

3.10 Linear Approximations and Differentials

When we first introduced the tangent line to a curve at a point, one characteristic was that this appears to be the line that is closest to the curve itself when one looks near that point: zooming in towards that point, the curve looks more and more like that tangent line.

This means that the tangent line can be useful as an approximation of the curve, and thus can be used to approximate the value of a function from its value at one point plus its derivative at that point.

The tangent line to $y = f(x)$ at $x = a$.

At the point $P(a, f(a))$ on the curve $y = f(x)$, the tangent line has slope $m = f'(a)$, and is

$$y = f(a) + f'(a)(x - a) \quad (1)$$

Since this line is close to the curve at least when x is close to a , this gives

$$f(x) \approx f(a) + f'(a)(x - a) \quad (2)$$

as the **linear approximation** or **tangent line approximation** of f at a . The linear function occurring here,

$$L(x) = f(a) + f'(a)(x - a) \quad (3)$$

is called the **linearization** of f at a . Note: $f(a)$ and $f'(a)$ are just numbers!

Example (A). At 10am, a car has traveled 200 miles since the start of the day and its speed is 55mph. Use the linearization of the function giving distance traveled in terms of time to estimate the total distance traveled by a slightly later time like 10:15am.

With $s = f(t)$ where t is time (in hours since midnight) and s is distance traveled in miles, we know that $f(10) = 200$ and $f'(10) = 55$. Thus the linearization at $t = 10$ is

$$L(t) = 200 + 55(t - 10)$$

For example, time 10:15 is $t = 10.25$, so the linear approximation of total distance traveled by then is

$$L(10.25) = 200 + 55(10.25 - 10) = 213.75 \text{ miles.}$$

Linearization: assuming constant rate of change

This procedure should be familiar: it is approximation by assuming that the speed stays constant at the given value of 55, or at least close to that speed.

More generally, linear approximation is the assumption that the rate of change is constant, or at least does not change much when the independent variable is changed by only small amount. This is the same idea as used when a population's size in the near future is estimated using the population's current size and rate of growth.

Linearization to evaluate functions approximately

Sometimes it is easy to find out about a function at one value of its argument, and we can use this information to approximate it at other arguments, where the exact calculation is harder.

Example (1 page 248). Find the linearization of $f(x) = \sqrt{x+3}$ at $a = 1$, and use this to approximate $\sqrt{3.98}$ and $\sqrt{4.05}$.

Example (2 page 249, a graphing calculator exercise). For which values of x is the above linear approximation

$$\sqrt{x+3} \approx L(x) = \frac{x}{4} + \frac{7}{4}$$

accurate to within 0.1?

 $\sin x \approx x$ for small angles

It is often useful to have an approximation of $\sin x$ for small angles. For example in optics, angles of only a few degrees are often involved, and then x is smaller than about 0.1 (in radians). Thus complicated optical formulas involving trig. functions and their inverses are accurately approximated by far simpler linear formulas.

This approximation is done with the linearization of $f(x) = \sin x$ at $a = 0$, where $f'(0) = \cos 0 = 1$. Since $\sin 0 = 0$, we get the very simple approximation

$$\sin x \approx L(x) = x.$$

This is one reason why radian measure is convenient!

Differentials: dx , dy , etc.

The Leibniz notation $\frac{dy}{dx}$ came from the intuition that the slope of curve $y = f(x)$ is given by the ratio of a very small difference dx in the value of argument x to the very small difference dy that this causes in the value of $y = f(x)$. These very small differences were called **differentials**, and the subject of derivatives is sometimes called **differential calculus**, meaning the subject of calculating with differentials.

This intuitive, approximate idea can be made more precise using linearization.

For a function $y = f(x)$,

- any change in the independent variable x is denoted dx , a **differential**. (Likewise dt if the independent variable is t , etc.)
- the resulting linear approximation of the change in the y value given by

$$dy = f'(x)dx$$

is called the differential dy . That is, $dy = L(x+dx) - L(x)$, the change in the value of the linearization of f at x .

Note that this also means that the ratio of the differentials is

$$\frac{dy}{dx} = f'(x)$$

so the Leibniz notation is now a genuine fraction!

Example (A revisited). In example A, the differential for the independent variable t is the change in time, $dt = 0.25$, and the resulting differential in the independent variable is

$$ds = s'(10)dt = 55 \cdot 0.25 = 13.57,$$

the estimated change in position.

Example (2 revisited). In example 2 with $f(x) = \sqrt{x+3}$ linearized at 1, the values 3.98 and 4.05 corresponding to differential in x of $dx = -0.02$ and $dx = 0.05$. The differential in the value of the function is

$$dy = \frac{dx}{2\sqrt{1+3}} = \frac{dx}{4},$$

giving $dy = -0.005$ and $dy = 0.0125$ respectively: the estimated changes in the value of the function from its value of 2 when $x = 3$.

Differential vs Difference

For the independent variable, a differential dx is the same thing as the difference denoted Δx : the change in the value of x . However for a dependent variable $y = f(x)$, $\Delta y = f(x + \Delta x) - f(x)$ is the exact change in the value of y , while $dy = f'(x)dx$ is the linear approximation of this change.

Example (3 page 250, modified). For $y = f(x) = x^3 + x^2 - 2x + 1$, compare the values of Δy and dy when x changes

- from 2 to 2.05
- from 1 to 0.99

Estimating the Effects of Measurement Error

One important use of differentials and linearization in experimental science is estimating the effect of error in measuring one quantity on the error in some other quantity computed from the measured value.

Example (4 page 251). The radius of a sphere has been measured and found to be 21 cm to the nearest mm, so the measurement error is at most 0.05 cm. Use differentials to estimate the maximum error in the value of the volume got by using this measured value for the radius.

The exact error is ΔV , where $V = \frac{4}{3}\pi r^3$. We approximate this with the differential $dV = 4\pi r^2 dr$, $= 4\pi(21)^2 dr$. The actual radius can vary from 21 by up to 0.05 in either direction, so all we know is that $-0.05 \leq dr \leq 0.05$, or $|dr| \leq 0.05$. This tells us that $-4\pi(21)^2 0.05 \leq dy \leq 4\pi(21)^2 0.05$ or

$$|dy| = \left| \frac{dV}{dr} \right| \cdot |dr| = 4\pi(21)^2 |dr| \leq 4\pi(21)^2 0.05 \approx 277 \text{ cm}^3.$$

Exact calculation gives $-276.4 \leq \Delta y \leq 277.7$, so this is a good error estimate.

Recommended exercises 3, 5, 9, 13, 15, 29, 33.