldots)$.
- If this formula involves more than one variable, seek equations relating these, and use them to *eliminate all but one independent variable*, giving *the quantity to be optimized as a function* of a single variable; say $Q(x)$.
- Determine the allowable values of the independent variable[s], and thus *determine the domain of the above function* ($Q(x)$).

- Using the various derivative tests above, or otherwise, *find the global optimum value* and the values of the independent variable[s] that give it.

  Do not forget that *the optimum might occur at an endpoint!*
- *Answer the Question.*

  The full answer often involves giving the values of all variables introduced in Step 3, and putting back in physical units.

Exercises

Study Calculus Volume 1, Section 4.7[2]; in particular Examples 33–35 and 37; Checkpoints 31–34 and 36; and a selection from Exercises 311–314, 315–318, 319–321, 322–326, 335–336 and 351–355.

Here the exercises are grouped in ranges by "question type", so start by trying one or two from each of the seven ranges; some suggested selections are Exercises 311, 316, 320, 322, 335 and 353.

4.8 L'Hôpital's Rule

References.

- OpenStax Calculus Volume 1, Section 4.8[1]
- *Calculus, Early Transcendentals* by Stewart, Section 4.4.

There are places in the graph of a function where simple evaluation can fail to show what is happening: argument values x where the formula gives a meaningless *indeterminate* form like "$0/0$", "∞/∞", "$0 \cdot \infty$", "$\infty - \infty$", "0^0", "1^∞" or "∞^0".

It can be useful to compute the *limits* at such points, and l'Hôpital's Rule often helps with these sort of limits. This rule can also be useful in exploring the sideways extremes of a graph, the limits as $x \to -\infty$ and $x \to \infty$.

When $f(x)$ gives "$0/0$" for some x values. With the function given by the formula $f(x) = \dfrac{\sin x}{x}$, the formula fails at $x = 0$, because substituting in $x = 0$ gives $0/0$, an **indeterminate form**.

It *cannot* be simplified to 0 (suggested by the zero numerator), or to infinity (suggested by the zero denominator), or to 1 (suggested by canceling equal factors).

An indeterminate form tells us nothing about what happens at that point!

A new idea is needed. Fortunately, in the above example, we have seen that this is a **removable discontinuity**: the domain of the function can be extended to include $x = 0$ while keeping function continuous, with a unique choice of the value given by the limit

$$f(0) = \lim_{x \to 0} \frac{\sin x}{x}, = 1.$$

Evaluating that limit was the hardest part of finding the derivative of sine, but now that we know a good collection of derivatives, computing such limits can be much easier, avoiding much algebra and trigonometry.

L'Hôpital's Rule for 0/0. For a limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ with $f(a) = g(a) = 0$ and both functions differentiable, the behavior for x values near a can be approximated with the linearizations

$$f(x) \approx f(a) + f'(a)(x - a) = f'(a)(x - a)$$

[2] openstax.org/books/calculus-volume-1/pages/4-7-applied-optimization-problems
[1] openstax.org/books/calculus-volume-1/pages/4-8-lhopitals-rule

$$g(x) \approx g(a) + g'(a)(x-a) = g'(a)(x-a).$$

This *suggests* that for $x \approx a$,

$$\frac{f(x)}{g(x)} \approx \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}.$$

The one hazard is that $g'(a)$ might also be zero. To avoid this, the precise result is

Theorem 4.8.1 L'Hôpital's Rule, basic form.
If $f(x)$ and $g(x)$ are differentiable at $x = a$ and $f(a) = g(a) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

if *both limits exist.*
So if f' and g' are continuous at a and $g'(a) \neq 0$, the limit is $f'(a)/g'(a)$.

Example 4.8.2

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x}{x} &= \lim_{x\to 0} \frac{(\sin x)'}{x'} = \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1 \\
\lim_{x\to 1} \frac{x-1}{\ln x} &= \lim_{x\to 1} \frac{1}{1/x} = 1 \\
\lim_{x\to 2} \frac{x-2}{x^2-4} &= \lim_{x\to 2} \frac{1}{2x} = \frac{1}{4} \\
\lim_{x\to 0} \frac{1-\cos x}{x^2} &= \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2} \\
\lim_{x\to 0} \frac{\tan x}{x^2} &= \lim_{x\to 0} \frac{\sec^2 x}{2x}, \text{ DNE: division by zero.}
\end{aligned}$$

□

Notes.

1. Keep going so long as *both* the top and bottom are zero at $x = a$, but evaluate as soon as either one is non-zero, even if the result is "no limit" or an infinite limit due to division by zero.
2. The top and bottom are differentiated separately, different from and easier than the quotient rule for derivatives!

Theorem 4.8.3 L'Hôpital's Rule extended to infinity.
L'Hôpital's Rule also applies to the cases of limits at infinity ($a = \infty$), one sided limits, and when each of $f(x)$ and $g(x)$ has an infinite limit at the relevant point. That is, it applies when an attempt to use the quotient rule for limits gives the nonsense result $0/0$ or ∞/∞.

Again, repeated application might be needed, so long as limits of the new top and bottom functions are both infinite or both zero, but you must stop as soon as this is no longer true.

Checkpoint 4.8.4 Evaluate $\displaystyle\lim_{x\to -\infty} \frac{4x^2-2x+5}{2x^2+x+1}$.

Checkpoint 4.8.5 Evaluate $\displaystyle\lim_{x\to \infty} \frac{e^x}{x^2}$.

Checkpoint 4.8.6 Evaluate $\displaystyle\lim_{x\to \infty} \frac{\ln x}{\sqrt[3]{x}}$.

Indeterminate Products $\infty \cdot 0$.

Another kind of indeterminate form that can arise in limits is a product $\infty \cdot 0$, as with $\lim_{x\to a} f(x)g(x)$ when $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$.

Simple examples like $\lim_{x\to 0}(5/x)x$, $\lim_{x\to 0}(1/x^2)x$ and $\lim_{x\to 0}(1/x)x^2$ show that the limit can be zero, infinity or anything in between; it can be negative too.

Fortunately, such limits can often be rewritten as a quotient where l'Hôpital's Rule applies: simply rewrite $f(x)g(x)$ as

$$\text{either} \quad \frac{f(x)}{1/g(x)} \quad \text{giving a form } \frac{\infty}{\infty}, \quad \text{or} \quad \frac{g(x)}{1/f(x)}, \quad \text{giving a form} \quad \frac{0}{0}.$$

The form ∞/∞ is usually more useful.

Example 4.8.7

$$\begin{aligned}
\lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+} \frac{\ln x}{1/x} \qquad \text{Simplify: rearrange in } \infty/\infty \text{ form} \\
&= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} \qquad \text{From l'Hôpital's Rule.} \\
&\text{The result is still } \infty/\infty\text{, so SIMPLIFY before trying anything fancier:} \\
&= \lim_{x\to 0^+} -x \\
&= 0.
\end{aligned}$$

□

Warning: if you do not simplify after the first use of l'Hôpital's Rule, and instead use l'Hôpital's Rule again, you will go on forever!

Indeterminate Differences: $\infty - \infty$.

If $\lim_{x\to a} f(x) = \infty$ and also $\lim_{x\to a} g(x) = \infty$, attempting to evaluate $\lim_{x\to a}(f(x) - g(x))$ can lead to another kind of indeterminate form, "$\infty - \infty$".

This can have any finite or infinite value, or no limit at all.

Sometimes, the function $f(x) - g(x)$ can be rewritten as a quotient giving an indeterminate form $0/0$, so that l'Hôpital's Rule can be tried. For example, this happens if $f(x)$ and $g(x)$ are both quotients with the same function in the denominator, with that function going to zero as $x \to a$.

Checkpoint 4.8.8 Compute $\lim_{x\to(\pi/2)^-}(\sec x - \tan x)$.

Indeterminate Powers: 0^0, ∞^0 and 1^∞.

A final kind of indeterminate form is with the limit as $x \to a$ of an exponential expression

$$y = [f(x)]^{g(x)}$$

where as $x \to a$, either

- $f(x) \to 0$ and $g(x) \to 0$ [type 0^0], or
- $f(x) \to \infty$ and $g(x) \to 0$ [type ∞^0], or
- $f(x) \to 1$ and $g(x) \to \pm\infty$ [type 1^∞].

