MATH 229: Vector Calculus with Chemical Applications

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Revised and expanded
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Some recent changes:

- Expanded the Tables of Integrals in §3.2
- Added $\cos\theta = x/r$ method to §1.1.3

These lecture notes are intended to serve as the content for MATH 229: Vector Calculus with Chemical Applications, a course offered in the Department of Mathematics at the College of Charleston.

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- D. Boucher
- T. Ivey
- K. Krantzman
- R. Lavrich
- K. Owens
- J. Truver

References are made to several other sources for additional reading, using the abbreviations

OSC-2 *Calculus, Volume 2*, the open source text from OpenStax.

OSC-3 Volume 3 of the above.

CET *Calculus, Early Transcendentals*, by James Stewart (eighth edition).

TCMB *The Chemistry Maths Book*, by Erich Steiner (second edition)

FCLA *A First Course in Linear Algebra*, an open source book by Robert Beezer.

Contents

Chapter 1

Coordinate Systems and Vectors

1.1 Real Numbers and Coordinate Systems

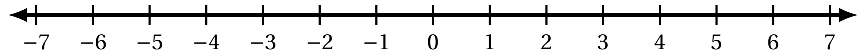
Objectives and Concepts:

- Real numbers are used to represent quantities along a continuum (continuous line). Every point on the real number line represents a distinct real number.
- Cartesian products of real numbers are used to describe space in two and higher dimensions.
- Coordinate systems are used to uniquely determine the position of a point or other object in a spatial region.
- There are several different coordinate systems that can be used to describe objects in two and three-dimensional space: Cartesian, polar/cylindrical, and spherical coordinates.

References: OSC-3 §1.3, §2.7; CET §§12.1, 10.3, 15.7, 15.8, Appendix B; TCMB §1.2 (p. 3), 2.2, 3.5, 10.2, 10.6.

1.1.1 Real Numbers

Definition: A **real number** is a value that represents a point on an infinitely long line called the number line or **real line**, where the points corresponding to integers are equally spaced. We use the symbol $\mathbb{R}$ to represent the set of all real numbers. The set of real numbers contain all rational numbers and irrational numbers. Every real number can be determined by a (possibly infinite) decimal representation.

An example of a number line: 

The real line represents a single dimension in physical space. Given any two points x_1 and x_2, the distance between the two points is $|x_1 - x_2|$.

1.1.2 The Cartesian Product and Multiple Dimensions

Definition: The **Cartesian product** of a set X and a set Y is the set of all **ordered pairs** (a, b) with $a \in X$ and $b \in Y$, i.e.,

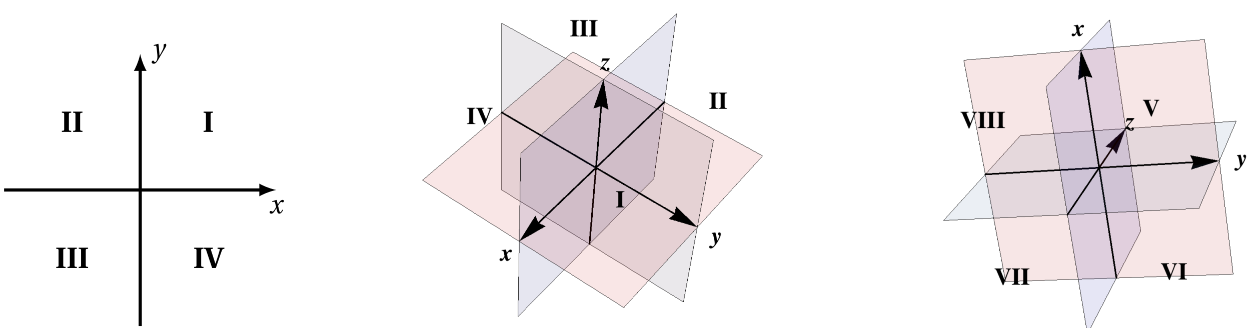
$$X \times Y = \{(a, b) \mid a \in X \text{ and } b \in Y\}.$$

We have that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. The Cartesian product of a set X with itself is often written as X^2. The Cartesian product of the sets $X_1, X_2, \ldots, X_n$ is the set

$$X_1 \times X_2 \times \cdots \times X_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in X_i, 1 \leq i \leq n\}.$$

The elements $(a_1, a_2, \ldots, a_n)$ are called n**-tuples**.

Thus we use $\mathbb{R}^2$ and $\mathbb{R}^3$ to represent the set of all ordered pairs and triples in two and three dimensional physical space, respectively.



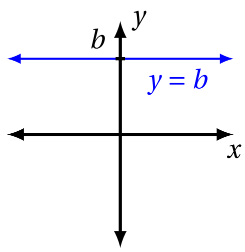
The **coordinate axes** in 2-D and 3-D are shown above. Most commonly the different **coordinates** are labeled with x, y, and z or with x_1, x_2, and x_3. For 3-D, the *right hand rule* is followed when assigning labels to axes: if you curl the fingers of your right hand from the positive x-axis to the positive y-axis, the positive z-axis is in the direction of your thumb. We can generalize the definition of the n-dimensional **Euclidean space** $\mathbb{R}^n$ to any positive integer n, but visualization of objects in 4-D and higher is somewhat challenging.

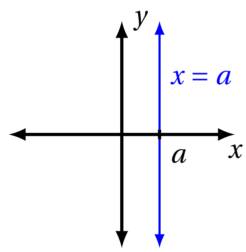
Concept Check: What is true of x, y and z values in the sixth octant?

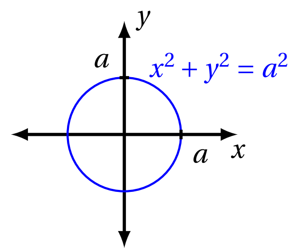
An equation involving one or more of the coordinate variables places a restriction on one of the coordinates - you can think of the xy-plane or xyz-frame as situations without any restrictions on the coordinate variables. On the other extreme, a point (a, b, c) in $\mathbb{R}^3$ is represented by the three equations $x = a$, $y = b$, and $z = c$.

When a variable is absent from an equation, it can take on any value.

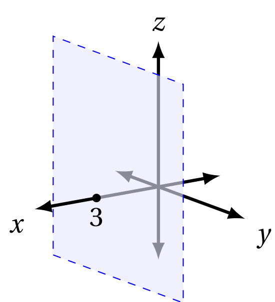
Equations in the xy-plane: The equation $x = a$ is a vertical line (there is no restriction on y) and the equation $y = b$ gives a horizontal line (no restriction on x). We often see equations such as $y = f(x)$ or $x = g(y)$ produce curves in the plane. Curves can also be implicitly defined, such as $x^2 + y^2 = 1$.

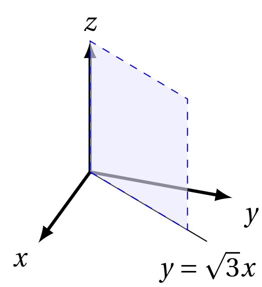


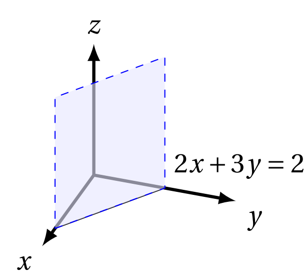




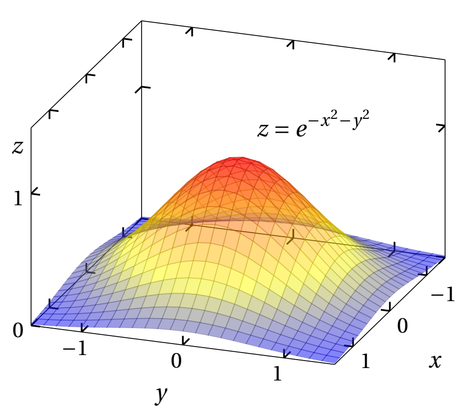
Equations in the xyz-frame: A single equation is used to describe a surface. For example, the equation $x = 3$ restricts x but y and z are free, so that equation represents a plane that is parallel to the yz-plane but contains the point $(3, 0, 0)$.

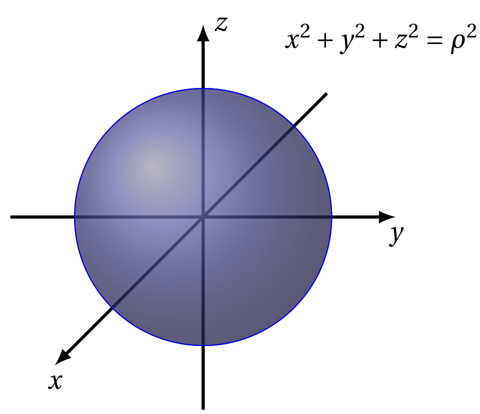






Other equations in x, y, and z can describe surfaces, even implicitly:

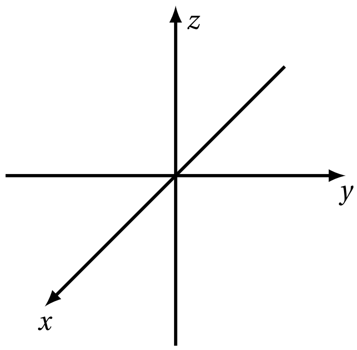




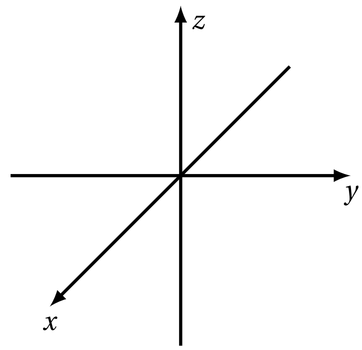
In trying to understand what a surface in three dimensions looks like, we can examine **traces** of the surface in planes parallel to the coordinate planes. A trace is formed by setting one of the variables in the equation to a particular constant and then analyzing the resulting 2-D curve.

Example 1: What surfaces are described by the following equations? Try to sketch them roughly.

$z = y^2$



$z^2 = x^2 + y^2$



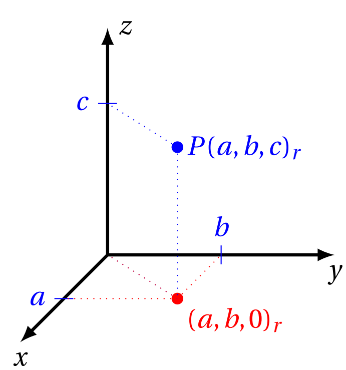
1.1.3 Coordinate Systems

References: OSC-3 §1.3, §2.7; CET §12.1; TCMB §§2.2, 3.5, 10.2, 10.6.

Given a point in space (most often $\mathbb{R}^2$ or $\mathbb{R}^3$) we describe its location using **coordinates**. We will describe three different coordinate systems for $\mathbb{R}^3$ - coordinate systems for $\mathbb{R}^2$ are obtained from these by simply omitting the third coordinate.

Cartesian Coordinates

In Cartesian coordinates, a point $P(a,b,c)$ in $\mathbb{R}^3$ is represented simply by its values along each of the coordinate axes. The Cartesian coordinate system is also called the *rectangular* coordinate system. We will use the subscript r to indicate that a point is written in Cartesian coordinates. The **projection** of the point $P(a,b,c)_r$ onto the xy-plane is the point $(a,b)_r$ in $\mathbb{R}^2$.



The distance between two points $P_1(x_1, y_1, z_1)_r$ and $P_2(x_2, y_2, z_2)_r$ is given by the formula

$$|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

The equation of a sphere with center $(h,k,l)_r$ and radius ρ is

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = \rho^2.$$

Example 2: Determine if the following points lie on a straight line:

$$P(0,-5,5)_r,\quad Q(1,-2,4)_r,\quad R(3,4,2)_r$$

Example 3: Give inequalities that describe the following regions in space:

a) The region between the yz-plane and the plane $x=4$.

b) The cylindrical region above the plane $z=-6$ and below the circle in the xy-plane centered at $(2,3,0)_r$ with radius 1.

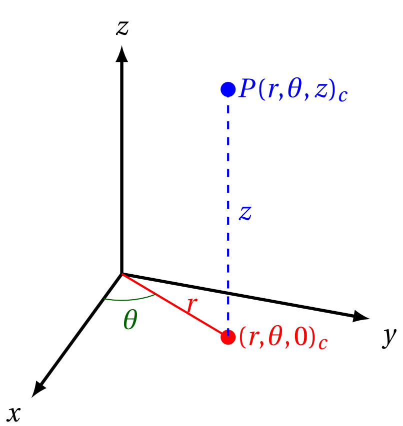
Cylindrical and Polar Coordinates

In many situations, it is physically or mathematically advantageous to describe a point or region in space with a different coordinate system. One such coordinate system for a point $P(x,y,z)$ in $\mathbb{R}^3$ is constructed on the following three pieces of information:

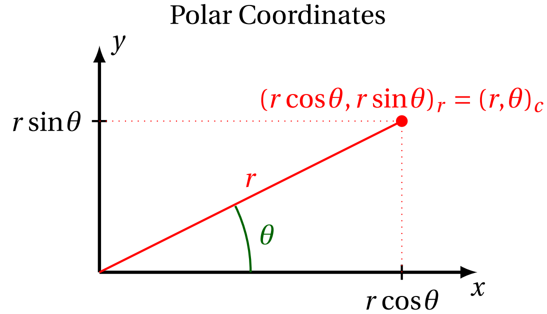
- r: the (signed) distance between the origin and the projection of P into the xy-plane $(x,y)_r$,
- θ: the angle between the positive x-axis and the line segment from the origin to the point $(x,y)_r$, and
- z: the height of the point above the xy-plane (the z in $P(x,y,z)_r$).

These three coordinates form the **cylindrical coordinate system** and a point is represented by the triple $(r,\theta,z)_c$. Cylindrical coordinates are essentially the **polar coordinate system** for $\mathbb{R}^2$ together with a Cartesian coordinate for z. We will use the subscript c to denote cylindrical coordinates (even in 2-D where they are polar coordinates).

Cylindrical Coordinates



Polar Coordinates



$$x=r\cos\theta \qquad y=r\sin\theta \qquad \tan\theta=\frac{y}{x}$$

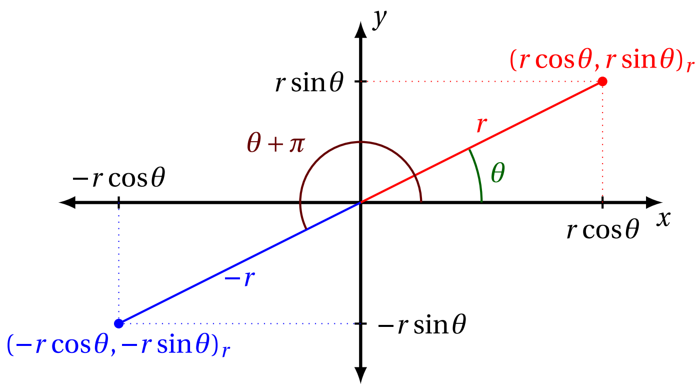
$$x^2+y^2=r^2$$

There are multiple possible values for the angle θ; the two most common choices are $-\pi<\theta\le\pi$ ("smallest magnitude"). and $0\le\theta|2\pi$ ("smallest positive"). When converting from cylindrical coordinates, it is also useful to note that $\cos\theta = x/r$, so that $\theta=\pm\arccos(x/r)$ with the that smallest magnitude option. Then the sign of y determines the sign of θ. Thus a procedure for converting from cartesian coordinates is to first compute r, and then use r and x to compute θ.

NOTE: You are expected to know the values of $\sin\theta$ and $\cos\theta$ where θ is any integer multiple of $\pi/6$ or $\pi/4$. From those you can determine the value of any other trigonometric function. You will also need to know how to evaluate inverse trig functions. See the trigonometric review materials available in OAKS.