All of these can be handled by converting the exponentiation into a product, by looking at the limit of the logarithm:

$$\ln y = \ln([f(x)]^g(x)) = g(x)\ln(f(x)),$$

and computing the limit of $\ln y$ gives the indeterminate form $0 \cdot (-\infty)$, or $0 \cdot \infty$, or $\pm\infty \cdot 0$ respectively for the three cases above. So one can try to evaluate this limit by the methods above for indeterminate products.

Note that if one succeeds, that gives the value of the limit of $\ln y$, so the last step is to exponentiate this to get the desired limit of y itself.

Checkpoint 4.8.9 $\lim_{x \to 0^+} x^x$

Checkpoint 4.8.10 $\lim_{x \to 0^+} (1 + \sin(4x))^{(\cot x)}$

Exercises

Study Calculus Volume 1, Section 4.8[2], Examples 38–41, 43 and 44, Checkpoints 37–40, 42 and 43, and a selection from Exercises 356–361, 362–366, 367–385, 387–389 and 391–395.

Here the exercises are grouped in ranges by "question type", so start by trying one or two from each of the ranges; some suggested selections are Exercises 357, 359, 363, 367, 371, 377, 379, 387, and 393.

4.9 Newton's Method (Omitted)

References.

- OpenStax Calculus Volume 1, Section 4.9[1]
- *Calculus, Early Transcendentals* by Stewart, Section 4.8.

This optional topic is not covered this semester, so these notes are just references and a brief study guide.

Exercises

Study Calculus Volume 1, Section 4.9[2]; in particular Examples 46 to 48, Checkpoints 45 to 47, and Exercises 407, 423 and 429.

4.10 Antiderivatives

References.

- OpenStax Calculus Volume 1, Section 4.10[1]
- *Calculus, Early Transcendentals* by Stewart, Section 4.9.

Probably the greatest use of calculus is in problems where one knows something about the derivatives of a function and wishes to learn about the function: going from knowledge about the rate of change of a quantity to knowing the quantity itself. For example, the laws of physics often describe acceleration (second derivative of position), from which position as a function of time is determined. Also in biological, chemical and economic models, rates of changes are often the measured or known information, from which we seek to make predictions of how a quantity will change over time.

We have already seen one simple but important example: *when the acceleration of a falling body is constant, its velocity is linear in time, and its position is a suitable quadratic function of time.*

[2] openstax.org/books/calculus-volume-1/pages/4-8-lhopitals-rule
[1] openstax.org/books/calculus-volume-1/pages/4-9-newtons-method
[2] openstax.org/books/calculus-volume-1/pages/4-9-newtons-method
[1] openstax.org/books/calculus-volume-1/pages/4-10-antiderivatives

Our interest here is the "opposite" of derivatives:

Definition 4.10.1 Antiderivative. A function F is called *an* **antiderivative** of another function f on interval I if $F' = f$ on that interval. ◊

For example, for function $f(x) = 3x^2$, the function $F(x) = x^3$ is an antiderivative, and so is the alternative $F(x) = x^3 + 7$, or indeed $F(x) = x^3 + C$ for any constant C.

Note the wording about ***an*** antiderivative of f rather than ***the*** antiderivative.

Position and Velocity from (Constant) Acceleration. When acceleration

$$a(t) = v'(t) = -9.8m/s^2,$$

one possible anti-derivative of a is

$$v(t) = -9.8t,$$

and since this $v(t)$ is $s'(t)$ for $s(t)$ the displacement, one possible antiderivative of v is displacement

$$s(t) = -4.9t^2.$$

But there are other antiderivatives: one can have

$$v(t) = -9.8t + v_0 \text{ for any constant } v_0\text{, the velocity when } t = 0.$$

Then this velocity has as one antiderivative the position function $s(t) = -4.9t^2 + v_0 t$, and again, adding any constant is allowed, giving the family of of possible position functions

$$s(t) = -4.9t^2 + v_0 t + s_0 \text{ for any constant } s_0\text{, the displacement when } t = 0.$$

Checkpoint 4.10.2 Find formulas for all possible antiderivatives of

- $f(x) = \sin x$
- $f(x) = x^n$, $n \neq -1$ a constant
- $f(x) = x^{-1} = 1/x$.

Hint. Be careful with the cases where the domain is not an interval because it excludes $x = 0$.

The "quirks" with domains like $x \neq 0$ illustrate why we usually work with derivatives and antiderivatives on *intervals*. This is natural; domains are intervals in most applications of antiderivatives (and indeed in most applications of calculus).

Checkpoint 4.10.3 Hunting and Collecting Antiderivatives (the start of an ongoing activity).

Every derivative formula gives the anti-derivative of a function, so this is how we start our collection of antiderivatives:

1. Write down all the the formulas you know for derivatives of basic functions, giving pairs $F(x)$, $f(x) = F'(x)$.
2. Multiply each by a constant if necessary so that the function $f(x)$ is as simple as possible.
3. Turn each pair around into a function-antiderivative pair $f(x)$, $F(x)$, and gather these in a table.
4. Add new antiderivatives to this table as we discover them.

This table will be useful for computing derivatives and anti-derivatives: keep it with your notes.

Theorem 4.10.4 Antiderivatives on an interval differ only by a constant.

If F and G are both antiderivatives of the same function f on the same interval I, they differ only by a constant:

$G(x) - F(x)$ is a constant C, so

$$G(x) = F(x) + C. \tag{4.10.1}$$

Proof. This follows from the Mean Value Theorem:

Firstly, the difference $G - F$ has derivative $(G - F)' = G' - F' = f - f = 0$ everywhere.

We saw in Section 4.4, p. 56 that the only function with derivative equal to zero everywhere on an interval is a constant, so $G(x) - F(x)$ is a constant: call it C. ■

As a result, the graphs of two different antiderivatives of f on an interval never pass through the same point, so once you know one point on the graph of the desired antiderivative, there is only one choice.

The Geometry of Antiderivatives. We have seen how derivatives relate to the "geometry" of a function; the shape or of the graph. This ideas is useful in the reverse direction too, using that fact that the *value* of $f(x)$ describes the *slope* of the antiderivative graph at x.

The above theorem says that *the graphs of any two different antiderivatives of the same function on an interval differ simply by a vertical shift.*

Checkpoint 4.10.5 Sketching Antiderivatives.

1. Sketch a few simple functions f like $f(x) = x^2$ or $f(x) = \sin x$, for which we know an antiderivative, and from the information there about the slope of the antiderivative, try to sketch an antiderivative $F(x)$, passing through the origin. Place the sketch of F directly below that of f.
2. Sketch the known antiderivatives and see how well you did.
3. Draw a graph of some function f for which you do not know a formula, and try to sketch several antiderivatives for it.

Rectilinear Motion (motion along a line). If we know the acceleration of an object moving on a line, its velocity is an antiderivative. Knowing the velocity at any one time allows one to choose the correct antiderivative:

Knowing acceleration at all times plus velocity at any one time determines velocity at all times.

In turn, knowing velocity tells as the position (an antiderivative of velocity) up to a constant and knowing position at one time determines the position function. Putting this all together,

*Knowing acceleration at all times plus position and velocity at any one time determines position at all times.*This is the basic form of the single most important mathematical contribution to physics in the last few centuries, and one of the main reasons why calculus is so important in physical science.

Exercises

Study Calculus Volume 1, Section 4.10[2]; including all the Examples and Checkpoints, and a selection from Exercises 465–469, 470–473, 474–489, 490–498, 499–503 and 504–508.

Here the exercises are grouped in ranges by "question type", so start by trying several from each of the ranges; some suggested selections are Exercises 465, 467, 469, 471, 477, 487, 491, 493, 499, 501 and 505.

Hint: It often helps to simplify the function first, and then use the list of derivatives and indefinite integrals in the online test.

[2] `openstax.org/books/calculus-volume-1/pages/4-10-antiderivatives`

Chapter 5

Integrals

References.

- OpenStax Calculus Volume 1, Chapter 5.[1]
- *Calculus, Early Transcendentals* by Stewart, Chapter 5.

5.1 Approximating Areas (and Distance Traveled)

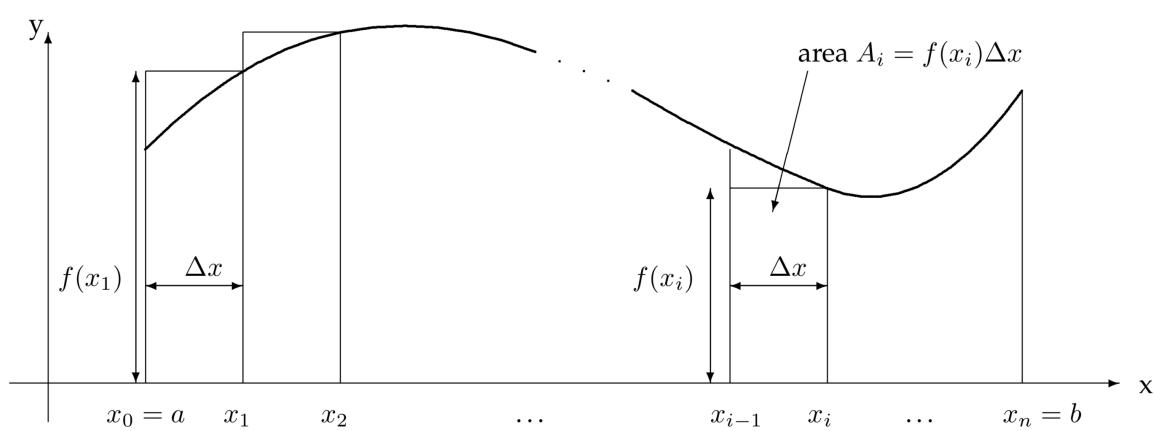
References.

- OpenStax Calculus Volume 1, Section 5.1[1]
- *Calculus, Early Transcendentals* by Stewart, Section 5.1.

One of the beauties of mathematics is that often problems that seem to be very different turn out to have very similar mathematical representations and solutions. Two such problems are

- finding the area of a region with curved boundary, and
- finding the distance traveled when we know how the velocity varies with time.

Problem 1: The Area of a Region with Curved Upper Boundary. We can compute the area shown in the figure where the upper boundary is the curve $y = f(x)$, the lower boundary is the x axis, the left boundary is the line $x = a$, and the right boundary is the line $x = b$ as follows:



[1]openstax.org/books/calculus-volume-1/pages/5-introduction
[1]openstax.org/books/calculus-volume-1/pages/5-1-approximating-areas

Divide the interval $[a, b]$ into n small subintervals each with length

$$\Delta x = \frac{b-a}{n}$$

and let $x_0, x_1, x_2, \ldots, x_n$ be the end points of the subintervals, so that

$$x_0 = a, x_i = a + i\Delta x, x_n = b$$

The area A_i above the i-th subinterval $x_{i-1} < x < x_i$ will be approximately the area of a rectangle with width Δx and height $f(x_i)$:

$$A_i \simeq f(x_i)\Delta x$$

Adding these up we get the total area is given (approximately) by

$$A = \sum_{i=1}^{n} A_i \simeq R_n := \sum_{i=1}^{n} f(x_i)\Delta x. \tag{5.1.1}$$

This is the so-called **right-hand endpoint rule**, because we use the value of $f(x)$ at the right-hand end of each sub-interval $[x_{i-1}, x_i]$ as the height of the rectangle over that interval. An alternative is to use the height at the left end of each interval, giving the **left-hand endpoint rule**

$$A \simeq L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x. \tag{5.1.2}$$

Note: never use the value at both the left and right endpoints!

The Exact Area, Using Limits. If we used more subintervals (larger n and thus smaller Δx), we could get a better approximation, because the rectangles would fit the true area closer over the shorter intervals. If we can find the limit as $n \to \infty$ and $\Delta x \to 0$ of these approximating sums, then we can find the area exactly:

$$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x, = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x \tag{5.1.3}$$

It also turns out that the approximations L_n and R_n lead to the same limit, so long as $f(x)$ is continuous. We could also use rectangles with heights given by the function's value at intermediate points in each interval, such as the middle points $a + h/2$, $a + 3h/2$ ... $b - h/2$. In fact, we use this formula to *define* area in this situation; without calculus and limits, area has only really been defined for polygons.

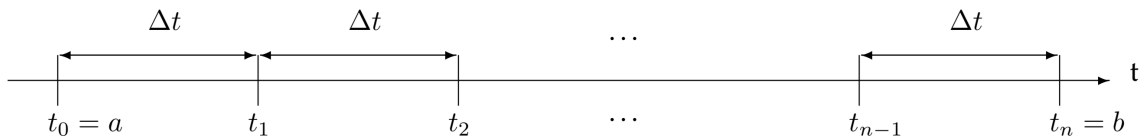
Later in this chapter we will learn how to evaluate such limits, at least for some functions f. However, the accurate approximations given by the sum formulas with small Δx are often also often useful in practice.

Problem 2: Displacement (net change in position) from Velocity. If we know a function that gives the velocity of an object at time t, that is we know $v = f(t)$ and we want to find the distance s that the object travels over a time interval $a \le t \le b$, we can proceed as follows: Divide the time interval $[a, b]$ into n small subintervals each with length

$$\Delta t = \frac{b-a}{n}$$

and let $t_0, t_1, t_2, \ldots, t_n$ be the end points of the subintervals, so that

$$t_0 = a, t_i = a + i\Delta t, t_n = b$$



Over the i-th subinterval $t_{i-1} < t < t_i$, the velocity can be approximated by its value at the start of that interval, $f(t_{i-1})$, so that we can use the familiar formula "distance = rate $\times$ time" to compute the distance s_i traveled (approximately) over that short time interval:

$$s_i \simeq f(t_{i-1})\Delta t. \tag{5.1.4}$$

Adding these up, the total distance traveled is given (approximately) by

$$s \simeq \sum_{i=0}^{n-1} f(t_i)\Delta t. \tag{5.1.5}$$

This is the *left-hand endpoint rule* (5.1.2) again, and again an alternative is to use the velocity at the end of each time interval; the *right-hand endpoint rule* (5.1.1).

From Approximations to the Exact Displacement. If we used more subintervals (larger n and thus smaller Δt), we could get a better approximation, because the velocity would be closer to being constant over the shorter intervals. If we can find the limit as $n \to \infty$ and $\Delta t \to 0$ of these approximating sums, then we can find the distance exactly.

$$s = \lim_{\Delta t \to 0} \sum_{i=1}^{n} f(t_i)\Delta t, = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i)\Delta t. \tag{5.1.6}$$

Comparing the Area and Distance Formulas. Note the similarity between Equation (5.1.3) for area under curve and Equation (5.1.6) for computing distance from a velocity function: this allows the same methods to be used for both the area and the distance problems, for both approximation and exact evaluation.

Indeed a great variety of the mathematical and scientific problems can be solved in terms of the same sort of "limit of a sum" formula, which makes evaluation of this quantity of great importance.

This is the topic for the rest of this course, and a major topic in Calculus II.

Exercises

Study Calculus Volume 1, Section 5.1[2]; in particular, if you are unfamiliar with the Σ notation for sums, the first part of that section should help. Study Exercises 15, 19, 23, 27, 29, and 43.

5.2 The Definite Integral

References.

- OpenStax Calculus Volume 1, Section 5.2[1]
- *Calculus, Early Transcendentals* by Stewart, Section 5.2.

The ``limit of sums'' formula seen in Section 5.1 for computing both distance traveled and area under a curve is also useful in many other cases, and the main goal of this chapter is to learn more

[2] `openstax.org/books/calculus-volume-1/pages/5-1-approximating-areas`
[1] `openstax.org/books/calculus-volume-1/pages/5-2-the-definite-integral`

about how to do this calculation in practice, without having to actually evaluate the sums or limits, but instead mostly using anti-derivatives.

In this section we make a careful statement of the quantity to be calculated, introduce some variants on the **Riemann sum** approximation of the area under a curve to make calculator approximations more accurate and efficient, and learn some properties akin to those for limits, derivatives and anti-derivatives: rules for sums, differences, constant multiples, etc.