Note that r itself is a *signed* distance - it can be positive or negative. A negative r can be interpreted two ways: $|r|$ is the distance from the origin at the angle of $\theta+\pi$, or $|r|$ is the distance from the origin *moving in the opposite direction* as θ. Recall that

$$\sin(\theta+\pi) = -\sin\theta \qquad \cos(\theta+\pi) = -\cos\theta$$



In most contexts the ambiguity about negative r is removed by defining $r = \sqrt{x^2+y^2}$. While θ is meaningful for any real number, is it also common to restrict $0 \le \theta \le 2\pi$. If $r \ge 0$ and $0 \le \theta < 2\pi$, then every point in $\mathbb{R}^3$ other than the origin has a unique set of cylindrical coordinates.

Example 4: Convert the following Cartesian coordinates to cylindrical coordinates (using a positive r):

a) $(-1, \sqrt{3}, \sqrt{3})_r$

b) $(-2, -2, -1)_r$

Example 5: Convert the following cylindrical coordinates to Cartesian coordinates:

a) $(2, \pi/3, -1)_c$

b) $(1, -\pi/4, 4)_c$

Example 6: Describe (in words) the surface in $\mathbb{R}^3$ that is given by the following cylindrical equations:

a) $r = 2$

b) $\theta = \pi/6$

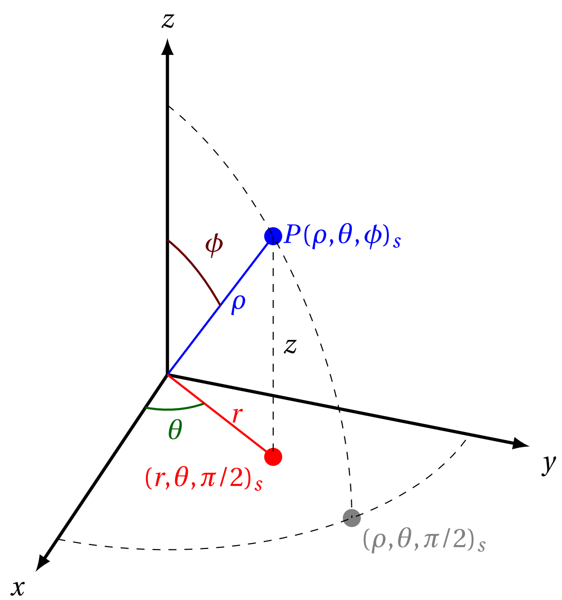
c) $z = 4 - r^2$

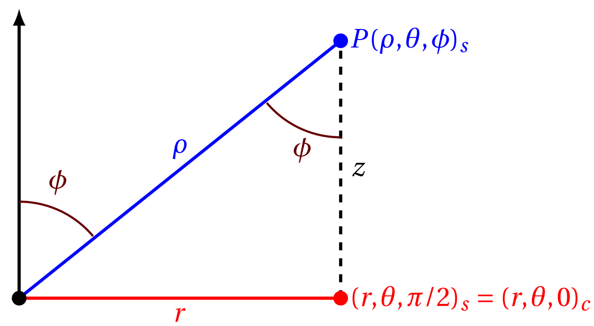
Spherical Coordinates

Spherical coordinates are much like polar coordinates for the xy-plane combined with polar coordinates for the θz-plane. The three components of the system are:

- ρ: the distance between the origin and P,
- θ: the angle between the positive x-axis and the line segment from the origin to the point $(x, y, 0)_r$, and
- ϕ: the angle between the z-axis and the line segment from the origin to the point $(x, y, z)_r$.

These three coordinates form the **Spherical coordinate system** and a point is represented by the triple $(\rho, \theta, \phi)_s$. We will use the subscript s to indicate that a point is written in spherical coordinates.





$$\rho^2 = r^2 + z^2 = x^2 + y^2 + z^2$$

$$z = \rho\cos\phi$$

$$r = \rho\sin\phi = \sqrt{x^2 + y^2}$$

$$x = \rho\sin\phi\cos\theta = r\cos\theta$$

$$y = \rho\sin\phi\sin\theta = r\sin\theta$$

Much like the case with cylindrical coordinates, we commonly restrict to $\rho \geq 0$ and either $-\pi < \theta \leq \pi$ or $0 \leq \theta < 2\pi$. The convention for ϕ is to restrict it to values between 0 and π. Also, akin the the strategy above for finding θ, note that $\cos\phi = z/\rho$, so that $\phi = \arccos(z/\rho)$. Thus a procedure for converting to spherical coordinates is to first compute ρ, and then use ρ and z to compute ϕ.

Example 7: Convert the following Cartesian coordinates to spherical coordinates:

a) $(1, -\sqrt{3}, 2\sqrt{3})_r$

b) $(-1, 1, \sqrt{6})_r$

Example 8: Convert the following spherical coordinates to Cartesian coordinates:

a) $(1, \pi, \pi/2)_s$

b) $(4, 3\pi/4, \pi/3)_s$

Concept Check: If a point satisfies $x > 0$, $y < 0$, and $z < 0$, what can you say about θ and ϕ?

The spherical coordinate system represents several geometric objects with very simple formulas. For example, $\rho = 2$ represents the sphere centered at the origin with radius 2. $\theta = \pi/4$ represents the half-plane that contains the line $y = x$ and is bounded by the z-axis. The equation $\phi = \pi/4$ represents the top half of the cone $z^2 = x^2 + y^2$.

Example 9: Describe (in words) the surface in $\mathbb{R}^3$ that is given by the following spherical equations:

a) $\rho = \sin\phi\sin\theta$

b) $\rho^2(\sin^2\phi\sin^2\theta + \cos^2\phi) = 9$

1.1 Review of Concepts

- Terms to know: real number, trace, Cartesian coordinates, rectangular coordinates, polar coordinates, cylindrical coordinates, spherical coordinates
- Know how to convert between the rectangular, cylindrical, and spherical coordinate systems.
- Know how to identify the general shape of a surface in three dimensions based on its equation.

1.1 Practice Problems

1. Convert the following Cartesian coordinates to cylindrical coordinates.

a) $(4\sqrt{3}, 4, -4)_r$ b) $(-5, 5, 6)_r$ c) $(0, 2, 0)_r$ d) $(4, -4\sqrt{3}, 6)_r$

2. Convert the following Cartesian coordinates to spherical coordinates.

a) $(1, \sqrt{3}, -2)_r$ b) $(1, -1, \sqrt{2})_r$ c) $(0, 3\sqrt{3}, 3)_r$ d) $(-5\sqrt{3}, 5, 0)_r$

3. Convert the following cylindrical coordinates to Cartesian coordinates.

a) $(4, \pi/6, 3)_c$ b) $(8, 3\pi/4, -2)_c$ c) $(5, 0, 4)_c$ d) $(7, \pi, -9)_c$

4. Convert the following cylindrical coordinates to spherical coordinates.

a) $(\sqrt{3}, \pi/6, 3)_c$ b) $(1, \pi/4, -1)_c$ c) $(2, 3\pi/4, 0)_c$ d) $(6, 1, -2\sqrt{3})_c$

5. Convert the following spherical coordinates to Cartesian coordinates.

a) $(5, \pi/6, \pi/4)_s$ b) $(7, 0, \pi/2)_s$ c) $(1, \pi, 0)_s$ d) $(2, 3\pi/2, \pi/2)_s$

6. Convert the following spherical coordinates to cylindrical coordinates.

a) $(5, \pi/4, 2\pi/3)_s$ b) $(1, 7\pi/6, \pi)_s$ c) $(3, 0, 0)_s$ d) $(4, \pi/6, \pi/2)_s$

7. An equation of a surface is given in Cartesian coordinates. Find an equation of the surface in cylindrical coordinates and spherical coordinates.

a) $z = 3$ b) $x^2 = 16 - z^2$ c) $x^2 + y^2 + z^2 = 9$ d) $2x + 3y + 4z = 1$

8. An equation of a surface is given in spherical or cylindrical coordinates. Express the equation in Cartesian coordinates.

a) (cyl) $r = 4\sin\theta$ c) (cyl) $\theta = \pi/4$ e) (sph) $\rho\sin\phi = 2\cos\theta$

b) (cyl) $r = 2\sec\theta$ d) (sph) $\phi = \pi/4$ f) (sph) $\rho = 2\sec\phi$

1.1 Exercises

1. Determine if the following statements are true or false, giving a brief explanation or counterexample in each case.

a) $(1,0,0)_r = (1,0,0)_c = (1,0,0)_s$.
b) $\{(\rho,\theta,\phi) \mid \phi = \pi/2\} = \{(r,\theta,z) \mid z = 0\} = \{(x,y,z) \mid z = 0\}$
c) Any point on the z-axis has more than one representation in both cylindrical and spherical coordinates.
d) The surface defined by $z = x^2$ in Cartesian coordinates is the same as the surface $z = r^2\cos^2\theta$ in cylindrical coordinates.
e) In $\mathbb{R}^3$, if a point lies in both the xy-plane and the xz-plane, then the point lies on the x-axis.
f) In cylindrical coordinates for a point, r also represents the distance from the point to the z-axis.
g) The surface $\theta = a$ (where a is a real number) in cylindrical coordinates is the same as the surface $\theta = a$ in spherical coordinates.
h) If $r > 0$ and θ in the cylindrical coordinates for a point satisfies $\pi/2 < \theta < 3\pi/2$, then x in the rectangular coordinates for that point satisfies $x < 0$.
i) If $\rho > 0$ and ϕ in the spherical coordinates for a point satisfies $0 < \phi < \pi/2$, then y in the rectangular coordinates for that point satisfies $y > 0$.
j) The spherical coordinates of any point in the sixth octant satisfy both $\pi/2 \le \theta \le \pi$ and $\pi/2 \le \phi \le \pi$.

2. Give the spherical equation that represents the cylinder $x^2 + y^2 = a^2$ for some $a > 0$.

3. Give the cylindrical equation that represents the Cartesian equation $y = mx + b$. What kind of surface is this?

4. Convert the following Cartesian coordinates to cylindrical and spherical coordinates:

a) $(1,0,0)_r$ b) $(0,1,-1)_r$ c) $(-1,-\sqrt{3},2)_r$

5. Convert the following cylindrical coordinates to Cartesian and spherical coordinates:

a) $(\sqrt{6},\pi/4,\sqrt{2})_c$ b) $(1,0,-1)_c$

6. Convert the following spherical coordinates to Cartesian and cylindrical coordinates:

a) $(4,\pi/3,2\pi/3)_s$ b) $(1,\pi,\pi)_s$

7. Write the following equations in cylindrical and spherical coordinates:

a) $x^2-2x+y^2+z^2=0$ b) $x+2y+3z=1$ c) $x^2+y^2=2y$

1.1 Answers to Practice Problems

1.

a) $(8,\pi/6,-4)_c$ b) $(5\sqrt{2},3\pi/4,6)_c$ c) $(2,\pi/2,0)_c$ d) $(8,5\pi/3,6)_c$

2.

a) $(2\sqrt{2},\pi/3,3\pi/4)_s$ b) $(2,7\pi/4,\pi/4)_s$ c) $(6,\pi/2,\pi/3)_s$ d) $(10,5\pi/6,\pi/2)_s$

3.

a) $(2\sqrt{3},2,3)_r$ b) $(-4\sqrt{2},4\sqrt{2},-2)_r$ c) $(5,0,4)_r$ d) $(-7,0,-9)_r$

4.

a) $(2\sqrt{3},\pi/6,\pi/6)_s$ b) $(\sqrt{2},\pi/4,3\pi/4)_s$ c) $(2,3\pi/4,\pi/2)_s$ d) $(4\sqrt{3},1,2\pi/3)_s$

5.

a) $(5\sqrt{6}/4,5\sqrt{2}/4,5\sqrt{2}/2)_r$ c) $(0,0,1)_r$

b) $(7,0,0)_r$ d) $(0,-2,0)_r$

6.

a) $(5\sqrt{3}/2, \pi/4, -5/2)_c$ b) $(0, 7\pi/6, -1)_c$ c) $(0, 0, 3)_c$ d) $(4, \pi/6, 0)_c$

7.

a) (cyl) $z = 3$, (sph) $\rho\cos\phi = 3$

b) (cyl) $r^2\cos^2\theta = 16 - z^2$, (sph) $\rho^2(\sin^2\phi\cos^2\theta + \cos^2\phi) = 16$

c) (cyl) $r^2 + z^2 = 9$, (sph) $\rho^2 = 9$

d) (cyl) $2r\cos\theta + 3r\sin\theta + 4z = 1$, (sph) $2\rho\sin\phi\cos\theta + 3\rho\sin\phi\sin\theta + 4\rho\cos\phi = 1$

8.

a) $x^2 + (y-2)^2 = 4$

b) $x = 2$

c) $y = x$

d) $z = \sqrt{x^2 + y^2}$

e) $(x-1)^2 + y^2 = 1$

f) $z = 2$

1.2 Review of Complex Numbers

Objectives and Concepts:

- Complex numbers have both real and imaginary components.
- Complex numbers have a planar representation and can be written in polar form.
- Euler's Formula and DeMoivre's Theorem give ways to compute exponentials of complex numbers.

References: TCMB §§1.7, 8.1–5

1.2.1 Complex Numbers

Definition: A **complex number** is a number of the form $a+bi$ where a and b are real numbers and the **imaginary unit** i has the property $i^2=-1$. The number a is known as the **real part** of $z=a+bi$, and b is the **imaginary part**. The notations $\Re(z)=a$ and $\Im(z)=b$ are sometimes used to represent the real and imaginary parts of z. Two complex numbers $z_1=a+bi$ and $z_2=c+di$ are equal if and only if $a=c$ and $b=d$. The set of all complex numbers is denoted $\mathbb{C}$. A complex number written as $a+bi$ is said to be in **standard form**.

Addition and subtraction of complex numbers is defined component-wise:

$$(a+bi)\pm(c+di)=(a\pm c)+(b\pm d)i.$$

Multiplication is performed in a standard way:

$$(a+bi)(c+di)=ac+adi+cbi+bdi^2=(ac-bd)+(ad+bc)i.$$

Division is not as straightforward - given an expression of the form $(a+bi)/(c+di)$, another definition is needed in order to write the quotient as another complex number.

Definition: The **complex conjugate** of $z=a+bi$ is the complex number $\overline{z}=a-bi$. We have

$$z\overline{z}=(a+bi)(a-bi)=a^2-abi+abi-b^2i^2=a^2+b^2.$$

Concept Check: Is the product $z\overline{z}$ ever complex?

To divide by a complex number, you write the quotient and then multiply by $\overline{z}/\overline{z}$:

$$\frac{a+bi}{c+di}=\frac{a+bi}{c+di}\cdot\frac{c-di}{c-di}=\frac{(a+bi)(c-di)}{(c+di)(c-di)}=\frac{(ac-bd)+(bc-ad)i}{c^2+d^2}=\frac{ac-bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i.$$

Some additional properties of conjugates:

- $\overline{w \pm z} = \overline{w} \pm \overline{z}$
- $\overline{wz} = \overline{w}\,\overline{z}$
- $\overline{z^n} = \overline{z}^n$

Example 1: Write the following expressions as a single complex number:

a) $\dfrac{1}{1-i}$

b) $\dfrac{3+2i}{1-4i}$

c) $\overline{2i(1-i)}$

> **Definition:** The **modulus** or **absolute value** of a complex number $z = a + bi$ is
> $$|z| = \sqrt{a^2 + b^2} = \sqrt{z\overline{z}}.$$

Although it was stated earlier that i satisfies $i^2 = -1$, this does not completely define i. Formally, i is defined as the **principal square root** of -1, i.e., $i = \sqrt{-1}$. This distinction is necessary because it is also true that $(-i)^2 = -1$. You have likely seen complex numbers before as a result of applying the quadratic formula. The definition of $i = \sqrt{-1}$ allows us to simplify the results of the quadratic formula. For example, the roots of $2x^2 + 2x + 1$ are

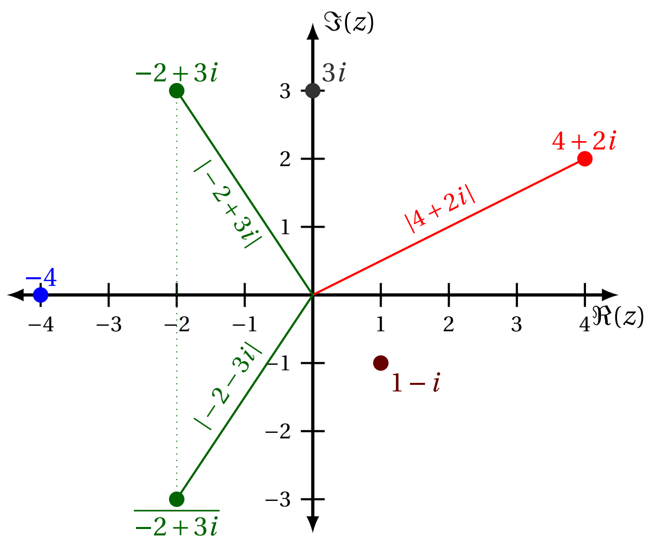
$$\frac{-2 \pm \sqrt{2^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} = \frac{-2 \pm \sqrt{-4}}{4} = \frac{-2 \pm i\sqrt{4}}{4} = -\frac{1}{2} \pm i\frac{1}{2} = \frac{1}{2}(-1 \pm i).$$

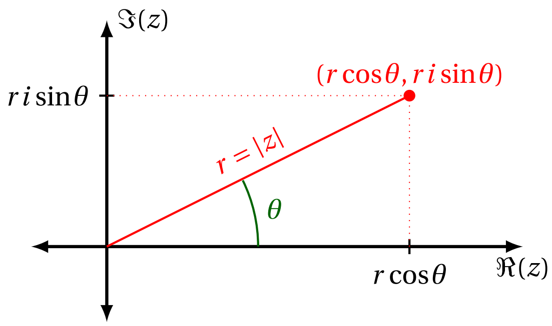
1.2.2 (‡) The Complex Plane

While real numbers are points on a continuous line, complex numbers are points in a plane called the **complex plane** (or **Argand plane**). The horizontal axis represents the real part $\Re$ of the number of the vertical axis represents the imaginary part $\Im$.

The modulus of z gives the distance from the origin to z.

From the plot it is evident that the conjugate $\overline{z}$ of z is simply the reflection of z across the real axis.





Thus, complex numbers can also be represented in **polar form**. Specifically, if we consider the coordinates of $z = a + bi$ to be (a, b), then a polar representation of z is

$$z = r(\cos\theta + i\sin\theta)$$

where $r \geq 0$. Here θ is known as the **argument** of z and is sometimes written $\theta = \arg(z)$.

Example 2: Find the polar form of the following complex numbers:

a) $1 + i$

b) $\sqrt{3} - i$

c) $3 + 4i$

Concept Check: Is the imaginary part of a complex number positive or negative if the argument of the number is $5\pi/6$?

The polar form also gives us another interpretation of arithmetic operations involving complex numbers. For $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, note that

$$\begin{aligned} z_1 z_2 &= [r_1(\cos\theta_1 + i\sin\theta_1)] \cdot [r_2(\cos\theta_2 + i\sin\theta_2)] \\ &= r_1 r_2 \left[\underbrace{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}_{=\cos(\theta_1+\theta_2)} + i\underbrace{(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)}_{=\sin(\theta_1+\theta_2)} \right] \\ &= r_1 r_2 (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)). \end{aligned}$$

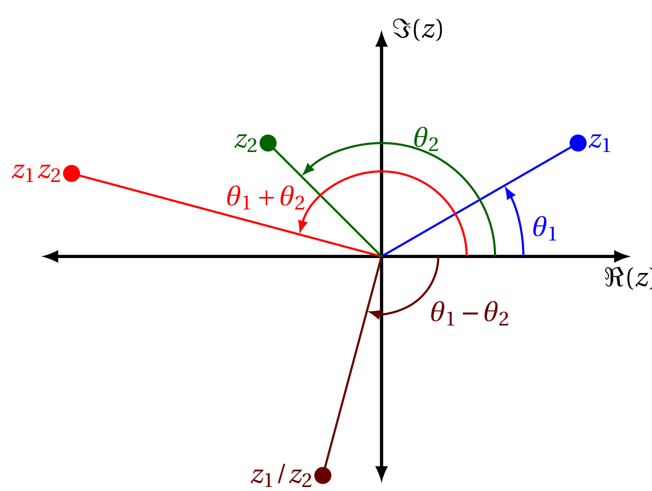
Thus we have

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)).$$

This means that when you multiply two complex numbers together, you multiply their radii and add their angles! A similar formula can be derived for dividing z_1 by a nonzero z_2.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)).$$

Also note that if $z_1 = 1$, then $r_1 = 1$ and $\theta_1 = 0$ and the even/odd properties of cosine and sine give an explicit formula for $1/z_2$.



Example 3: For $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$, use their polar representations to calculate $z_1 z_2$ and z_1/z_2.

1.2.3 (‡) DeMoivre's Theorem and Euler's Formula

The rule for products of complex numbers via polar representation can be extended to the following theorem:

DeMoivre's Theorem: If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

Example 4: Compute $\left(2\sqrt{3}+2i\right)^5$.

The last result from complex numbers that we will use heavily is Euler's formula for computing e^z when $z = a+bi$. This formula can be derived using *infinite series*, a topic we will discuss later on in the course.

Euler's Formula: $e^{ix} = \cos x + i\sin x$

If $z = a+bi$, then

$$e^z = e^{a+bi} = e^a e^{bi} = e^a(\cos b + i\sin b).$$

This is often used to evaluate expressions such as $e^{i\theta}$ where θ is the polar angle.

Example 5: Write $e^{i\pi}$ and $e^{-4i\pi/3}$ in standard form.

1.2.4 (‡) Roots of Complex Numbers

While i itself is defined to be the principal square root of -1, the square root of a complex number itself needs to be formally defined:

Definition: The **square root of a complex number** z is a number w such that $w^2 = z$.

There are two square roots of a complex number (just as with real numbers), the first one is given by, if $z = re^{i\theta}$,

$$\sqrt{z} = \sqrt{re^{i\theta}} = \sqrt{r}e^{i\theta/2}.$$

The other square root is found by realizing that $z = re^{i\theta} = re^{i(\theta+2\pi)}$:

$$\sqrt{z} = \sqrt{re^{i(\theta+2\pi)}} = \sqrt{r}e^{i(\theta+2\pi)/2} = \sqrt{r}e^{i(\pi+\theta/2)}.$$

Example 6: Find the square roots of $3e^{i\pi/2}$.

The n distinct nth roots of z are found in the same way for any natural number n: they are the solutions w of $w^n = z$.

Concept Check: Is the number i^i real or complex?

1.2 Review of Concepts

- Terms to know: complex number, standard form, polar form, imaginary unit, real part, imaginary part, complex plane, complex conjugate, modulus, argument
- Know how to perform basic arithmetic with complex numbers, including finding the modulus, division, and roots.
- Know how to find the polar representation of complex numbers, and how to use the polar representation to multiply and divide complex numbers.
- Know how to use DeMoivre's Theorem to compute exponents of complex numbers.

1.2 Practice Problems

1. Find wz, w/z, and $1/z$ for $w = -2+11i, z = 2-i$.

2. Let $z = a+bi$. Find, in terms of a and b,

a) $\Re(z^4) - (\Re(z^2))^2$ b) $\Re(z/\overline{z})$ c) $\Im(1/\overline{z}^2)$

3. Represent $z = 2-2\sqrt{3}i$ in polar form and find z^3.

1.2 Exercises

1. Determine if the following statements are true or false, giving a brief explanation or counterexample in each case.

a) In the complex plane, the complex conjugate of z is the reflection of z across the imaginary axis.
b) For any complex number z, $|z| = \sqrt{\Re(z)^2 + \Im(z)^2}$.
c) For any complex number z, the product $z\overline{z}$ is always real.
d) If $\pi < \arg(z) < 2\pi$, then the real part of z cannot be positive.
e) The imaginary part of a complex number z is given by $\Im(z) = \dfrac{z-\overline{z}}{2i}$.

2. Which of the following is the conjugate of the standard form of the complex number $(\sqrt{3}+i)^4$?

A) $8-8\sqrt{3}i$ B) $-8-8\sqrt{3}i$ C) $-8+8\sqrt{3}i$ D) $8+8\sqrt{3}i$

3. Find wz, w/z, and $1/z$ for each:

a) $w = 1 + \sqrt{3}i, z = \sqrt{3} + i$

b) $w = -3 - 3i, z = 4\sqrt{3} + 4i$

4. Find the standard form of $z + w$ if $z = 4e^{3i}$ and $w = 5e^{2i}$.

5. Find the complex conjugate of $z = (x + iy)^2 - 4e^{ixy}$.

6. Compute $(1 - i)^8$.

7. If $z = \left(\dfrac{\sqrt{3} - i}{2 + 2i}\right)^2$, find $\Re(z)$, $\Im(z)$, and the polar representation of z.

8. Give a general formula for the three cube roots of $z = e^{i\theta}$ and use it to find the three cube roots of $1 - i$.

9. Use Euler's formula to derive the following formulas for $\sin x$ and $\cos x$:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

1.2 Answers to Practice Problems

1. For $w = -2 + 11i, z = 2 - i$,

$$wz = (-2 + 11i)(2 - i) = -4 - 11i^2 + 2i + 22i = 7 + 24i.$$

$$\frac{w}{z} = \frac{-2 + 11i}{2 - i} \cdot \frac{2 + i}{2 + i} = \frac{-4 + 11i^2 - 2i + 22i}{5} = \frac{-15 + 20i}{5} = -3 + 4i.$$

$$\frac{1}{z} = \frac{1}{2 - i} \cdot \frac{2 + i}{2 + i} = \frac{2 + i}{5}.$$

2. a)

$$\begin{aligned}
\Re(z^4) - (\Re(z^2))^2 &= \Re((a + bi)^4) - (\Re((a + bi)^2)^2 \\
&= \Re(a^4 + 4a^3bi + 6a^2b^2i^2 + 4ab^3i^3 + b^4i^4) - \left(\Re(a^2 + 2abi + b^2i^2)\right)^2 \\
&= \Re(a^4 + 4a^3bi - 6a^2b^2 - 4ab^3i + b^4) - \left(\Re(a^2 + 2abi - b^2)\right)^2 \\
&= a^4 - 6a^2b^2 + b^4 - \left(a^2 - b^2\right)^2 = a^4 - 6a^2b^2 + b^4 - \left(a^4 - 2a^2b^2 + b^4\right) = -4a^2b^2.
\end{aligned}$$

b) $\Re(z/\overline{z}) = \Re\left(\dfrac{a + bi}{a - bi}\right) = \Re\left(\dfrac{a + bi}{a - bi} \cdot \dfrac{a + bi}{a + bi}\right) = \Re\left(\dfrac{a^2 + 2abi - b^2}{a^2 + b^2}\right) = \dfrac{a^2 - b^2}{a^2 + b^2}.$

c) There are at least two ways to find this. First, note that

$$\begin{aligned}\Im(1/\overline{z}^2) &= \Im\left(\frac{1}{a^2-2abi+b^2i^2}\right) = \Im\left(\frac{1}{(a^2-b^2)-2abi}\cdot\frac{(a^2-b^2)+2abi}{(a^2-b^2)+2abi}\right)\\ &= \Im\left(\frac{(a^2-b^2)+2abi}{(a^2-b^2)^2+(2ab)^2}\right) = \frac{2ab}{(a^2-b^2)^2+(2ab)^2}\\ &= \frac{2ab}{a^4-2a^2b^2+b^4+4a^2b^2} = \frac{2ab}{a^4+2a^2b^2+b^4} = \frac{2ab}{(a^2+b^2)^2}.\end{aligned}$$

We also have

$$\Im\left(\frac{1}{\overline{z}^2}\right) = \Im\left(\frac{1}{\overline{z}^2}\cdot\frac{z^2}{z^2}\right) = \Im\left(\frac{z^2}{(\overline{z}z)^2}\right) = \Im\left(\frac{z^2}{(|z|^2)^2}\right) = \Im\left(\frac{a^2-b^2+2abi}{(a^2+b^2)^2}\right) = \frac{2ab}{(a^2+b^2)^2}.$$

3. We have that if $z = 2-2\sqrt{3}i$, then $r = |z| = \sqrt{2^2+(2\sqrt{3})^2} = \sqrt{4+12} = 4$ and therefore $\cos\theta = \frac{2}{4} = \frac{1}{2}$ and $\sin\theta = -\frac{\sqrt{3}}{2}$. Thus θ is in the fourth quadrant and we have $\theta = \frac{5\pi}{3}$. Then $z = 4\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$, so

$$z^3 = 4^3\left(\cos\left(3\cdot\frac{5\pi}{3}\right) + i\sin\left(3\cdot\frac{5\pi}{3}\right)\right) = 64\,(\cos 5\pi + i\sin 5\pi) = -64.$$

1.3 Vectors and Vector Operations

Objectives and Concepts:

- Vectors are quantities that represent magnitude and direction, and they are independent of position. Every point in $\mathbb{R}^n$ can be represented by a position vector. Vectors are computed using Cartesian coordinates.
- Every vector in $\mathbb{R}^3$ can be written as a linear combination of the standard basis vectors $\vec{\boldsymbol{\imath}}$, $\vec{\boldsymbol{\jmath}}$, $\vec{\boldsymbol{k}}$.
- Vectors are often used to represent forces and moments in applications.

References: OSC-3 §§2.1, 2.2; CET §12.2; TCMB §§16.1–3,5.

1.3.1 Vectors and Scalars

Definition: A **vector** is a quantity that has both magnitude and direction. Geometrically a vector is usually represented with an arrow. A (real) vector with n **components** is an element of the set $\mathbb{R}^n$. Vectors are typically represented using Cartesian coordinates. A **scalar** is an element of $\mathbb{R}$ (i.e., a scalar is a real number).

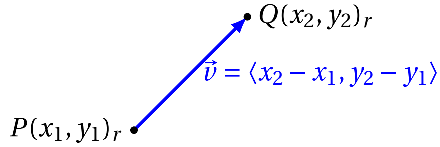
We typically use boldface letters ($\boldsymbol{v}$) or an arrow above a symbol ($\vec{v}$) to denote vectors. Any two points in $\mathbb{R}^n$ can be used to define a vector. A vector $\vec{v}$ is determined by its components $v_1, v_2, \ldots, v_n$ and we often use the following notations to write $\vec{v}$:

$$\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle, \qquad \text{or} \qquad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

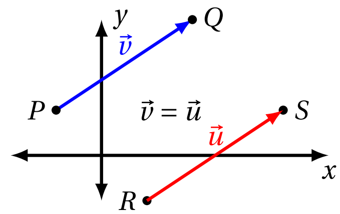
If $P(x_1, y_1, z_1)_r$ and $Q(x_2, y_2, z_2)_r$ are two points in $\mathbb{R}^3$, then the vector $\overrightarrow{PQ}$ is given by

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

In this case $\vec{v}$ represents the **displacement** from point P to point Q.