A key calculus strategy: first approximate, then find a limit. There are many other problems that can be calculated by the above process of

- *approximating* a quantity by a sum of function values times a small interval width Δx, and then
- finding the *exact quantity* as the limit as the number of function values used goes to ∞ and Δx goes to 0.

Thus we need a name and notation for it:

Definition 5.2.1 Definite Integral, right-hand rule version. For $f(x)$ is a continuous function on the interval $a \le x \le b$, the **definite integral** of $f(x)$ over the interval $[a, b]$, denoted $\int_a^b f(x)\,dx$, is the numerical value given by the limit

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x, = \lim_{\Delta x\to 0} \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$, so $x_0 = a$ and $x_n = b$. $\Diamond$

Note that the notation

- modifies the ``Greek S'' $\Sigma$ to become the ``elongated S'' $\int$, and
- changes the ``Greek D'' $\Delta$ in $\Delta x$ to the ``small d'' in dx, to indicate that the limit was taken as $n \to \infty$ (i.e. $\Delta x \to 0$).

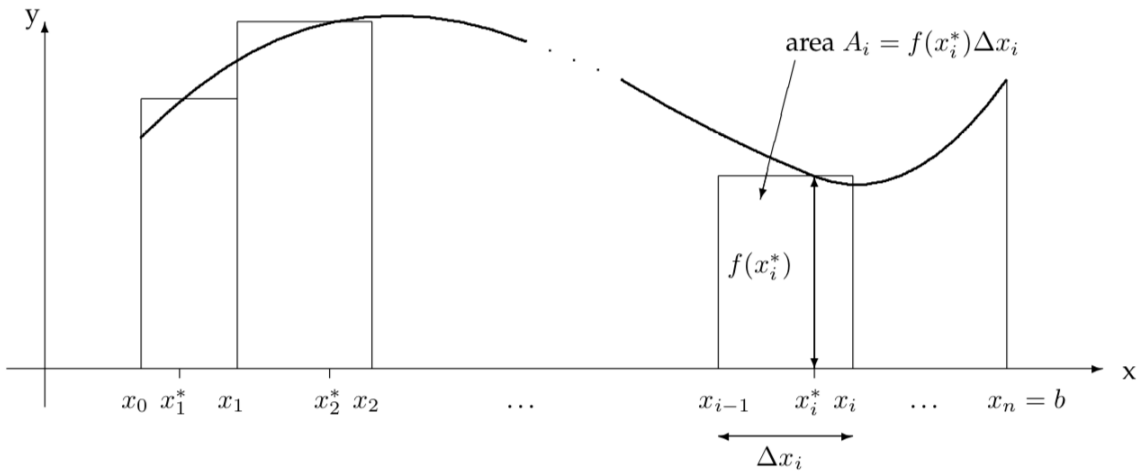
Other Choices for the Rectangle Heights and Widths. The sums of areas of rectangles used above to approximate the area under the curve is called a **Riemann Sum**, but the choice of using intervals of *equal width* with the height of each rectangle being the height of the curve at the *right endpoint* of each interval is not the only possibility: it was used partly because it makes the notation easiest. The intervals can instead vary in width, and the heights can instead be computed at other points x_i^* in each interval, like the left endpoints or the midpoints, or a different choice in each interval. Of these options, using the mid-point of each interval is intuitively the best choice, and this in fact can be proven to be the most accurate in some sense, to be seen in Calculus 2.

The most general form of the approximation for area under the curve allows for the interval $[a, b]$ to be divided into possibly unequal intervals by x values $a = x_0 < x_1 \cdots < x_n = b$, with widths

$$\Delta x_1 = x_1 - x_0, \ldots, \Delta x_i = x_i - x_{i-1}, \ldots, \Delta x_n = x_n - x_{n-1},$$

taking any point x_i^* within each sub-interval $[x_{i-1}, x_i]$ to get the height of a rectangle on that sub-interval. Then the approximate area under the curve is the general *Riemann sum Approximation*

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 \cdots f(x_n^*)\Delta x_n.$$



It can be shown that even with varying intervals widths and choices of where in each interval to compute the rectangle height, the approximations all get close to the same value when the widths of all the subintervals are very small (no rectangle width Δx_i bigger than some maximum width Δx), so long as $f(x)$ is continuous on $[a, b]$. The proof is omitted here; it is seen is advanced calculus courses.

This gives the most general definition:

Definition 5.2.2 Definite Integral, with all Riemann Sum Approximations. If $f(x)$ is a continuous function on the interval $a \le x \le b$, with the x_i, Δx_i and x_i^* as above and $\Delta x_i \le \Delta x$, then

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

$\Diamond$

The Mid-point Rule. Of these approximations, using the mid-point of each interval is intuitively the best choice. % and this in fact can be proven to be the most accurate in some sense. %, to be seen in Calculus 2. It is still simplest to use n intervals of equal width $h = \Delta x = (b-a)/n$, which gives the n**-point midpoint rule approximation**

$$\int_a^b f(x)\,dx \approx M_n = h\sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right) = h\sum_{i=1}^{n} f(a+(i-1/2)h),$$

where $x_i = a + ih$ ($x_0 = a, x_1 = a + h$, etc.)

This sum can be evaluated on calculators with something like

```
sum(seq(f(a+(i-0.5)*h),i,1,n))*h
```

or the slightly quirky but easier to type version

```
sum(seq(f(x),x,a+h/2,b,h))*h
```

This uses x values $a + h/2$, $a + 3h/2$ and so on, continuing so long as the value is less than b. Actually the last value used is $b - h/2$, but using the upper limit of b is safer; if you use $b - h/2$, a slight rounding error can cause that last x value to be omitted!

Properties of the Definite Integral. Thinking of definite integrals as areas under curves or displacements given by velocities, the following facts are intuitive. We will soon see a simple way to verify them, using anti-derivatives.

1. $\int_a^b c\,dx = c(b-a)$ where c is any constant.
2. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ where c is any constant.
3. $\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
4. $\int_a^b f(x)-g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
5. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
6. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
7. $\int_a^a f(x)\,dx = 0$

Comparison Properties of the Definite Integral.

1. If $f(x) \geq 0$ for $a \leq x \leq b$, then
$$\int_a^b f(x)\,dx \geq 0.$$
2. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then
$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$
3. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then
$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

The last is a cousin of the Mean Value Theorem.

Exercises

Study Calculus Volume 1, Section 5.2[2]; in particular Examples 8 to 13, Checkpoints 8 to 12, and Exercises 61, 65, 73, 75, 79, 81, 89, 91, 93, 99, 101 and 107.

5.3 The Fundamental Theorem of Calculus

References.

- OpenStax Calculus Volume 1, Section 5.3[1]
- *Calculus, Early Transcendentals* by Stewart, Section 5.3.

The Fundamental Theorem of Calculus relates derivatives to definite integrals, giving an easy way to evaluate many definite integrals using antiderivatives. One half is the formula

$$\int_a^b f(x)\,dx = F(b) - F(a), \tag{5.3.1}$$

[2] openstax.org/books/calculus-volume-1/pages/5-2-the-definite-integral
[1] openstax.org/books/calculus-volume-1/pages/5-3-the-fundamental-theorem-of-calculus

true when F is *any* antiderivative of f on the interval $[a, b]$: $F' = f$.

This equation can for example be corroborated for some simple definite integrals whose values are clear from geometry:

Example 5.3.1 For $f(x) = c$, any constant, all antiderivatives have the form $cx + C$ (C another constant), so Equation (5.3.1) says

$$\int_a^b c\,dx = F(b) - F(a) = (cb + C) - (ca + C) = c(b - a),$$

the expected area of the rectangle of width $b - a$, height c under this curve. □

Example 5.3.2 For $f(x) = x$, antiderivatives have the form $x^2/2 + C$ (C a constant),

$$\int_a^b x\,dx = F(b) - F(a) = \left(\frac{b^2}{2} + C\right) - \left(\frac{a^2}{2} + C\right) = \frac{b^2}{2} - \frac{a^2}{2} = \frac{a+b}{2}(b - a)$$

which is the area of the trapezoid under this line $y = x$: the difference of the areas of two right triangular regions, or the width of the trapezoid times its average height. (Case $a = 0$ is the area $1/2 \cdot b \cdot b$ of a right triangle of width b, height b.) □

Next, one that requires a far less obvious anti-derivative:

Checkpoint 5.3.3

1. Verify that $f(x) = \sqrt{1 - x^2}$ has antiderivative $F(x) = \dfrac{x\sqrt{1 - x^2} + \arcsin(x)}{2}$.
2. Sketch a graph of $y = \sqrt{1 - x^2}$ on interval $[-1, 1]$.
3. Use this graph to explain why the value of $2\int_{-1}^{1} \sqrt{1 - x^2}dx$ should be π.
4. Verify this by evaluating this integral, using FTC, Equation (5.3.1).