Two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are **equal** if and only if $u_i = v_i$ for all $i = 1, \ldots, n$. Vectors are *independent of position* - this means that two vector with the same components are equal, regardless of where they may be situated in space.



Definition: The **magnitude** or **length** of a vector $\vec{v}$ is denoted by $|\vec{v}|$ and is given by

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

Example 1: Find the magnitude of the vector $\langle -2, 4, 1 \rangle$

It is also common to use $\|\vec{v}\|$ to also represent the magnitude of a vector, and it is also known as the **norm** of $\vec{v}$. The norm of a vector can be conceptually extended to several different settings. The **zero vector** $\vec{0} = \langle 0, 0, \ldots, 0 \rangle$ is the only vector with a magnitude of 0.

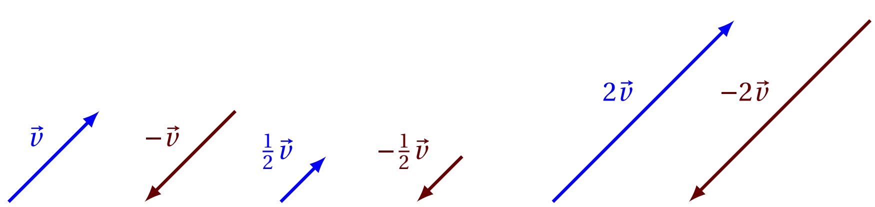
1.3.2 Vector Operations

Scalar Multiplication of a Vector: Let $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ be a vector and let a be a scalar. The **scalar multiple** $a\vec{v}$ is given by

$$a\vec{v} = \langle av_1, av_2, \ldots, av_n \rangle.$$

If $\vec{v}$ is any vector and a is any scalar, then $|a\vec{v}| = |a|\,|\vec{v}|$.

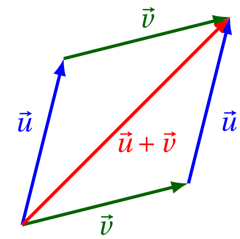
Multiplication by a positive scalar does not change the direction of a vector, however multiplication by a negative scalar *reverses the direction of the vector*. In this way it is easy to see that two vectors are **parallel** if they are scalar multiples of each other. (Sometimes, to distinguish that vectors point in the "opposite direction", we say that two vectors are **anti-parallel** if one is a negative multiple of the other.)



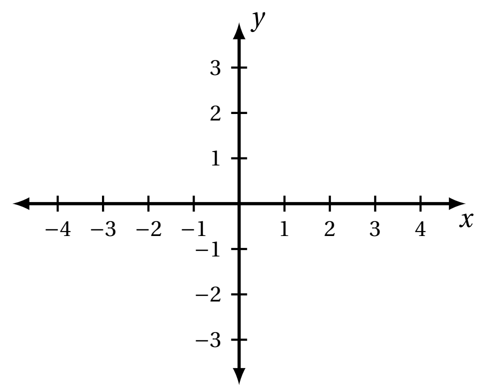
Vector Addition: Let $\vec{u} = \langle u_1, u_2, \dots, u_n \rangle$ and $\vec{v} = \langle v_1, v_2, \dots, v_n \rangle$ be vectors. The sum of $\vec{u}$ and $\vec{v}$ is given by

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, \dots, u_n + v_n \rangle.$$

Vectors are added component-wise. Graphically, vector addition can be represented by placing the vectors *head-to-tail* and drawing the resulting vector. It does not matter which vector is placed first as can be seen in the figure on the right.



Example 2: Let $\vec{u} = \langle 4, 2 \rangle$ and $\vec{v} = \langle -1, -3 \rangle$. Compute and draw the vector $\vec{u} + \vec{v}$ on the provided axes.



Definition: Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ be vectors and let $a_1, a_2, \dots, a_m$ be scalars. The expression

$$a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_m \vec{v}_m$$

is a **linear combination** of the $\vec{v}_i$ and the a_i.

Definition: A **unit vector** is a vector with magnitude 1. For any vector $\vec{v}$, the vector $\dfrac{\vec{v}}{|\vec{v}|}$ is a unit vector in the same direction as $\vec{v}$. This vector is sometimes denoted $\hat{n}$, and this hat notation often replaces the arrow to advertise that a vector is of unit length.

Example 3: Given $\vec{u} = \langle 2, 0, -1 \rangle$, $\vec{v} = \langle 1, 1, 1 \rangle$, and $\vec{w} = \langle 4, 2, 3 \rangle$, find the following:

a) $3\vec{u} - 2\vec{v} + \vec{w}$

b) $-\vec{u} + \vec{v} - \vec{w}$

c) a unit vector in the same direction as $\vec{v} - 2\vec{u}$.

Concept Check: How many unit vectors are in the same direction as a vector $\vec{v}$?

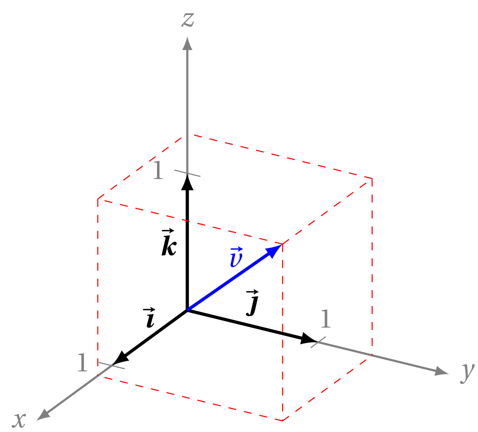
1.3.3 The Standard Basis

Definition: The **standard basis vectors** (or **elementary basis vectors**) are given by

$$\begin{aligned} \vec{\imath} &= \langle 1,0,0\rangle \\ \vec{\jmath} &= \langle 0,1,0\rangle \\ \vec{k} &= \langle 0,0,1\rangle \end{aligned}$$

Any vector $\vec{v}$ in $\mathbb{R}^3$ can be written as a linear combination of the standard basis vectors:

$$\vec{v} = \langle a_1, a_2, a_3\rangle = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}.$$



The components of a vector $\vec{v}$ in $\mathbb{R}^3$ are sometimes denoted using a subscript that represents the variable of the direction: $\vec{v} = \langle v_x, v_y, v_z\rangle$. The standard basis vectors give us a way to express any vector as a **linear combination** of unit vectors.

Example 4: Let $\vec{u} = 2\vec{\imath} - \vec{k}$, $\vec{v} = -\vec{\imath} + 4\vec{\jmath} + 2\vec{k}$, and $\vec{w} = \vec{\jmath} + 2\vec{k}$.

a) Find $\vec{u} + \vec{v} - \vec{w}$.

b) For what value(s), if any, of c does the vector $2\vec{u} - \vec{v} + c\vec{w}$ lie in the xy-plane?

Properties of Vectors: Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors and let a and b be scalars. Then we have

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutativity of +)
2. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ (associativity of +)
3. $\vec{u} + \vec{0} = \vec{u}$ (additive identity)
4. $\vec{u} + (-\vec{u}) = \vec{0}$ (additive inverse)
5. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$ (left distribution of scalar multiplication)
6. $(a+b)\vec{u} = a\vec{u} + b\vec{u}$ (right distribution of scalar multiplication)
7. $a(b\vec{u}) = (ab)\vec{u}$ (associativity of scalar multiplication)
8. $1\vec{u} = \vec{u}$ (scalar multiplication identity)

1.3.4 Applications of Vectors

For any point in space, there is a unique vector associated with that point:

> **Definition:** Let $P(x, y, z)$ be a point in $\mathbb{R}^3$. Then the vector $\vec{v} = x\vec{\imath} + y\vec{\jmath} + z\vec{k} = \langle x, y, z\rangle$ is the **position vector** of P.

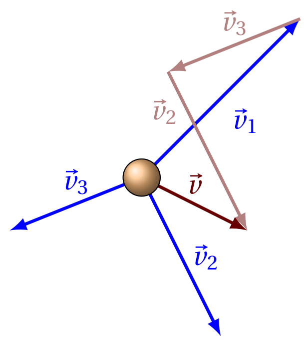
The **velocity** of an object is a quantity that is often represented by a vector. If the velocity of an object is given by the vector $\vec{v}$, then $|\vec{v}|$ is the **speed** of the object.

Concept Check: Is the temperature of an ideal gas a vector quantity or scalar quantity? What about the force of gravity acting on the Cooper River Bridge? The mass of all of the vehicles on the Cooper River Bridge?

Forces and Static Equilibrium

Vectors can be used to represent forces acting on an object (particle, rigid body, etc.). When multiple forces act on an object, the **net force** or **resultant force** is the sum of all forces acting on the object. An object that is being acted upon by the forces $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N$ is said to be in **static equilibrium** if the net force is zero, i.e., if

$$\sum_{i=1}^{N} \vec{v}_i = \vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_N = \vec{0}.$$



The resultant concept can also be applied to objects in motion. For example, consider a plane flying with velocity $\vec{v}$ through a wind that has a velocity $\vec{w}$. Then the **true course** or **track** of the plane is the direction of the resultant $\vec{v} + \vec{w}$. The **ground speed** of the plane is the magnitude of the resultant.

Center of Mass

The **center of mass** of an object is the unique point in an object or system which can be used to describe the system's response to external forces. When the system consists of many particles, the position vectors of each particle are used to find the center of mass of a system. Even if the particles themselves have volume (are not simply point masses), the position vector of the center of mass of each particle are used.

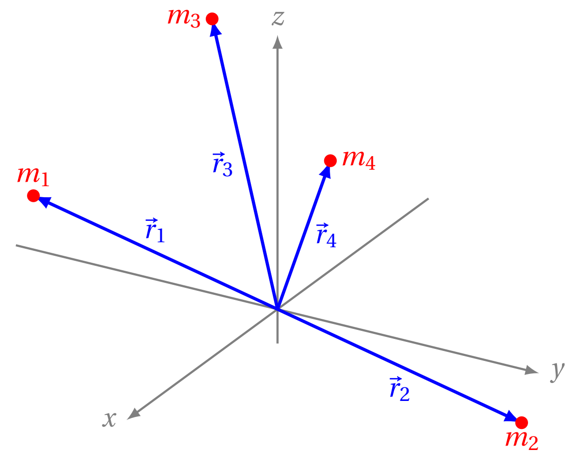
Consider a system of N particles in space at positions $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$ with masses $m_1, m_2, \ldots, m_N$. The position vector of the center of mass of the system is then given by

$$\begin{aligned}\vec{r} &= \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N) \\ &= \frac{1}{M}\sum_{i=1}^{N} m_i\vec{r}_i,\end{aligned}$$

where

$$M = \sum_{i=1}^{N} m_i$$

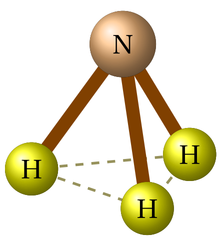
is the sum of the masses.



$$\vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + m_4\vec{r}_4}{m_1 + m_2 + m_3 + m_4}$$

Note: When finding the center of mass of a set of objects in space, the calculations can often be simplified by first calculating the centers of mass of a subset or subsets of the objects, treating those subsets themselves as a point mass in the final calculation.

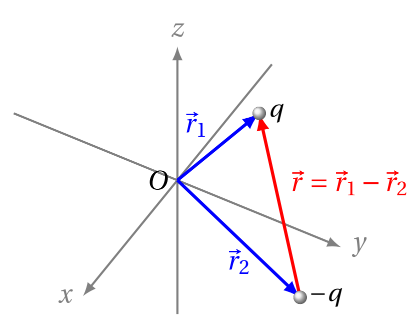
Example 5: An ammonia molecule (NH_3) has three hydrogen atoms, which form a equilateral triangle as the base of a pyramid. At the top of the pyramid is the nitrogen atom. The length between any two hydrogen atoms is $\ell_{\mathrm{H}} = 9.4 \times 10^{-11}$m, the length from nitrogen to any of the hydrogen atoms is $\ell_{\mathrm{N}} = 10.14 \times 10^{-11}$m. Let m_{N} be the mass of nitrogen and m_{H} be the mass of hydrogen (the ratio $m_{\mathrm{N}}/m_{\mathrm{H}}$ is 13.8964). Establish a coordinate system for the molecule and compute its center of mass in that coordinate system.



Dipole Moments

A **dipole** consists of two equal and opposite point charges. Dipoles can be characterized by their **dipole moment**, a vector quantity. For the simple two-particle electric dipole given here, the electric dipole moment $\vec{\mu}$ points from the negative charge $-q$ towards the positive charge q, and has a magnitude equal to the strength of each charge times the separation between the charges. Essentially, we have

$$\vec{\mu} = q\vec{r}_1 - q\vec{r}_2 = q(\vec{r}_1 - \vec{r}_2) = q\vec{r}.$$



A system of N charges $q_1,\ldots,q_N$ with position vectors $\vec{r}_1,\ldots,\vec{r}_N$ has the dipole moment

$$\vec{\mu} = q_1\vec{r}_1 + q_2\vec{r}_2 + \cdots + q_N\vec{r}_N = \sum_{i=1}^{N} q_i\vec{r}_i,$$

where the origin O is used as the point of reference (the origin has a position vector of $\vec{0}$). The quantity depends on the position of the reference point if the total charge $Q = \sum_{i=1}^{N} q_i$ is not zero. For a general reference point R with position vector $\vec{R}$, the dipole moment of the system of charges with respect to R is

$$\vec{\mu}(\vec{R}) = \sum_{i=1}^{N} q_i(\vec{r}_i - \vec{R}) = \left(\sum_{i=1}^{N} q_i\vec{r}_i\right) - \vec{R}\sum_{i=1}^{N} q_i = \vec{\mu}(\vec{0}) - Q\vec{R},$$

as $\vec{r}_i - \vec{R}$ is the position vector of charge q_i relative to the point R. Finally, a system of electric dipoles with moments $\vec{\mu}_1, \vec{\mu}_2, \ldots, \vec{\mu}_N$ has total dipole moment

$$\vec{\mu} = \vec{\mu}_1 + \vec{\mu}_2 + \cdots + \vec{\mu}_N.$$

Example 6: Three charges $q_1 = 3$, $q_2 = -1$, and $q_3 = 1$ are located at points $(2,2,1)_r$, $(2,-2,3)_r$, and $(0,-4,-3)_r$, respectively.

a) Find the dipole moment of the system of charges with respect to the origin.

b) Find the position of the point R such that the dipole moment with respect to R is the zero vector.

1.3 Review of Concepts

- Terms to know: vector, component, scalar, displacement, magnitude, norm, unit vector, zero vector, scalar multiple, linear combination, standard basis vectors, elementary basis vectors, position vector, velocity, speed, resultant force, static equilibrium, center of mass, dipole, dipole moment.
- Know how to perform vector addition and scalar multiplication, and how to compute the unit vector in the direction of a given vector.
- Know how to represent vectors using the standard basis vectors.
- Know how to apply formulas to compute resultant forces, centers of mass, and dipole moments.

1.3 Practice Problems

1. Given $\vec{u} = \langle 1,2,3\rangle$, $\vec{v} = \langle -2,3,-4\rangle$, and $\vec{w} = \langle 0,4,-1\rangle$, find

a) $\vec{u} - 2\vec{v} + 3\vec{w}$

b) $|\vec{u} + \vec{v}|$

c) $|\vec{u}| + |\vec{v}|$

d) a unit vector in the direction of $\vec{w} - \vec{v}$.

2. Given that $\vec{v}$ is a vector in $\mathbb{R}^3$ and $|\vec{v}| = 3$, find all values of k such that $|k\vec{v}| = 5$.

3. Find $\vec{u}$ and $\vec{v}$ if $\vec{u} + 2\vec{v} = 3\vec{\imath} - \vec{k}$ and $3\vec{u} - \vec{v} = \vec{\imath} + \vec{\jmath} + \vec{k}$.

4. Find a force $\vec{v}$ such that the resultant force of $\vec{v}$, $\vec{u} = 2\vec{\imath} - 3\vec{\jmath} + \vec{k}$, and $\vec{w} = 2\vec{\jmath} - \vec{k}$ is the zero force.

1.3 Exercises

1. Which of the following quantities are vectors and which are scalars?

a) The mass of a water molecule.

b) The center of mass of a water molecule.

c) The bond length of a water molecule.

d) The dipole moment of a water molecule.

e) The charge of an electron.

f) The temperature of a gas.

g) The pressure of a gas.

h) The velocity of a gas.

2. Suppose a molecule has three atoms. After establishing a coordinate system (in the xy-plane), you determine that atom 1 is located at the point $(1,1)$ and has a mass of 2 units, atom 2 is located at the point $(-1,1)$ and has a mass of 4 units, and atom 3 is located at the point $(1,-1)$ and has a mass of 8 units. Where do you expect the center of mass of this molecule to be?

A) Quadrant I   B) Quadrant II   C) Quadrant III   D) Quadrant IV   E) The Origin

3. Find two distinct unit vectors that are parallel to the vector $\vec{v} = 4\vec{\imath} - 2\vec{\jmath} + \frac{1}{2}\vec{k}$.

4. Find $4\vec{u} - \vec{v}$ and $|\vec{u} + 3\vec{v}|$ if $\vec{u} = \langle -2, -3, 0\rangle$ and $\vec{v} = \langle 1, 2, 1\rangle$.

5. Determine if the following statements are true or false, giving a brief explanation or counterexample in each case.

a) The magnitude of a sum of vectors is the sum of the magnitudes of the vectors.

b) Given a nonzero scalar a and vectors $\vec{u}$ and $\vec{v}$, the vector equation $a\vec{w} + \vec{u} = \vec{v}$ always has a unique solution $\vec{w}$.

c) It is possible for two vectors that point in the same direction to have different magnitudes.

d) It is possible for two vectors that point in different directions to have the same magnitudes.

6. An object is acted upon by the forces $\vec{F}_1 = \langle 10, 6, 3\rangle$ and $\vec{F}_2 = \langle 0, 4, 9\rangle$. Find the force $\vec{F}_3$ that must act on the object so that the sum of their forces is zero (so the object is in static equilibrium).

7. Find a vector parallel to $\vec{v} = \langle 3, -2, 6\rangle$ with length 10.

8. A molecule of carbon monoxide (CO) consists of a single carbon atom (with mass $m_C = 12.0107$amu) and a single oxygen atom (with mass $m_O = 15.9994$amu). The two atoms are $\ell = 1.11 \times 10^{-10}$m apart. Establish a coordinate system and find the center of mass. Explain your process.

9. Three charges $q_1 = -2$, $q_2 = 2$, and $q_3 = -1$ are located at points $(3,0,-2)_r$, $(1,1,1)_r$, and $(0,5,-2)_r$, respectively.

a) Find the dipole moment of the system of charges with respect to the origin.

b) Find the position of the point R such that the dipole moment with respect to R is the zero vector.

1.3 Answers to Practice Problems

1. $\vec{u} = \langle 1,2,3\rangle$, $\vec{v} = \langle -2,3,-4\rangle$, $\vec{w} = \langle 0,4,-1\rangle$

a) $\vec{u} - 2\vec{v} + 3\vec{w} = \langle 1,2,3\rangle - 2\langle -2,3,-4\rangle + 3\langle 0,4,-1\rangle = \langle 1+4+0, 2-6+12, 3+8-3\rangle = \langle 5,8,8\rangle$

b) $|\vec{u}+\vec{v}| = |\langle 1,2,3\rangle + \langle -2,3,-4\rangle| = |\langle -1,5,-1\rangle| = \sqrt{1+25+1} = 3\sqrt{3}$

c) $|\vec{u}| + |\vec{v}| = |\langle 1,2,3\rangle| + |\langle -2,3,-4\rangle| = \sqrt{1+4+9} + \sqrt{4+9+16} = \sqrt{14} + \sqrt{29}$

d) a unit vector in the direction of $\vec{w} - \vec{v}$. Now $\vec{w} - \vec{v} = \langle 0,4,-1\rangle - \langle -2,3,-4\rangle = \langle 2,1,3\rangle$. Then $|\vec{w}-\vec{v}| = |\langle 2,1,3\rangle| = \sqrt{14}$. Thus a unit vector in the same direction as $\vec{w}-\vec{v}$ is

$$\frac{1}{|\vec{w}-\vec{v}|}(\vec{w}-\vec{v}) = \frac{1}{\sqrt{14}}\langle 2,1,3\rangle = \left\langle \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle.$$

2. Since

$$|k\vec{v}| = |k\langle v_1, v_2, v_3\rangle| = |\langle kv_1, kv_2, kv_3\rangle| = \sqrt{k^2 v_1^2 + k^2 v_2^2 + k^2 v_3^3} = |k||\vec{v}| = 3|k|,$$

we have that $|k|$ must be $5/3$, so $k = \pm 5/3$.

3. $\vec{u} + 2\vec{v} = 3\vec{\imath} - \vec{k}$ and $3\vec{u} - \vec{v} = \vec{\imath} + \vec{\jmath} + \vec{k}$ so $\vec{u} = 3\vec{\imath} - \vec{k} - 2\vec{v}$, which implies

$$\vec{\imath} + \vec{\jmath} + \vec{k} = 3\left(3\vec{\imath} - \vec{k} - 2\vec{v}\right) - \vec{v} = 9\vec{\imath} - 3\vec{k} - 6\vec{v} - \vec{v} = -7\vec{v} + 9\vec{\imath} - 3\vec{k}.$$

Thus

$$\vec{v} = -\frac{1}{7}\left(-8\vec{\imath} + \vec{\jmath} + 4\vec{k}\right) = \left\langle \frac{8}{7}, -\frac{1}{7}, -\frac{4}{7} \right\rangle,$$

which implies

$$\vec{u} = 3\vec{\imath} - \vec{k} - 2\vec{v} = \left\langle 3 - \frac{16}{7}, \frac{2}{7}, -1 + \frac{8}{7} \right\rangle = \left\langle \frac{5}{7}, \frac{2}{7}, \frac{1}{7} \right\rangle.$$

4. We must have $\vec{v} + \vec{u} + \vec{w} = \vec{0}$, so

$$\vec{v} = -\vec{u} - \vec{w} = -\left(2\vec{\imath} - 3\vec{\jmath} + \vec{k}\right) - \left(2\vec{\jmath} - \vec{k}\right) = -2\vec{\imath} + \vec{\jmath}.$$

1.4 The Scalar Product (a.k.a "Dot Product", a.k.a. "Inner Product")

Objectives and Concepts:

- The scalar product (or dot product, or inner product) of two vectors is a scalar measure of how much the vectors point in the same direction.
- The scalar product can be used to compute projections of one vector onto another.
- Two vectors are orthogonal if their scalar product is zero.

References: OSC-3 §2.3; CET §12.3, TCMB §16.5.

1.4.1 The Scalar Product (a.k.a "Dot Product", a.k.a. "Inner Product") and Orthogonality

Definition: Let $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ be vectors in $\mathbb{R}^n$. The **scalar product** of $\vec{u}$ and $\vec{v}$ (also knowns as the **dot product** or **inner product**) is the scalar quantity

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i.$$

Example 1: Compute the following scalar products:

a) $(\vec{\boldsymbol{\imath}} + \vec{\boldsymbol{\jmath}} + \vec{\boldsymbol{k}}) \cdot (\vec{\boldsymbol{\imath}} - 2\vec{\boldsymbol{\jmath}} + 2\vec{\boldsymbol{k}})$

b) $\vec{\boldsymbol{\imath}} \cdot \vec{\boldsymbol{\jmath}}$

c) $\langle 4, -3, 1 \rangle \cdot \langle 2, 2, -2 \rangle$

d) $\langle 1, 2, 3 \rangle \cdot \langle -1, -2, -3 \rangle$

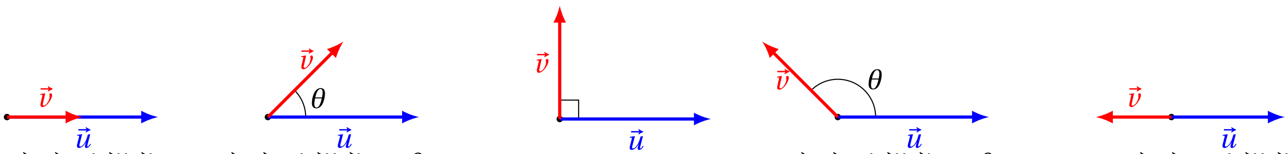
e) $\langle 1, 2, 3 \rangle \cdot \langle 1, 2, 3 \rangle$

The scalar product of $\vec{u}$ and $\vec{v}$ can be thought of as a measure of how much the two vectors point in the same direction. Indeed, the scalar product can be given in terms of the angle θ between two *nonzero* vectors:

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$$

Concept Check: Since we know $-1 \le \cos\theta \le 1$, what does that say about the scalar product of $\vec{u}$ and $\vec{v}$?

This also means that the sign of the scalar product immediately tells you if the angle between the two vectors is less than $\pi/2$ or greater than $\pi/2$. The figures below give an example of the usefulness of the formula $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$:



$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|$ $\quad$ $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta > 0$ $\quad$ $\vec{u}\cdot\vec{v} = 0$ $\quad$ $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta < 0$ $\quad$ $\vec{u}\cdot\vec{v} = -|\vec{u}||\vec{v}|$

Several other interesting observations can be drawn:

- If $\vec{u}$ and $\vec{v}$ are unit vectors, $\vec{u}\cdot\vec{v} = \cos\theta$.
- $\vec{u}\cdot\vec{u} = |\vec{u}||\vec{u}|\cos 0 = |\vec{u}|^2$ so $|\vec{u}| = \sqrt{\vec{u}\cdot\vec{u}}$.
- $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$ so $\theta = \cos^{-1}\left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$.

Example 2: Find the angle between the vectors $\vec{u} = \sqrt{3}\vec{\imath} - \vec{\jmath}$ and $\vec{v} = -\vec{\imath} + \sqrt{3}\vec{\jmath}$.

Concept Check: What is the scalar product of two vectors that form a right angle?

> **Definition:** The vectors $\vec{u}$ and $\vec{v}$ are **orthogonal** if and only if $\vec{u}\cdot\vec{v} = 0$.

Example 3: Is there a value of c such that the vectors $\vec{u} = 2\vec{\imath} - 3\vec{\jmath} + \vec{k}$ and $\vec{v} = \vec{\imath} + 2\vec{\jmath} + c\vec{k}$ are orthogonal?

Concept Check: How many different unit vectors are orthogonal to the vector $\vec{\imath}$?

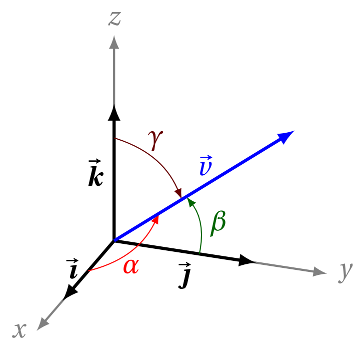
Orthogonality is a slight generalization of the concept of perpendicular lines - in $\mathbb{R}^2$ and $\mathbb{R}^3$ it is certainly the case that two nonzero vectors that are orthogonal are perpendicular, so they have an angle of $\pi/2$ between them. However, the concept of orthogonality can be applied in settings where it isn't as easy to think about objects geometrically.

> **Properties of the Scalar Product:** Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^n$ and let $a \in \mathbb{R}$. Then we have:
>
> 1. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ (commutativity of $\cdot$)
> 2. $a(\vec{u}\cdot\vec{v}) = (a\vec{u})\cdot\vec{v} = \vec{u}\cdot(a\vec{v})$ (associativity of scalar multiplication)
> 3. $\vec{u}\cdot\vec{0} = 0$ (orthogonality of $\vec{0}$)
> 4. $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$ (distributivity of $\cdot$)

Concept Check: If $\vec{u}$ is orthogonal to $\vec{v}$, what is $|\vec{u}+\vec{v}|^2$?

Concept Check: Is the expression $\vec{u}\cdot(\vec{v}\cdot\vec{w})$ meaningful? Why or why not? Is the scalar product associative?

> **Definition:** The **direction angles** α, β, and γ of a vector $\vec{v}$ in $\mathbb{R}^3$ are the angles between $\vec{v}$ and $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$, respectively. The **direction cosines** are the cosines of the angles α, β, and γ.



The direction cosines are easy to calculate. For example, we have

$$\cos\alpha = \frac{\vec{v}\cdot\vec{\imath}}{|\vec{v}||\vec{\imath}|} = \frac{v_1}{|\vec{v}|}.$$

This means we can write $\vec{v}$ as

$$\vec{v} = \langle v_1, v_2, v_3\rangle = \langle |\vec{v}|\cos\alpha, |\vec{v}|\cos\beta, |\vec{v}|\cos\gamma\rangle.$$

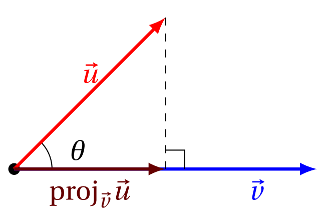
Thus we have $\vec{v} = |\vec{v}|\langle\cos\alpha, \cos\beta, \cos\gamma\rangle$.

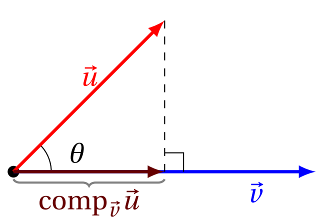
Concept Check: What is a reasonable range of values that the direction angles α, β, γ can take on?

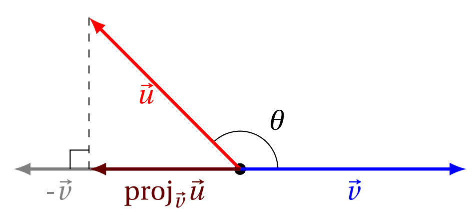
1.4.2 Vector Projections

The scalar product can be used to determine how much of a given vector lies in the same direction as another vector. This is known as the **projection** of a vector onto another.

> **Definition:** The **scalar projection** (or **component**) of $\vec{u}$ along $\vec{v}$ is the signed magnitude of the component of $\vec{u}$ in the direction of $\vec{v}$:
> $$\text{comp}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = |\vec{u}|\cos\theta.$$
> The **vector projection** of $\vec{u}$ onto $\vec{v}$ is the component of $\vec{u}$ along $\vec{v}$ times the unit vector in the direction of $\vec{v}$:
> $$\text{proj}_{\vec{v}}\vec{u} = \left(\text{comp}_{\vec{v}}\vec{u}\right)\frac{\vec{v}}{|\vec{v}|} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}.$$







Concept Check: What is the projection of a vector onto itself?

Example 4: Find the component and vector projection of $\vec{u} = \langle 3, 6, -2\rangle$ onto $\vec{v} = \langle 1, 2, 3\rangle$.

Concept Check: Can the scalar projection of $\vec{u}$ along $\vec{v}$ ever be larger than the length of $\vec{u}$?

Another way to think about projections is to consider them a way to "decompose" a vector into terms of other vectors. For example, every time we use the $\vec{\boldsymbol{\imath}}, \vec{\boldsymbol{\jmath}}, \vec{\boldsymbol{k}}$ notation to describe a vector $\vec{u}$, we are really decomposing $\vec{u}$ into the sum of the projections of $\vec{u}$ onto the standard basis vectors.