The moral here is that we would like to know how to find many more anti-derivatives, like the one seen above.

Getting Antiderivatives from Definite Integrals. To understand why the above result is true, we do something a bit more ambitious: using the definite integral over intervals of variable width $[a, x]$ for x between a and b, so that the value of the definite integral depends on the choice of x. This gives a function of x, which turns out to be an antiderivative of f.

That is, we define a function $g(x)$, $a \le x \le b$ by

$$g(x) = \int_a^x f(t)\,dt$$

Let us compute the derivative of g, using the definition

$$g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h}.$$

The numerator in the difference quotient is

so

$$\frac{g(x + h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

Intuitively, the value of $f(t)$ over the small interval is close to $f(x)$, so the area given by the interval is close to that of a rectangle of height $f(x)$, width h. That is, the integral here is approximately $f(x)h$, so that the difference quotient is approximately $f(x)$, and then the limit gives

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt = \lim_{h\to 0} \frac{1}{h} f(x)h = f(x).$$

This can be shown more carefully using the Extreme Value Theorem and the comparison properties of definite integrals.

So as claimed, $g(x) = \displaystyle\int_a^x f(t)\,dt$ is an antiderivative of f:

$$\frac{d}{dx} \int_a^x f(t)\,dt = f(x).$$

Getting Definite Integrals from Antiderivatives. If F is any antiderivative of f on interval $[a, b]$, it differs from the above antiderivative g only by an added constant, so $F(x) = \displaystyle\int_a^x f(t)\,dt + C$. Thus

The last is true because the variable name used, t or x, has no effect on the value of a definite integral, which is a number, not a function of x or of t.For F any antiderivative of f on interval $[a, b]$,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The difference here arises so often that it is useful to have a short-hand for it:

$$\Big[F(x)\Big]_a^b = F(b) - F(a), \quad = \begin{array}{r} F(b) \\ -F(a) \end{array}$$

I sometimes use the "vertical" form at right above to keep straight which term is added and which subtracted.

Integration and differentiation as inverse processes. The two parts of the Fundamental Theorem of Calculus can be summarized by the idea that integration and differentiation are like inverses:

- Computing the integral of a function f [to upper limit x] and then differentiating the result gets you back to where you started: function f.
- Differentiating a function F (getting $f = F'$) and then integrating over an interval $[a, x]$ the result gets you back to where you started: function F (up to adding a constant.)

Exercises

Study Calculus Volume 1, Section 5.3[2]; in particular Theorems 4 and 5, Examples 17, 18, 20 and 21; Checkpoints 16, 17 and 19; and Exercises 149, 153, 155, 157, 161, 171, 177, 179, 183, 190, 191 and 195.

For further practice, look at several exercises from each of the following ranges: 148–159, 160–163, 170–189, 190–193, and 194–197.

[2] `openstax.org/books/calculus-volume-1/pages/5-3-the-fundamental-theorem-of-calculus`

5.4 Integration Formulas and the Net Change Theorem

References.

- OpenStax Calculus Volume 1, Section 5.4[1]
- *Calculus, Early Transcendentals* by Stewart, Section 5.4.

Now that we have seen the connection between antiderivatives and definite integrals, it is convenient to recast antiderivatives in terms of integrals, and use the notation of integrals when calculating with antiderivatives. Thus, just as $\int_a^x f(t)\,dt$ gives one antiderivative of f (a different one for each different choice of a) we denote the general antiderivative by dropping the specific choice a, and simplifying a bit:

Definition 5.4.1 The **Indefinite Integral of f with respect to x** is the most general function $F(x)$ having $F'(x) = f(x)$, including an arbitrary added constant. This is denoted

$$\int f(x)\,dx$$

The function inside this expression is called the **integrand**.

The differential "dx" is essential! For example, we can verify that

$$\int x\,t\,d\mathbf{x} = x^2 t/2 + C \qquad \text{while} \qquad \int x\,t\,d\mathbf{t} = xt^2/2 + C.$$

$\diamondsuit$

Example 5.4.2

$$\int x^2\,dx = \frac{x^3}{3} + C$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \sec^2 x\,dx = \tan x + C$$
$$\int \tan x\,dx = \log|\sec x| + C$$
$$\int \ln x\,dx = x\ln x - x + C$$
$$\int 2xe^{x^2}dx = e^{x^2} + C$$

$\square$

How do we know that these are correct?

Differentiate the formula at right and verify that this gives the integrand, the function inside the integral expression at left.

Checkpoint 5.4.3

Rather than list numerous sums and constant multiples, the list can start with the two general "combining" rules that we have for *sums*, *differences* and *constant multiples* of antiderivatives, rephrased as facts about integrals:

$$\int cf(x)\,dx = c\int f(x)\,dx \quad \text{for any constant c.}$$

[1] `openstax.org/books/calculus-volume-1/pages/5-4-integration-formulas-and-the-net-change-theorem`

$$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

But beware: we have ***no rules*** for *products* or *quotients* or *compositions* of functions.

Simplify first! As usual, it often helps to simplify the function as much as possible before looking for antiderivatives, both by using the above rules to break up sums and differences and extract constant factors and by using other algebraic rules and trigonometric facts.

Connection to Definite Integrals. The Fundamental Theorem of Calculus gives

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

and we denote the difference here with the shorthand forms

$$\Big[F(x)\Big]_a^b = F(x)\Big|_a^b = F(b) - F(a)$$

so we can now use the indefinite integral notation for the antiderivative:

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b = \int f(x)\,dx\Big|_a^b$$

(I prefer always using matching left and right brackets to avoid any possible ambiguity, but some texts use the right bracket only.)

Integrals of Derivatives and the Net Change Theorem. The indefinite integral a function is the *general* antiderivative, so the indefinite integral of the derivative f' of function f is the general antiderivative of the derivative. The original function f itself is one such antiderivative, so all that remans is to add an arbitrary constant:

$$\int_a^b f'(x)\,dx = f(b) - f(a), = \Big[f(x)\Big]_a^b$$

This says that the definite integral of the rate of change of a quantity gives the net change in the quantity.

For $f(t)$ a Velocity, Displacement is Net Change. For example, if function f gives position and the independent variable is time, the rate of change is velocity, $v(t) = f'(t)$, so the definite integral of velocity from a to b is the *net* change in position between times a and b, the *displacement*, not the *total* distance traveled:

The *displacement* between times a and b is $\displaystyle\int_a^b v(t)\,dt = \int_a^b f'(t)\,dt = f(b) - f(a)$.

This is what we saw in Problem 2 of Section 5.1, p. 72, motivating the idea of the definite integral. Geometrically, this is the difference between the area under the positive part of the graph of $v = f'$ and the area below the negative part.

Total Distance Traveled is Total Change. On the other hand, the total distance traveled is the "total change" of position, given by integrating the rate of change of position without regard to direction: this is *speed*, which is the magnitude of the velocity, $|v(t)|$.

The *total distance traveled* between times a and b is $\displaystyle\int_a^b |v(t)|\,dt = \int_a^b |f'(t)|\,dt$.

Geometrically, this is the total area between the graph of v and the t-axis, adding area above and below the axis. It is not given by the simple formula $f(b) - f(a)$, so how can it be computed?

The answer is to break the integral up into several integrals over several intervals such that on each interval, $v = f'$ is either positive throughout or negative throughout. Then each integral is of the form either $\int_c^d f'(t)\,dt$ or $\int_c^d -f'(t)\,dt$, and so each can be evaluated easily by the Net Change Theorem, as either $f(d) - f(c)$ or $f(c) - f(d)$. Adding these positive pieces gives the total distance traveled.