Using the vector projection, we can decompose $\vec{u}$ into a sum of a vector parallel to $\vec{v}$ and a vector orthogonal to $\vec{v}$:

$$\vec{u} = \underbrace{\text{proj}_{\vec{v}}\vec{u}}_{\text{parallel to } \vec{v}} + \underbrace{\vec{u} - \text{proj}_{\vec{v}}\vec{u}}_{\text{orthogonal to } \vec{v}} = \underbrace{\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}}_{\text{parallel to } \vec{v}} + \underbrace{\left(\vec{u} - \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}\right)}_{\text{orthogonal to } \vec{v}}$$

Example 5: Write the vector $\vec{u} = \langle 2, -1, 4\rangle$ as the sum of a vector parallel to $\vec{v} = \langle 2, 0, 1\rangle$ and a vector orthogonal to $\vec{v}$.

1.4.3 Applications

As the scalar product gives a quantification of how much of a given vector lies in the direction of another, it is useful in several applications. A classic example of this is the physical concept of work.

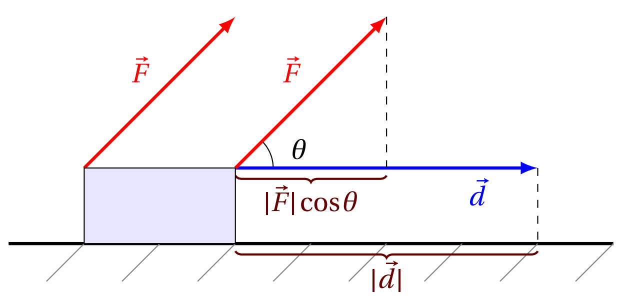
Work

Definition: Work is the result of the application of a force $\vec{F}$ across a displacement $\vec{d}$.

When the force and displacement have the same direction, the work is simply the product of the magnitudes, $W = |\vec{F}||\vec{d}|$. However, if the force is applied at a direction that is not the same as $\vec{d}$, then a vector projection is used to compute the amount of work done. In this case, the work done is given by the scalar product $W = \vec{F}\cdot\vec{d}$.

In the figure on the right, a force in the direction of $\vec{F}$ is applied to at the upper left corner of a rectangular object. The force moves the object across a frictionless surface a displacement $\vec{d}$. The work done by this action is given by

$$W = \vec{F}\cdot\vec{d} = |\vec{F}||\vec{d}|\cos\theta.$$



Example 6: How much work is performed by pulling an object 50ft across a horizontal (frictionless) floor with a constant force of 50lb at an angle of 30° above the horizontal? If instead the 50lb force was being applied horizontally (i.e., if the force was parallel to the floor), how far would the same amount of work move the object?

Dipoles in Electric Fields

Recall that a dipole moment $\vec{\mu}$ is a vector that represents the product of magnitude of charges and the distance of separation between the charges. If we apply an electric field $\vec{E}$ (also a vector quantity) to a dipole moment, then the potential energy of interaction V is given by $V = -\vec{\mu} \cdot \vec{E}$.

Example 7: Calculate the potential energy of interaction between the system of charges $q_1 = 2$, $q_2 = -3$, and $q_3 = 1$ with position vectors $\vec{r}_1 = \langle 3, -2, 1\rangle$, $\vec{r}_2 = \langle 0, 1, 2\rangle$, and $\vec{r}_3 = \langle 0, 2, 1\rangle$, respectively, and the applied electric field $\vec{E} = -2\vec{\boldsymbol{k}}$.

1.4 Review of Concepts

- Terms to know: scalar product, dot product, angle between two vectors, orthogonal, direction angles, direction cosines, projection, vector projection, scalar projection, component,
- Know how to compute the scalar product of and angle between two vectors, and be able to describe a vector in terms of its direction angles and direction cosines.
- Know how to compute the scalar projection and vector projection of one vector onto another.
- Know how to decompose a vector into the sum of a vector parallel and a vector orthogonal to another vector.

1.4 Practice Problems

1. Given $\vec{u} = \langle 1, 1, 1\rangle$, $\vec{v} = \langle -2, 1, 2\rangle$, and $\vec{w} = \langle 0, \sqrt{3}, -1\rangle$, find

a) $\vec{u} \cdot \vec{v}$

b) $\vec{v} \cdot \vec{w}$

c) $\text{proj}_{\vec{v}} \vec{u}$

d) $\text{proj}_{\vec{w}} \vec{v}$

e) the cosine of the angle between $\vec{u}$ and $\vec{w}$

f) $\vec{u}$ as the sum of a vector parallel to $\vec{v}$ and a vector orthogonal to $\vec{v}$.

g) the direction cosines of the vector $\vec{w}$.

2. Find a vector of length 8 in the same direction as $\vec{v} = \langle 2, -1, -2\rangle$.

3. Find r such that the vector from the point $A(1, -1, 3)$ to the point $B(3, 0, 5)$ is orthogonal to the vector from A to the point $P(r, r, r)$.

1.4 Exercises

1. Determine if the following statements are true or false, giving a brief explanation or counterexample in each case.

a) If $\vec{u}$ and $\vec{v}$ are vectors, then the the product of the magnitudes of $\vec{u}$ and $\vec{v}$ is less than or equal to the absolute value of the scalar product of $\vec{u}$ and $\vec{v}$.

b) When $\vec{u}\cdot\vec{v} > 0$, the angle between $\vec{u}$ and $\vec{v}$ is less than $\pi/2$.

c) If $\vec{u} = \langle u_1, u_2, u_3\rangle$ is a vector, then the vector $\vec{w} = \vec{u} - u_2\vec{\boldsymbol{j}}$ is orthogonal to $\vec{\boldsymbol{j}}$.

d) If $\vec{u} = \langle u_1, u_2, u_3\rangle$ is a vector in $\mathbb{R}^3$, then there are exactly two unit vectors that are orthogonal to $\vec{u}$.

e) Two vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $|\vec{u}+\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2$.

2. Find all values of c such that the vectors $\vec{u} = \langle c, -6, 2\rangle$ and $\vec{v} = \langle c^2, c, c\rangle$ are orthogonal.

3. Find two vectors that are orthogonal to $\langle 0,1,1\rangle$ and to each other.

4. Find another vector that has the same projection onto $\vec{v} = \langle 1,1,1\rangle$ as $\vec{u} = \langle 1,2,3\rangle$.

5. Express $\vec{u} = \langle -1,2,3\rangle$ as the sum of a vector parallel to $\vec{v} = \langle 2,1,1\rangle$ and a vector orthogonal to $\vec{v}$.

6. Find the direction angles of the vector $\vec{u} = \vec{\boldsymbol{\imath}} - \vec{\boldsymbol{j}} + \sqrt{2}\vec{\boldsymbol{k}}$.

7. If $(\vec{a} - \vec{b})\cdot\vec{c} = 0$, is it true that $\vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}$? Explain why or why not.

8. Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors. Which of the following expressions are meaningful and which are not? Give the object that results (scalar, vector, etc.) or explain why it is not meaningful.

a) $(\vec{u}\cdot\vec{v})\vec{w}$

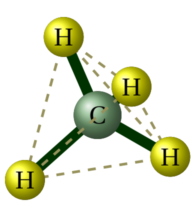
b) $|\vec{u}|(\vec{v}\cdot\vec{w})$

c) $\vec{u}\cdot\vec{v}+\vec{w}$

d) $\vec{u}\cdot(\vec{v}+\vec{w})$

e) $|\vec{u}|\cdot(\vec{v}+\vec{w})$

9. A molecule of methane, CH_4, is structured with the four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid. The **bond angle** is the angle formed by the H-C-H combination - it is the angle between the lines that join the carbon atom to two of the hydrogen atoms. Find the bond angle (in degrees) of methane. (Hint: set the hydrogen atoms at the vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,1)$, then the centroid is at $(1/2,1/2,1/2)$.)



1.4 Answers to Practice Problems

1. $\vec{u} = \langle 1,1,1\rangle$, $\vec{v} = \langle -2,1,2\rangle$, $\vec{w} = \langle 0,\sqrt{3},-1\rangle$,

a) $\vec{u}\cdot\vec{v} = \langle 1,1,1\rangle\cdot\langle -2,1,2\rangle = -2+1+2 = 1$

b) $\vec{v}\cdot\vec{w} = \langle -2,1,2\rangle\cdot\langle 0\sqrt{3},-1\rangle = \sqrt{3}-2$

c) $\mathrm{proj}_{\vec{v}}\vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \dfrac{1}{9}\vec{v} = \left\langle -\dfrac{2}{9},\dfrac{1}{9},\dfrac{2}{9}\right\rangle$

d) $\mathrm{proj}_{\vec{w}}\vec{v}\dfrac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w} = \dfrac{\sqrt{3}-2}{4}\left\langle 0,\sqrt{3},-1\right\rangle = \left\langle 0,\dfrac{3-2\sqrt{3}}{4},-\dfrac{\sqrt{3}-2}{4}\right\rangle$

e) the cosine of the angle between $\vec{u}$ and $\vec{w}$ is given by

$$\cos\theta = \frac{\vec{u}\cdot\vec{w}}{|\vec{u}||\vec{w}|} = \frac{\sqrt{3}-1}{2\sqrt{3}}.$$

f) $\vec{u}$ as the sum of a vector parallel to $\vec{v}$ and a vector orthogonal to $\vec{v}$:

$$\vec{u} = \underbrace{\mathrm{proj}_{\vec{v}}\vec{u}}_{\text{parallel to }\vec{v}} + \underbrace{\vec{u}-\mathrm{proj}_{\vec{v}}\vec{u}}_{\text{orthogonal to }\vec{v}} = \left\langle -\frac{2}{9},\frac{1}{9},\frac{2}{9}\right\rangle + \left\langle \frac{11}{9},\frac{8}{9},\frac{7}{9}\right\rangle$$

g) the direction cosines of the vector $\vec{w}$: note that $|\vec{w}| = 2$ so

$$\vec{w} = |\vec{w}|\langle\cos\alpha,\cos\beta,\cos\gamma\rangle = 2\left\langle 0,\frac{\sqrt{3}}{2},-\frac{1}{2}\right\rangle.$$

Note that this implies $\alpha = \pi/2$, $\beta = \pi/6$, and $\gamma = 2\pi/3$.

2. We have that $|\vec{v}| = \sqrt{2^2+(-1)^2+(-2)^2} = \sqrt{9} = 3$. Any (positive) scalar multiple of $\vec{v}$ points in the same direction as $\vec{v}$, so a vector $\vec{w}$ that has a magnitude of 8 is given by

$$\vec{w} = 8\frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{16}{3},-\frac{8}{3},-\frac{16}{3}\right\rangle.$$

3. $\vec{u} = \overrightarrow{AB} = \langle 2,1,2\rangle$, $\vec{v} = \overrightarrow{AP} = \langle r-1,r+1,r-3\rangle$. If $\vec{u}\cdot\vec{v} = 0$, we have

$$\vec{u}\cdot\vec{v} = 2(r-1)+1(r+1)+2(r-3) = 2r-2+r+1+2r-6 = 5r-7 = 0$$

so $r = 7/5$.

1.5 The Vector Product (a.k.a "Cross Product")

Objectives and Concepts:

- The vector (cross) product of two vectors is a vector that represents the direction and magnitude of rotation from one vector to the next.
- The vector product of two vectors is orthogonal to both vectors.
- Vector products are important in a wide variety of applications, including torque, angular velocity, and angular momentum.

References: OSC-3 §2.4; CET §12.4, TCMB §16.6.

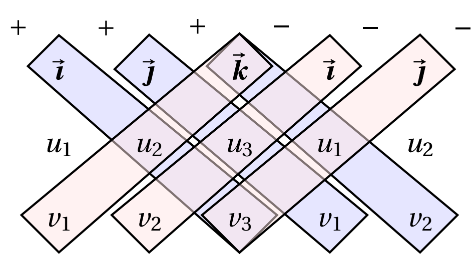
1.5.1 The Vector Product

Definition: Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be vectors in $\mathbb{R}^3$. The **vector product** (or **cross product**) of $\vec{u}$ and $\vec{v}$ is the vector given by

$$\vec{u} \times \vec{v} = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle.$$

Note: the vector product is defined only for vectors in $\mathbb{R}^3$ - there is an analogous quantity for vectors in $\mathbb{R}^2$ (although it is interpreted as a scalar) and it can be generalized to higher dimensions, but in our context a vector product will only be computed for vectors in $\mathbb{R}^3$.

The vector product is somewhat difficult to remember at first - one device that may be helpful is to write the three standard basis vectors in a row, then write the components of $\vec{u}$ in a row under the standard basis, then write the components of $\vec{v}$ under $\vec{u}$. Then copy the first two columns to the right of the third column, so there are 5 columns total. The vector product of $\vec{u}$ and $\vec{v}$ is then computed by multiplying the entries along the diagonals, adding the left-to-right diagonals and subtracting the right-to-left diagonals.



$$\vec{u} \times \vec{v} = u_2 v_3 \vec{\imath} + u_3 v_1 \vec{\jmath} + u_1 v_2 \vec{k} \\ - u_2 v_1 \vec{k} - u_3 v_2 \vec{\imath} - u_1 v_3 \vec{\jmath}$$

Example 1: Compute the following vector products:

a) $(\vec{\imath} + \vec{\jmath} + \vec{k}) \times (\vec{\imath} - 2\vec{\jmath} + 2\vec{k})$

b) $\langle -1,-3,1\rangle \times \langle 1,2,-2\rangle$

c) $\langle 1,2,-2\rangle \times \langle -1,-3,1\rangle$

Note that the last two examples show something interesting - the vector product depends on the order of the vectors! In fact, we have that $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$. Thus the vector product is *anticommutative*. Below are some other properties of the vector product.

Properties of the Vector Product: Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^3$ and let $a \in \mathbb{R}$. Then we have:

1. $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$ (anticommutativity of $\times$)
2. $(a\vec{u})\times\vec{v} = a(\vec{u}\times\vec{v}) = \vec{u}\times(a\vec{v})$ (associativity of scalar multiplication)
3. $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v}+\vec{u}\times\vec{w}$ (left distribution of $\times$ across $+$)
4. $(\vec{u}+\vec{v})\times\vec{w} = \vec{u}\times\vec{w}+\vec{v}\times\vec{w}$ (right distribution of $\times$ across $+$)

Also, by using vectors of zeros and ones, it is easy to show the following:

$$\begin{array}{ccc}
\vec{\imath}\times\vec{\jmath} = \vec{k} & \vec{\jmath}\times\vec{k} = \vec{\imath} & \vec{k}\times\vec{\imath} = \vec{\jmath} \\
\vec{\jmath}\times\vec{\imath} = -\vec{k} & \vec{k}\times\vec{\jmath} = -\vec{\imath} & \vec{\imath}\times\vec{k} = -\vec{\jmath} \\
\vec{\imath}\times\vec{\imath} = \vec{0} & \vec{\jmath}\times\vec{\jmath} = \vec{0} & \vec{k}\times\vec{k} = \vec{0}
\end{array}$$

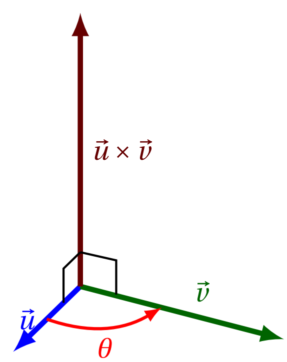
Using these identities and properties, the vector product of the vectors $\vec{u}$ and $\vec{v}$ can be written as:

$$\begin{aligned}
\vec{u}\times\vec{v} &= \left(u_1\vec{\imath}+u_2\vec{\jmath}+u_3\vec{k}\right)\times\left(v_1\vec{\imath}+v_2\vec{\jmath}+v_3\vec{k}\right) \\
&= u_1v_1\underbrace{\vec{\imath}\times\vec{\imath}}_{=\vec{0}}+u_1v_2\underbrace{\vec{\imath}\times\vec{\jmath}}_{=\vec{k}}+u_1v_3\underbrace{\vec{\imath}\times\vec{k}}_{=-\vec{\jmath}}+u_2v_1\underbrace{\vec{\jmath}\times\vec{\imath}}_{=-\vec{k}}+u_2v_2\underbrace{\vec{\jmath}\times\vec{\jmath}}_{=\vec{0}}+u_2v_3\underbrace{\vec{\jmath}\times\vec{k}}_{=\vec{\imath}} \\
&\quad +u_3v_1\underbrace{\vec{k}\times\vec{\imath}}_{=\vec{\jmath}}+u_3v_2\underbrace{\vec{k}\times\vec{\jmath}}_{=-\vec{\imath}}+u_3v_3\underbrace{\vec{k}\times\vec{k}}_{=\vec{0}} \\
&= (u_2v_3-u_3v_2)\vec{\imath}+(u_3v_1-u_1v_3)\vec{\jmath}+(u_1v_2-u_2v_1)\vec{k}.
\end{aligned}$$

Geometrically, the vector product of two vectors $\vec{u}$ and $\vec{v}$ results in a vector $\vec{u}\times\vec{v}$ that is orthogonal to *both* $\vec{u}$ and $\vec{v}$. The direction of the vector is given by the **right-hand rule** - as the fingers of your right hand curl from $\vec{u}$ to $\vec{v}$, your thumb points in the direction of $\vec{u}\times\vec{v}$. The magnitude of the vector product vector is

$$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$$

where $0 \le \theta \le \pi$.



If $\vec{u}$ and $\vec{v}$ are parallel, then $\vec{u} \times \vec{v} = \vec{0}$. If $\vec{u}$ and $\vec{v}$ are orthogonal, then $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|$.

The vector product can then be used to compute a vector that is orthogonal to two given vectors.

Example 2: Find a unit vector that is orthogonal to both $\vec{u} = \langle 2, 3, -5\rangle$ and $\vec{v} = \langle -1, 1, -1\rangle$.

Example 3: Find the magnitude of $\vec{u} \times \vec{v}$ if $\vec{u} = \langle 1, -1, 3\rangle$ and $\vec{v} = \langle -2, 0, -1\rangle$.

Concept Check: What is the relationship between $\vec{u} \cdot \vec{v}$ and $|\vec{u} \times \vec{v}|$?

Concept Check: If $\vec{u} \cdot \vec{v} = 0$ and $\vec{u}$ and $\vec{v}$ are nonzero vectors, is it possible for $\vec{u} \times \vec{v} = \vec{0}$?

1.5.2 The Scalar Triple Product

Definition: The **scalar triple product** of the vectors $\vec{u}, \vec{v}$, and $\vec{w}$ is the scalar quantity $\vec{u} \cdot (\vec{v} \times \vec{w})$.

Though this appears to treat the first factor $\vec{u}$ differently than the others, this is not in fact true; instead all are treated equally, and all that matters is the order of the three factors, as shown by multiplying out to get the following explicit formula:

Theorem: For the vectors $\vec{u} = \langle u_1, u_2, u_3\rangle$, $\vec{v} = \langle v_1, v_2, v_3\rangle$ and $\vec{w} = \langle w_1, w_2, w_3\rangle$, the scalar triple product is

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = u_1 v_2 w_3 + u_2 v_3 w_1 + u_3 v_1 w_2 - u_1 v_3 w_2 - u_2 v_1 w_3 - u_3 v_2 w_1$$

From this we get

Properties of the Scalar Triple Product: For vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$:

1. It does not matter where the parentheses are: $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
2. "Rotating" the factors does not change the value: $\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{v} \cdot (\vec{w} \times \vec{u}) = \vec{w} \cdot (\vec{u} \times \vec{v})$
3. Changing the order negates the value: $\vec{u} \cdot (\vec{w} \times \vec{v}) = -\vec{u} \cdot (\vec{v} \times \vec{w})$, $\vec{v} \cdot (\vec{u} \times \vec{v}) = -\vec{u} \cdot (\vec{v} \times \vec{w})$ and so on.

Also, the value is zero if and only if the three vectors lie in a common plane.

The equations follow (with a bit of thought!) from the above explicit formula for this triple product; the final property holds because being zero means that $\vec{u}$ is orthogonal to the product $\vec{v} \times \vec{w}$ which is in turn orthogonal to both $\vec{v}$ and $\vec{w}$.

The sign of the scalar triple product is related to whether, loosely speaking, the three vectors are arranged in "right-handed" of "left-handed" order. For example, the right-handed triple $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$ has scalar triple product 1, but swapping any two of those gives a left-handed triple and a scalar triple product of -1, whereas "rotating" them to order $\vec{\jmath}, \vec{k}, \vec{\imath}$ keeps the triple right handed and the product's value is still 1.

We will see more about the scalar triple product in Sections 2.3 and 2.4, in relation to the **determinant** and the concept of **linear dependence**.

1.5.3 (†) The Vector Triple Product

Definition: The **vector triple product** of the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ is the vector $\vec{u} \times (\vec{v} \times \vec{w})$.

The vector triple product has applications in orthogonal sets of vectors, rotational reference frames, and curvature of paths in space. The vector triple product also has several interesting properties.

Properties of the Vector Triple Product: Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^3$ and let $a \in \mathbb{R}$. Then we have:

1. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$
2. $(\vec{u} \times \vec{v}) \times \vec{w} = -\vec{w} \times (\vec{u} \times \vec{v}) = -(\vec{w} \cdot \vec{v})\vec{u} + (\vec{w} \cdot \vec{u})\vec{v}$
3. $\vec{u} \times (\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u}) + \vec{w} \times (\vec{u} \times \vec{v}) = \vec{0}$
4. $(\vec{u} \times \vec{v}) \times \vec{w} = \vec{u} \times (\vec{v} \times \vec{w}) - \vec{v} \times (\vec{u} \times \vec{w})$

The fourth property above indicates that $\times$ is not associative in general. The first property above is often written with vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ as $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$ and the mnemonic "BAC minus CAB" may be helpful in remembering the formula.

1.5.4 (†) A Few Other Identities

Some other useful identities involving the vector product are

$$(\vec{u} \times \vec{v}) \cdot (\vec{u} \times \vec{v}) = |\vec{u} \times \vec{v}|^2 = |\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2,$$

which relates the dot and vector products (you can solve for $\vec{u} \cdot \vec{v}$ to get another definition of the dot product), and

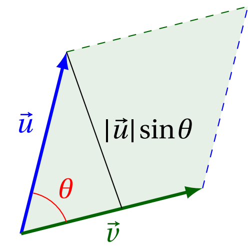
$$(\vec{t} \times \vec{u}) \cdot (\vec{v} \times \vec{w}) = (\vec{t} \cdot \vec{v})(\vec{u} \cdot \vec{w}) - (\vec{t} \cdot \vec{w})(\vec{u} \cdot \vec{v}),$$

which is sometimes referred to as Lagrange's identity.

1.5.5 Applications of the Vector Product

Area of a Parallelogram

There are several geometric applications of the vector product. First, the magnitude of $\vec{u} \times \vec{v}$ gives the area of the parallelogram formed by $\vec{u}$ and $\vec{v}$.



Area $= |\vec{v}||\vec{u}| \sin\theta$

Example 4: Find the area of the parallelogram formed by the vectors $\vec{u} = \langle 1, 0, 1 \rangle$ and $\vec{v} = \langle 2, 4, 2 \rangle$.

Concept Check: What is the area of the triangle formed by the points P, Q, and R in $\mathbb{R}^3$?

Volume of a Parallelepiped

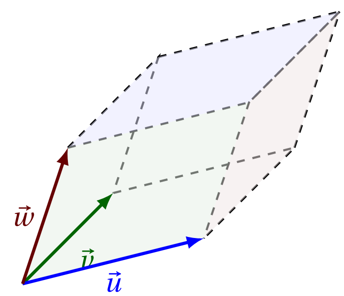
The absolute value of the scalar triple product gives the volume of a parallelepiped formed by the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$.

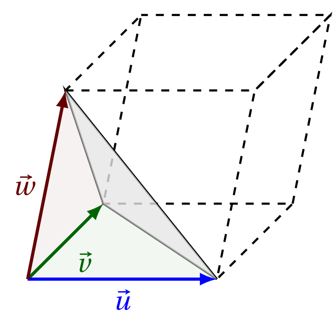
Example 5: Find the volume of the parallelepiped formed by the vectors $\vec{u} = \langle 1, 0, 1 \rangle$, $\vec{v} = \langle 2, 4, 2 \rangle$, and $\vec{w} = \langle 7, 2, -4 \rangle$.

Note that it does not matter what order the three vectors are written when forming the parallelepiped:

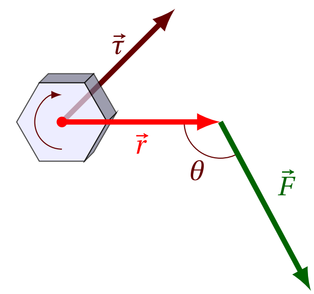
$$|\vec{u} \cdot (\vec{v} \times \vec{w})| = |\vec{v} \cdot (\vec{u} \times \vec{w})| = |\vec{w} \cdot (\vec{u} \times \vec{v})|$$

The tetrahedron formed by the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ has volume 1/6 of the volume of the corresponding parallelepiped.





Torque



Torque, or **moment of force**, is the tendency of a force to rotate an object about an axis. Just as a force is a push or a pull, a torque can be thought of as a twist to an object. Consider a rigid body situated at the origin of a coordinate system. If a force $\vec{F}$ acts on a rigid body at a point whose position is given by $\vec{r}$, then the rotational force exerted on the body is the **torque** $\vec{\tau}$ given by $\vec{\tau} = \vec{r} \times \vec{F}$. The torque vector has a magnitude of

$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = |\vec{r}||\vec{F}|\sin\theta.$$

Since $|\vec{F}|$ has units of Newtons (N), or kg-m/s^2, the magnitude of torque has units of Newton-meters (N-m).

Example 6: A wrench 30cm long lies along the positive y-axis and grips a bolt at the origin. A force $\vec{F}$ is applied in the direction of $\langle 0, 3, -4\rangle$ at the end of the wrench. Find the magnitude of the force ($|\vec{F}|$) required to produce 100 N-m of torque on the bolt.

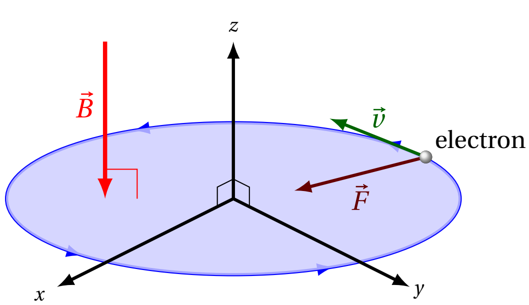
Magnetic Force on Moving Charges

When an electric charge q that is in motion with velocity $\vec{v}$ (for example, current in a wire or an electron rotating about a nucleus) is subjected to a magnetic field $\vec{B}$, the resulting force $\vec{F}$ acting on the charge is given by $\vec{F} = q(\vec{v} \times \vec{B})$. The standard unit of magnetic charge is the Tesla (T) that is defined by 1T=1kg/(C-s) where C is the unit of charge in coulombs and s is time in seconds. Thus, since

$$|\vec{F}| = |q||\vec{v}||\vec{B}|\sin\theta,$$

the magnitude of magnetic force has units of kg-m/s^2.

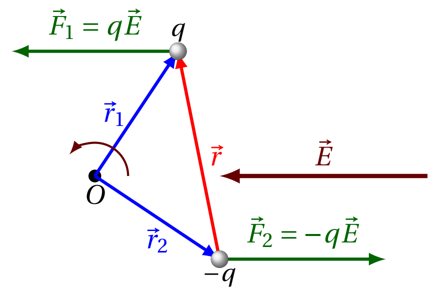
Example 7: An electron with mass 9.1×10^{-31}kg and a charge of -1.6×10^{-19} Coulombs travels in a circular path in a magnetic field of 0.05 Tesla that is orthogonal to the path of the electron. Assume the electron loses no energy and the force $\vec{F}$ acts as a centripetal force with magnitude $|\vec{F}| = m|\vec{v}|^2/r$ where r is the radius of the orbit. If the radius of the path is 0.002m, what is the speed of the electron? (Note that the centripetal force vector $\vec{F}$ is orthogonal to the particle velocity vector $\vec{v}$ in a circular orbit. In the figure below, both $\vec{v}$ and $\vec{F}$ are parallel to the xy-plane).



Dipole Rotation

The electric dipole $\vec{\mu} = q\vec{r}$ in an electric field $\vec{E}$ experiences a torque $\vec{\tau} = \vec{\mu} \times \vec{E}$ that tends to align the dipole along the direction of the electric field. In the figure to the right, the total torque about the origin O is

$$\vec{\tau} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 = q(\vec{r}_1 - \vec{r}_2) \times \vec{E} = \vec{\mu} \times \vec{E}.$$



Example 8: Calculate the torque experienced by the system of charges $q_1 = 2$, $q_2 = -3$, and $q_3 = 1$ at positions $\vec{r}_1 = \langle 3, -2, 1\rangle$, $\vec{r}_2 = \langle 0, 1, 2\rangle$, and $\vec{r}_3 = \langle 0, 2, 1\rangle$, respectively, in the electric field $\vec{E} = -\vec{\boldsymbol{k}}$.

1.5 Review of Concepts

- Terms to know: vector product, cross product, anticommutative, right-hand rule, scalar triple product, vector triple product, parallelepiped, torque, moment of force, dipole rotation.
- Know how to compute the vector product of two vectors in $\mathbb{R}^3$ and use it to compute areas and volumes of associated geometric objects.
- Know how to use the vector product in various application areas.

1.5 Practice Problems

1. Given $\vec{u} = \langle 1, 1, 1\rangle$, $\vec{v} = \langle -2, 1, 2\rangle$, and $\vec{w} = \langle 0, \sqrt{3}, -1\rangle$, find

a) $\vec{u} \times \vec{v}$

b) $\vec{v} \times \vec{w}$

c) $\vec{v} \times \operatorname{proj}_{\vec{v}} \vec{u}$

d) $|\vec{w} \times \vec{v}|$

e) a unit vector orthogonal to $\vec{u}$ and $\vec{\boldsymbol{\imath}} - \vec{\boldsymbol{\jmath}} + \vec{\boldsymbol{k}}$

f) the angle between $\vec{w}$ and $\langle \sqrt{2}, -2, 0\rangle$.

2. Find the area of the triangle formed by the points $P(0, 2, -1)$, $Q(-3, 2, 1)$, and $R(2, 0, -3)$.

1.5 Exercises

1. Let $\vec{t}$, $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^3$. Which of the following expressions are meaningful and which are not? Give the object that results (scalar, vector, etc.) or explain why it is not meaningful.

a) $(\vec{u} \cdot \vec{v}) \times \vec{w}$

b) $(\vec{u} \times \vec{v})\vec{w}$

c) $|\vec{u}|(\vec{v} \times \vec{w})$

d) $(\vec{u} \cdot \vec{v})\vec{t} + \vec{w}$

e) $(\vec{u} + \vec{t}) \times (\vec{v} + \vec{w})$

f) $\vec{t} \times (\vec{u} \cdot \vec{v})\vec{w}$

2. Suppose that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^3$ and that $\vec{u} \neq \vec{0}$. Determine if the statements below are true or false, giving a brief explanation or counterexample.

a) If $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$, then $\vec{v} = \vec{w}$.

b) If $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, then $\vec{v} = \vec{w}$.

c) If $\vec{u} \times \vec{v} = \vec{0}$ then $\vec{u}$ and $\vec{v}$ must be the same vector.