Exercises

Study Calculus Volume 1, Section 5.4[2]; in particular Theorem 6, Examples 23–26, 28 and 29; Checkpoints 21, 22 and 24; and Exercises 207, 209, 211 and 223.

For further practice, look at several exercises from each of the following ranges: 207-212 and 223-228.

5.5 Substitution

References.

- OpenStax Calculus Volume 1, Section 5.5[1]
- *Calculus, Early Transcendentals* by Stewart, Section 5.5.

So far we are rather limited in our ability to calculate antiderivatives and integrals because, unlike with derivatives, knowing indefinite integrals for two functions does not in general allow us to calculate the indefinite integral of their product, quotient, or composition. However, we can find a rule that will help up with ***some*** products and compositions, using the same strategy that lead us to our first few antiderivatives: take a fact about derivatives and "invert" it.

Surprisingly, it is the Chain Rule that is most useful, because the derivative of a composition is a certain product, and thus running it backwards gives an antiderivative for that product: ***almost*** a product rule for indefinite integrals.

Getting some integrals involving products, quotients and compositions. To get an idea of how the Substitution Rule will work, let us first get a few examples of integrals of products by working backwards from some derivatives.

$$\frac{d}{dx}(\sin x)^3 = 3(\sin x)^2 \frac{d}{dx}(\sin x) = 3\sin^2 x \cos x \text{ , so dividing by } 3,$$

$$\int \sin^2 x \cos x\,dx = \frac{1}{3}(\sin x)^3 + C \tag{5.5.1}$$

$$\frac{d}{dx}\ln(\cos x) = \frac{1}{\cos x}\frac{d}{dx}(\cos x) = -\frac{\sin x}{\cos x} \text{ , so}$$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\ln(\cos x) + C \tag{5.5.2}$$

Reversing a Chain Rule Calculation. Let us look at the first calculation above in reverse order. Factor $\cos x$ has antiderivative $\sin x$, which also appears in the other factor, so

$$\sin^2 x \cos x = (\sin x)^2 \frac{d}{dx}(\sin x)$$

[2] openstax.org/books/calculus-volume-1/pages/5-4-integration-formulas-and-the-net-change-theorem
[1] openstax.org/books/calculus-volume-1/pages/5-5-substitution

Using the name u for this repeated term $\sin x$, this is $u^2 \dfrac{du}{dx}$.

Since u^2 has antiderivative $u^3/3$,

$$u^2 \frac{du}{dx} = \frac{d}{du}\left(\frac{u^3}{3}\right)\frac{du}{dx}.$$

This is what the Chain Rule gives for

$$\frac{d}{dx}\left(\frac{u^3}{3}\right), = \frac{d}{dx}\frac{(\sin x)^3}{3}.$$

Thus $\sin^2 x \cos x$ has antiderivative $\dfrac{(\sin x)^3}{3}$.

Reversing a Chain Rule Calculation. In this calculation, the antiderivatives for $\cos x$ and u^2 have been combined to get this new antiderivative. In terms of indefinite integrals, we have used the two simple indefinite integrals

$$\int \cos x \, dx = \sin x + C$$

and

$$\int u^2 du = \frac{u^3}{3} + C$$

plus the Chain Rule $\dfrac{d}{dx} f(u) = f'(u)\dfrac{du}{dx}$ to get

$$\int \sin^2 x \cos x \, dx = \frac{\sin^3 x}{3} + C.$$

The Substitution Rule. Suppose that we seek the (indefinite) integral of a function of the special "composition-product-derivative" form $f(g(x))g'(x)$, and we know an antiderivative F for f. Then the Chain Rule gives

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x) = f(g(x))g'(x)$$

so $F(g(x))$ is an antiderivative of this function, and

$$\int f(g(x))g'(x)\, dx = F(g(x)) + C.$$

On the other hand, if we define $u = g(x)$ then $g'(x) = \dfrac{du}{dx}$, $F(g(x)) = F(u)$, and $\int f(u)\, du = F(u) + C$. Combining these results

$$\int f(g(x))g'(x)\, dx = \int f(u)\frac{du}{dx}\, dx = \int f(u)\, du = F(u) + C. \tag{5.5.3}$$

In practice, the emphasis is on choosing the new quantity u and changing to it as the variable, which involves getting a differential du in the integral formula in place of dx. To be precise,

Theorem 5.5.1 Substitution Rule. *If $u = g(x)$ is differentiable with range covering some interval I, and function f is continuous on that interval I, then*

$$\int f(g(x))g'(x)\, dx = \int f(u)\, du$$

That is, one can effectively make a substitution with the differential formula $du = \dfrac{du}{dx}dx$ inside integral formulas, and this helps so long as the rest of the formula can also be expressed entirely in terms of the new variable u.

Note well: for this substitution method to be useful, one must completely convert to u from x, not have a mix of both variables in the transformed integral.

Choosing u. The key in practice is finding a suitable choice of u, and there may be more than one worth trying.

One strategy is to use the quantity inside a composition as u, since u is the "inside" part of the Chain Rule.

This composition should then multiply some other factor containing just the derivative of u.

Another strategy is that to seek some quantity such that both its and its derivative appear in the integrand,

with the derivative simply multiplying the rest of the integrand.

Checkpoint 5.5.2 Find $\int x^3 \cos(x^4+2)\,dx$.

Start by thinking of options for u.

Checkpoint 5.5.3 Find $\int \cos x \sin x\,dx$.

Start by seeking several possible options for u.

Checkpoint 5.5.4 Find $\int \sqrt{2x+1}\,dx$.

Here there is no product, just a composition, so what do we do about the need for factor du/dx?

It still helps to try the inside of a composition as u.

Checkpoint 5.5.5 Find $\int \frac{x}{\sqrt{1-4x^2}}dx$.

Checkpoint 5.5.6 $\int e^{3x}\,dx$

Checkpoint 5.5.7 $\int \sqrt{1+x^2}\,x^5\,dx$

Checkpoint 5.5.8 $\int \tan x\,dx$

Substitution in Definite Integrals. Often the easiest way to deal with definite integrals is to first seek an indefinite integral, and then use the FTC. However with substitution, this involves the step of converting back from a function of new variable u to the original variable x, and it may be easier to avoid that by converting everything to the new variable, including the limits of integration.

Theorem 5.5.9 The Substitution Rule for Definite Integrals. *If $u = g(x)$ is differentiable with range covering some interval I, g' is continuous, and function f is continuous on that interval I, then*

$$\int_{x=a}^{x=b} f(g(x))g'(x)\,dx = \int_{u=c}^{x=d} f(u)\,du, \quad \textit{with } c = g(a), d = g(b). \tag{5.5.4}$$

That is, for F an antiderivative of f,

$$\int_{x=a}^{x=b} f(g(x))g'(x)\,dx = \Big[F(u)\Big]_{u=g(a)}^{u=g(b)}. \tag{5.5.5}$$

I write the integral limits as "$x = a$", "$u = b$" and so on to emphasize that u must *completely* displace x, in *three* places:

- in the formula for the integrand, $f(u)$ replaces $f(g(x))$;
- in the differential, du replaces $\frac{du}{dx}dx = g'(x)dx$; and

- in the limits of integration, $c = g(a)$ and $d = g(b)$ replace a and b.

Checkpoint 5.5.10 $\displaystyle\int_0^4 \sqrt{2x+1}\,dx$

Checkpoint 5.5.11 $\displaystyle\int_1^2 \frac{1}{(3-5x)^2}\,dx, = \int_1^2 \frac{dx}{(3-5x)^2}$

Checkpoint 5.5.12 Evaluate $\displaystyle\int_1^e \frac{\ln x}{x} dx$.

Try it both ways: using the above formula , and by first finding the indefinite integral as a function of x and then using the FTC.

Short-cuts From Symmetry. For even and odd functions integrated over a symmetric interval $[-a, a]$, the intervals simplify:

- (a) If $f(x)$ is odd $\big[f(-x) = -f(x)\big]$ then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.
- (b) If $f(x)$ is even $\big[f(-x) = f(x)\big]$ then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

Checkpoint 5.5.13 Evaluate $\displaystyle\int_{-2}^{2} x^6 + 1\ dx$.

Checkpoint 5.5.14 Evaluate $\displaystyle\int_{-1}^{1} \frac{\tan x}{1+\sec^2 x} dx$.

Exercises

Study Calculus Volume 1, Section 5.5[2]; in particular Theorem 7, the *Problem Solving Strategy* that follows it, Examples 30–33 (and maybe 34 and 35), Checkpoints 25–28, (and maybe 29 and 30), and one or several exercises from each of the following ranges: 256–260, 261–270, 271–287 and 292–297; Some suggested selections are Exercises 257, 261, 265, 271, 275, 281, 293, 297.

As noted above, for definite integrals one can either do it as described there (Theorem 8, Examples 34 and 35, Checkpoints 29 and 30) or (a) first get the indefinite integral $\int f(x)dx = F(x) + C$ using substitution and then (b) use FTC: $\int_a^b f(x)dx = F(b) - F(a)$.

5.6 Integrals Involving Exponential and Logarithmic Functions — Summary

References.

- OpenStax Calculus Volume 1, Section 5.6[1]

This section of the OpenStax text just introduces a couple useful new indefinite integrals, and then gives some example and practicee of using them in combination with substitutions; these notes just provide a brief study guide to that.

The main new integrals here are:

$$\int e^x\ dx = e^x + C$$

[2] openstax.org/books/calculus-volume-1/pages/5-5-substitution
[1] openstax.org/books/calculus-volume-1/pages/5-6-integrals-involving-exponential-and-logarithmic-fu

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \text{ and}$$
$$\int \ln x\,dx = x\ln(x) - x + C = (x-1)\ln x$$

along with

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

already seen.

Exercises

Study Calculus Volume 1, Section 5.6[2]; in particular Examples 37, 38, 39, 41, 44, 45, 47, 48, the Checkpoints that immediately follow each of those Examples, and a few Exercises from each of the ranges 320–325, 328–339, and 355–357.

5.7 Integrals Resulting in Inverse Trigonometric Functions — Summary

References.

- OpenStax Calculus Volume 1, Section 5.7[1]

As with Section 5.6, p. 85 , this section of the OpenStax text just introduces a few useful indefinite integrals, and then gives some example and practice with using them in combination with substitutions; often simple ones of the form $u = ax$; these notes just provide a brief guide to that.

The two very useful integrals here are

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin(x/a) + C, \quad a > 0, \text{and}$$
$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\arctan(x/a) + C$$

The third one

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}(|x/a|) + C, \quad a > 0$$

is also occasionally useful, but less often.

Exercises

Study Calculus Volume 1, Section 5.7[2]. All Examples and Checkpoint items are worth looking at; Then do a few Exercises from each of the ranges 391–394, 397–400, and 411–414.

[2] openstax.org/books/calculus-volume-1/pages/5-6-integrals-involving-exponential-and-logarithmic-fu
[1] openstax.org/books/calculus-volume-1/pages/5-7-integrals-resulting-in-inverse-trigonometric-funct
[2] openstax.org/books/calculus-volume-1/pages/5-7-integrals-resulting-in-inverse-trigonometric-funct

Appendix A

Study Guide

(Updated December 19, 2023)

Some suggested strategies when studying for tests and the final exam:

1. Review previous quizzes and tests, rework questions that you did not get completely correct, and use this to identify topics and sections where you most need to review.
2. Cycle through the sections covered repeatedly, rather than trying to master everything in a section before moving on.
3. When you start studying a section, first read through the class notes and corresponding part of the OpenStax online text.
4. Then attempt each of the indicated examples on paper, initially not looking at the solutions provided.
5. Once you have worked an example, check the text's solutions.
6. Once you feel that you understand an example, try the Checkpoint item that immediately follows, if there is one.
7. With the recommended Exercises, try a few at a time— if there are ranges of Exercises indicated, initially attempt at most one from each range; return to others when you cycle back to the section (see Item 2, p. 87).

Chapter 2, *Limits*.

- Calculus Volume 1, Section 2.1[1], In particular Example 1, as usual the Checkpoint immediately following that (also 1 in this case), and Exercises 4, 5, 6, 16 and 17.
- Calculus Volume 1, Section 2.2[2], In particular Examples 4, 5, 7, 8, 9, 10 and 11, as usual the Checkpoints immediately following those (4, 6, 7, 8, 9 and 10 in this case), and Exercises 30, 31, 35, 36, 37, 46–49, 77 and 79.
- Calculus Volume 1, Section 2.3[3], All Examples and Checkpoints are worth studying, and Exercises 83, 85, 89, 91, 93, 97, 107, 111, 119, 121, 127, and 128.
- Calculus Volume 1, Section 2.4[4], All Examples and Checkpoints are worth studying, and Exercises 133, 137, 141, 147, 150, 151, 154, 157, 163, and 165.
- Calculus Volume 1, Section 2.5[5], In particular Examples 39, 41, 43 and 44; the Checkpoints immediately following those (28 and 30), and Exercises 177, 184, 185, 187, and 191.

[1] openstax.org/books/calculus-volume-1/pages/2-1-a-preview-of-calculus
[2] openstax.org/books/calculus-volume-1/pages/2-2-the-limit-of-a-function
[3] openstax.org/books/calculus-volume-1/pages/2-3-the-limit-laws
[4] openstax.org/books/calculus-volume-1/pages/2-4-continuity
[5] openstax.org/books/calculus-volume-1/pages/2-5-the-precise-definition-of-a-limit

Chapter 3, *Derivatives*.

- Calculus Volume 1, Section 3.1[6]; in particular Examples 1,2, 3, 5, 6 and 9, Checkpoint items 1, 3 and 4, and Exercises 1, 7, 11, 13, 15, 25, 37, 39, 41 and 51.
- Calculus Volume 1, Section 3.2[7]: all Examples and Checkpoint items are worth reviewing, and Exercises 55, 57, 65, 67, 79, 80 and 96.
- Calculus Volume 1, Section 3.3[8]: all Examples, Checkpoint items 12 to 19 and Exercises 107, 109, 111, 119, 122, 127, 129, 130, 131, 133, 142, 143 and 147.
- Calculus Volume 1, Section 3.4[9]: Examples 34 to 36, Checkpoint item 22, and Exercises 151, 159 and 165.
- Calculus Volume 1, Section 3.5[10]; in particular Examples 39–44, Checkpoint items 25–30, and Exercises 175, 178, 181, 182, 191, 197 and 206.
- Calculus Volume 1, Section 3.6[11] Examples 48, 48, 50, 52 and 53, all Checkpoint items, and Exercises 215, 217, 219, 221, 224, 229, 233, 235, 245, 251 and 257.
- Calculus Volume 1, Section 3.7[12]; Examples 61–67, Checkpoint items 43–46, and Exercises 265, 267, 269, 271, 279, and 291.

  Hint for Exercise 279. One approach is to use the "equation solving" strategy of making the inverse function disappear: solve for $\sin(y) = x^2$ and then differentiate each side of that equation.
- Calculus Volume 1, Section 3.8[13] Examples 68, 69, 71 and 72, both Checkpoint items, and Exercises 301, 303, 305, 307, 311, 316, 325, and 329.
- Calculus Volume 1, Section 3.9[14] Examples 74, 75, 77, 78, 81 and 82, Checkpoint 54, and Exercises 333, 339, 347, 351 and 353.

  We in particular emphasize the last topic of *Logarithmic Differentiation*, using the strategy of simplifying functions of the form $\log(\dots)$ using the laws of logarithms like $\log(ab) = \log(a) + \log(b)$.

Chapter 4, *Applications of Derivatives*.

- Calculus Volume 1, Section 4.1[15]: all Examples and Checkpoints and Exercises 1, 3, 5, 7, 9, 17, and 25.
- Calculus Volume 1, Section 4.2[16] all Examples and Checkpoints and a few Exercises from each of the ranges 50–55, 62–67, 68–71, 72–77, 78–83, 84–86; for example, Exercises 49, 51, 52, 57, 69, 73, 79 and 84.
- Calculus Volume 1, Section 4.3[17]: in particular the *Problem Solving Strategy*, both Examples and Checkpoints, and a few Exercises from each of the ranges 91–98, 100–103, 104–107, 108–117, 118–128 and 129–134. (Some suggested selections are Exercises 91, 93, 97, 101, 107, 109, 119 and 129.)

[6] openstax.org/books/calculus-volume-1/pages/3-1-defining-the-derivative
[7] openstax.org/books/calculus-volume-1/pages/3-2-the-derivative-as-a-function
[8] openstax.org/books/calculus-volume-1/pages/3-3-differentiation-rules
[9] openstax.org/books/calculus-volume-1/pages/3-4-derivatives-as-rates-of-change
[10] openstax.org/books/calculus-volume-1/pages/3-5-derivatives-of-trigonometric-functions
[11] openstax.org/books/calculus-volume-1/pages/3-6-the-chain-rule
[12] openstax.org/books/calculus-volume-1/pages/3-7-derivatives-of-inverse-functions
[13] openstax.org/books/calculus-volume-1/pages/3-8-implicit-differentiation
[14] openstax.org/books/calculus-volume-1/pages/3-9-derivatives-of-exponential-and-logarithmic-functio
[15] openstax.org/books/calculus-volume-1/4-1-related-rates
[16] openstax.org/books/calculus-volume-1/pages/4-2-linear-approximations-and-differentials
[17] openstax.org/books/calculus-volume-1/pages/4-3-maxima-and-minima

- Calculus Volume 1, Section 4.4[18]. Pay particular attention the *Corollaries of the Mean Value Theorem* in the second half: Theorems 6, 7 and 8: these will be extremely useful for applications later in this chapter.

  Study Examples 14 and 15, Checkpoint 14, and a selection from Exercises 148–150, 152–156, 161–166, 167–169, 182–184, and 190–193.

  Here I group the exercises in ranges by "question type", so start by trying one or two from each of the six ranges. For example, some suggested selections are Exercises 149, 153, 161, 169, 182 and 192.
- Calculus Volume 1, Section 4.5[19]: the *Problem Solving Strategy*, the *First Derivative Test*, the *Second Derivative Test* all Examples and Checkpoints, and a selection from Exercises 194–200, 201–205, 206–210, 211–215, 216–220, 221–223 and 224–230.

  Some suggested selections are Exercises 199, 201, 203, 213, 215, 217, 223, 225, 229.
- Calculus Volume 1, Section 4.6[20] Examples 21–26, 28, 29 and 31, Checkpoints 20, 23–25, 27, 28 and 30, (We omit *oblique asymptotes*, so skip Example 30 and Checkpoint 29) and a selection from Exercises 251–255, 256–260, 261–270, 271–281, 285–288 and 294–298.

  Here the exercises are grouped in ranges by "question type", so start by trying one or two from each of the seven ranges; some suggested selections are Exercises 251, 256, 257, 259, 261, 263, 265, 267, 271, 279, 281, 285, 306 and 307.
- Calculus Volume 1, Section 4.7[21] Examples 33–35 and 37, Checkpoints 31–34 and 36, and a selection from Exercises 311–314, 315–318, 319–321, 322–326, 335–336 and 351–355.

  Here the exercises are grouped in ranges by "question type", so start by trying one or two from each of the seven ranges; some suggested selections are Exercises 311, 316, 320, 322, 335 and 353.
- Calculus Volume 1, Section 4.8[22] Examples 38–41, 43 and 44, Checkpoints 37–40, 42 and 43, and a selection from Exercises 356–361, 362–366, 367–385, 387–389 and 391–395.

  Here the exercises are grouped in ranges by "question type", so start by trying one or two from each of the ranges; some suggested selections are Exercises 357, 359, 363, 367, 371, 377, 379, 387, and 393.
- Calculus Volume 1, Section 4.10[23] all the Examples and Checkpoints and a selection from Exercises 4465–469, 470–473, 474–489, 490–498, 499–503 and 504–508.

  Here the exercises are grouped in ranges by "question type", so start by trying several from each of the ranges; some suggested selections are Exercises 465, 467, 469, 471, 477, 487, 491, 493, 499, 501 and 505.

  Hint: It often helps to simplify the function first, and then use the list of derivatives and indefinite integrals in the online test.

Chapter 5, *Integrals*.

- Calculus Volume 1, Section 5.1[24] If you are unfamiliar with the Σ notation for sums, the first part of that section should help. Study Example 4, Checkpoint 4, and Exercises 15, 19, 23, 27, 29, and 43.
- Calculus Volume 1, Section 5.2[25] Examples 8–13, Checkpoints–12, and Exercises 61, 65, 73, 75, 79, 81, 89, 91, 93, 99, 101 and 107.

[18] openstax.org/books/calculus-volume-1/pages/4-4-the-mean-value-theorem
[19] openstax.org/books/calculus-volume-1/pages/4-5-derivatives-and-the-shape-of-a-graph
[20] openstax.org/books/calculus-volume-1/pages/4-6-limits-at-infinity-and-asymptotes
[21] openstax.org/books/calculus-volume-1/pages/4-7-applied-optimization-problems
[22] openstax.org/books/calculus-volume-1/pages/4-8-lhopitals-rule
[23] openstax.org/books/calculus-volume-1/pages/4-10-antiderivatives
[24] openstax.org/books/calculus-volume-1/pages/5-1-approximating-areas
[25] openstax.org/books/calculus-volume-1/pages/5-2-the-definite-integral

- Calculus Volume 1, Section 5.3[26] Theorems 4 and 5, Examples 17, 18, 20 and 21; Checkpoints 16, 17 and 19; and Exercises 147, 149, 153, 155, 157, 161, 171, 177, 179, 183, 190, 191 and 195.
- Calculus Volume 1, Section 5.4[27] Theorem 6, Examples 23–26, 28 and 29, Checkpoints 21, 22 and 24, and Exercises 207, 209, 211 and 223.
- Calculus Volume 1, Section 5.5[28]: Theorem 7, the *Problem Solving Strategy* that follows it, Examples 30–33 (and maybe 34 and 35), Checkpoints 25–28, (and maybe 29 and 30), and one or several exercises from each of the following ranges: 256–260, 261–270, 271–287 and 292–297; Some suggested selections are Exercises 257, 261, 265, 271, 275, 281, 293, 297.

  Note that, for definite integrals one can either do it as described there (Theorem 8, Examples 34 and 35, Checkpoints 29 and 30) or (a) first get the indefinite integral $\int f(x)dx = F(x) + C$ using substitution and then (b) use FTC: $\int_a^b f(x)dx = F(b) - F(a)$.
- Calculus Volume 1, Section 5.6[29] Examples 37, 38, 39, 41, 44, 45, 47, 48, the Checkpoints that immediately follow each of those, and a few Exercises from each of the ranges 320–325, 328–339 and 355–357.
- Calculus Volume 1, Section 5.7[30] All Examples and Checkpoint items are worth looking at; and study a few Exercises from each of the ranges 391–394, 397–400, and 411–414.

[26] openstax.org/books/calculus-volume-1/pages/5-3-the-fundamental-theorem-of-calculus
[27] openstax.org/books/calculus-volume-1/pages/5-4-integration-formulas-and-the-net-change-theorem
[28] openstax.org/books/calculus-volume-1/pages/5-5-substitution
[29] openstax.org/books/calculus-volume-1/pages/5-6-integrals-involving-exponential-and-logarithmic-fu
[30] openstax.org/books/calculus-volume-1/pages/5-7-integrals-resulting-in-inverse-trigonometric-funct

Appendix B

Some Tables

- OpenStax Calculus Volume 1, Appendix B: Table of Derivatives.[1]The relevant ones are up to *Exponential and Logarithmic Functions*; you may ignore the ones towards the end involving hyperbolic functions.
- OpenStax Calculus Volume 1, Appendix A: Table of Integrals.[2]Only a few of these are are relevant; up to number 16. (The rest are for use in Calculus 2, MATH 220). Use this list to test yourself on those basic ones.

[1] openstax.org/books/calculus-volume-1/pages/b-table-of-derivatives
[2] openstax.org/books/calculus-volume-1/pages/a-table-of-integrals