3. Determine if the following are true or false, giving a brief explanation or counterexample.

a) If $\vec{u} \times \vec{v} = \vec{0}$ and $\vec{u} \cdot \vec{v} = 0$, then either $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$.

b) $\vec{u} \times (\vec{u} \times \vec{v}) = \vec{0}$.

c) $|\vec{u} \times \vec{v}|$ is less than both $|\vec{u}|$ and $|\vec{v}|$.

d) For any vector $\vec{v} \in \mathbb{R}^3$, $\vec{v} \cdot (\vec{v} \times \vec{v}) = 0$.

e) The magnitude of $\vec{u} \times \vec{v}$ is equal to the absolute value of $\vec{u} \cdot \vec{v}$.

4. If $\vec{u}$ and $\vec{v}$ are unit vectors with an angle between them of $\pi/3$, what is $|\vec{u} \times \vec{v}|$?

5. Find the area of the triangle with vertices $A(1,2,3)$, $B(5,1,5)$, and $C(2,3,3)$.

6. Find two unit vectors orthogonal to both $\vec{\boldsymbol{\imath}} + \vec{\boldsymbol{\jmath}} + \vec{\boldsymbol{k}}$ and $2\vec{\boldsymbol{\imath}} + \vec{\boldsymbol{k}}$.

7. A particle with a positive unit charge $(q = 1)$ enters a constant magnetic field $\vec{B} = \vec{\boldsymbol{\imath}} + \vec{\boldsymbol{\jmath}}$ with a velocity $\vec{v} = 20\vec{\boldsymbol{k}}$. Find the magnitude and direction of the magnetic force on the particle.

8. A charge q moving with velocity $\vec{v}$ in the presence of both a magnetic field $\vec{B}$ and an electric field $\vec{E}$ experiences a total force of $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$ called the **Lorentz force**. Calculate the force acting on the charge $q = 3$ moving with velocity $\vec{v} = \langle 2,3,1\rangle$ in the presence of an electric field $\vec{E} = 2\vec{\boldsymbol{\imath}}$ and magnetic field $\vec{B} = 3\vec{\boldsymbol{\jmath}}$.

1.5 Answers to Practice Problems

1. $\vec{u} = \langle 1,1,1\rangle$, $\vec{v} = \langle -2,1,2\rangle$, and $\vec{w} = \langle 0,\sqrt{3},-1\rangle$

a) $\vec{u} \times \vec{v} = \langle 1,1,1\rangle \times \langle -2,1,2\rangle = \langle 1,-4,3\rangle$.

b) $\vec{v} \times \vec{w} = \langle -2,1,2\rangle \times \langle 0,\sqrt{3},-1\rangle = \langle -1-2\sqrt{3},-2,-2\sqrt{3}\rangle$

c) $\vec{v} \times \text{proj}_{\vec{v}}\vec{u} = \vec{0}$ because the vector $\text{proj}_{\vec{v}}\vec{u}$ is a scalar multiple of $\vec{v}$.

d) Using part b):

$$\begin{aligned}|\vec{w} \times \vec{v}| = |\vec{v} \times \vec{w}| &= |\langle -1-2\sqrt{3},-2,-2\sqrt{3}\rangle| \\ &= \sqrt{(-1-2\sqrt{3})^2 + (-2)^2 + (-2\sqrt{3})^2} = \sqrt{1+4\sqrt{3}+12+4+12} = \sqrt{29+4\sqrt{3}}.\end{aligned}$$

e) A vector orthogonal to $\vec{u}$ and $\vec{\boldsymbol{\imath}} - \vec{\boldsymbol{\jmath}} + \vec{\boldsymbol{k}}$ is given by

$$\langle 1,1,1\rangle \times \langle 1,-1,1\rangle = \langle 2,0,-2\rangle,$$

and a unit vector in the same direction is

$$\frac{1}{|\langle 2,0,-2\rangle|}\langle 2,0,-2\rangle = \frac{1}{\sqrt{8}}\langle 2,0,-2\rangle = \left\langle \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right\rangle.$$

f) The angle between $\vec{w}$ and $\langle \sqrt{2}, -2, 0\rangle$ can be found using the dot product:

$$\cos\theta = \frac{\vec{w}\cdot\langle\sqrt{2},-2,0\rangle}{|\vec{w}||\langle\sqrt{2},-2,0\rangle|} = \frac{-2\sqrt{3}}{2\sqrt{6}} = \frac{-\sqrt{3}}{\sqrt{6}} = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

Thus the angle between the two vectors is $3\pi/4$.

2. Find the area of the triangle formed by the points $P(0,2,-1)$, $Q(-3,2,1)$, and $R(2,0,-3)$.

The area of the triangle formed by the points P, Q, and R will be given by $\frac{1}{2}|\vec{u}\times\vec{v}|$ where $\vec{u} = \overrightarrow{PQ}$ and $\vec{v} = \overrightarrow{PR}$ (actually, any two nonparallel vectors formed by the vertices of the triangle will work). Now $\vec{u} = \langle -3,0,2\rangle$ and $\vec{v} = \langle 2,-2,-2\rangle$, so

$$\vec{u}\times\vec{v} = \langle -3,0,2\rangle \times \langle 2,-2,-2\rangle = \langle 4,-2,6\rangle.$$

Then the area of the triangle is

$$\frac{1}{2}|\vec{u}\times\vec{v}| = \frac{1}{2}|\langle 4,-2,6\rangle| = \frac{1}{2}\sqrt{16+4+36} = \frac{1}{2}\sqrt{56} = \sqrt{14}.$$

1.6 Lines and Planes

Objectives and Concepts:

- Lines in two or three dimensions are determined by a point on the line and a direction. We can use a vector equation, parametric equations, or symmetric equations to represent a line.
- In $\mathbb{R}^2$, two distinct lines are either parallel or they intersect at a point. In $\mathbb{R}^3$, two lines may be parallel, skew(†), or intersecting.
- Planes in three dimensions are determined by a point on the plane and a normal vector. We can use a vector equation or a scalar equation to represent a plane.
- Parallel planes have parallel normal vectors. Two nonparallel planes will intersect in a line. The angle between two planes is the angle between their normal vectors.

References: OSC-3 §2.5; CET §12.5.

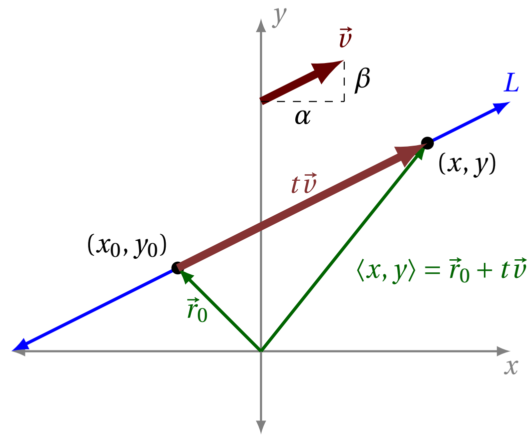
1.6.1 Lines in $\mathbb{R}^2$

We have known for some time that a line in $\mathbb{R}^2$ is determined by a point on the line and the **slope** of the line. Let α represent the change in x and β represent the corresponding change in y of the slope (if the line is vertical then $\alpha = 0$). The slope $m = \beta/\alpha$ can be interpreted as a direction vector: $\vec{v} = \langle \alpha, \beta \rangle$. If $\alpha \neq 0$ and the line passes through the point (x_0, y_0), then we know its equation is given by

$$y - y_0 = m(x - x_0) = \frac{\beta}{\alpha}(x - x_0),$$

and if also $\beta \neq 0$ this can also be written in the **symmetric form**

$$\frac{x - x_0}{\alpha} = \frac{y - y_0}{\beta}.$$



Two other useful forms come from clearing the denominators in the above, giving $\beta(x - x_0) - \alpha(y - y_0) = 0$, or defining $a = \beta, b = -\alpha, d = ax_0 + by_0 = \beta x_0 - \alpha y_0$,

$$a(x - x_0 + b(y - y_0) = 0 \text{ or } ax + by = d. \tag{1.1}$$

Vectors give some other nice ways to describe a line. Any point on the line (x, y) lies somewhere in the direction of the vector $\vec{v}$ from point (x_0, y_0) (so we call $\vec{v}$ the **direction vector** of the line). In other words, there is some scalar multiple of $\vec{v}$ such that the position vector of the point (x,y) is the sum of the scaled $\vec{v}$ and the position vector $\vec{r}_0$ of the known point (x_0,y_0). Thus there is some value of a **parameter** t (a scalar) such that the position vector for the point (x,y) is given by

$$\langle x,y\rangle = \vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle x_0,y_0\rangle + t\langle \alpha,\beta\rangle = \langle x_0+\alpha t, y_0+\beta t\rangle.$$

In this case, the equation $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ is known as the **vector equation** of the line L, and the two equations $x = x_0+\alpha t$, $y = y_0+\beta t$ are the **parametric equations** of L. Note that if $\alpha = 0$, then the line is vertical, so in this case the x-coordinate of any point on the line is x_0. This yields the vector equation $\vec{r}(t) = \langle x_0,y_0\rangle + t\langle 0,\beta\rangle$, and thus the parametric equations are $x = x_0$, $y = y_0+\beta t$. If $\beta = 0$, then y is always y_0, and the vector equation is $\vec{r}(t) = \langle x_0+\alpha t, y_0\rangle$, and the parametric equations are $x = x_0+\alpha t$, $y = y_0$.

Also, defining $\vec{n} = \langle a,b\rangle$, Eq. (1.1) can be written as

$$(\vec{r} - \vec{r}_0)\cdot\vec{n} = 0,$$

so that vector $\vec{n}$ is "orthogonal" or **normal** to the line.

Definition: Let L represent the line in $\mathbb{R}^2$ passing through (x_0,y_0) with direction vector $\vec{v} = \langle\alpha,\beta\rangle$. Then the following equations represent all points on L:

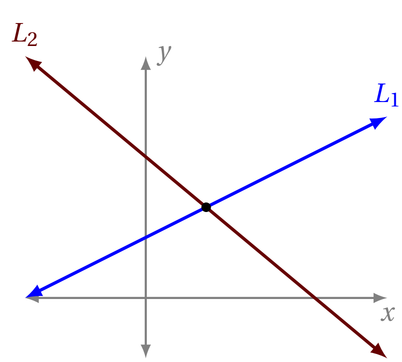
Symmetric Equations:	$\dfrac{x-x_0}{\alpha} = \dfrac{y-y_0}{\beta}$
Vector Equation:	$\vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle x_0,y_0\rangle + t\langle\alpha,\beta\rangle$
Parametric Equations:	$x = x_0+\alpha t, y = y_0+\beta t$
Scalar Equation:	$a(x-x_0)+b(y-y_0) = 0$, where $a = \beta$, $b = -\alpha$

with the obvious restriction that the first form requires that both α and β are non-zero.

Example 1: Find the symmetric, vector, and parametric equations of the line passing through the points $(-2,3)$ and $(4,1)$. Then find an alternate *parametrization* of the line (i.e., another set of parametric equations that describe the same line).

Concept Check: How many different parametrizations does a line have?

Two lines in $\mathbb{R}^2$ are either parallel (i.e., have the same slope) or they intersect at a single point. The point of intersection (x,y) must lie on both lines, so the coordinates of the point of intersection must satisfy the equations of both lines. However, if using the parametric or vector equations of the lines, note that the values of t at the point of intersection may be different for each. **Thus, if using parametric equations or vector equations to solve for a point of intersection, then you may want to use a different variable (e.g. s) for the parameter of one of the lines.**

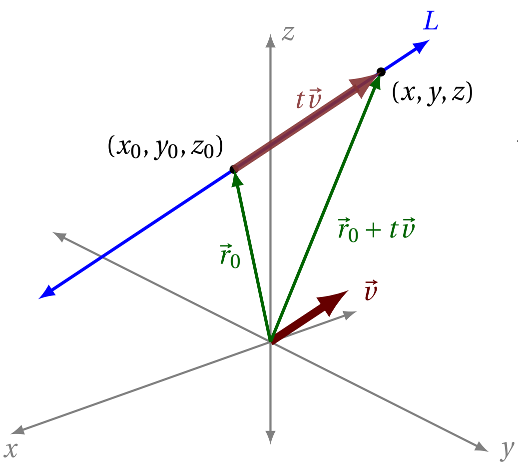


Example 2: Find the coordinates of the point of intersection of the lines L_1 and L_2 given by

$$L_1: \vec{r}_1(t) = \langle 3-t, 2+t\rangle, \quad \text{and} \quad L_2: \vec{r}_2(s) = \langle 4+2s, -3s\rangle.$$

1.6.2 Lines in $\mathbb{R}^3$

Lines in three spatial dimensions are described in the same way as they are in $\mathbb{R}^2$. One major difference however is that the concept of slope does not readily extend to $\mathbb{R}^3$ - the change in y with respect to x is no longer sufficient to describe the direction of the line. Thus in order to describe a line, we must have a point (x_0, y_0, z_0) on the line and a vector $\vec{v} = \langle \alpha, \beta, \gamma\rangle$ that gives the direction. The numbers α, β, γ are known as the **direction numbers** of the line.



Symmetric Equations: $\dfrac{x-x_0}{\alpha} = \dfrac{y-y_0}{\beta} = \dfrac{z-z_0}{\gamma}$

Vector Equation: $\vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle x_0, y_0, z_0\rangle + t\langle \alpha, \beta, \gamma\rangle$

Parametric Equations: $x = x_0 + \alpha t, y = y_0 + \beta t, z = z_0 + \gamma t$

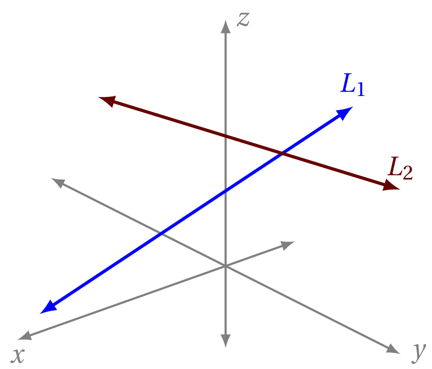
Note that if one of the direction numbers is 0 the symmetric equations will be slightly different. For example, if $\alpha = 0$, then L would be represented by the symmetric equations

$$x = x_0, \qquad \frac{y-y_0}{\beta} = \frac{z-z_0}{\gamma}.$$

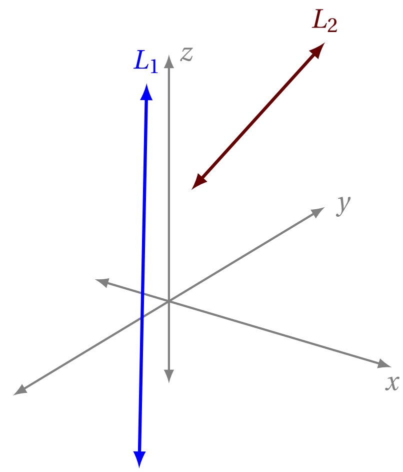
Example 3: Find the symmetric, vector, and parametric equations of the line through the points $(6,1,-3)$ and $(2,4,5)$.

While it is certainly the case that parallel lines do not intersect, **in $\mathbb{R}^3$ there are nonparallel lines that do not intersect**.

Definition: (†) Two lines L_1 and L_2 in $\mathbb{R}^3$ are **skew** if they are not parallel and do not intersect.



The plot on the left gives the perspective from the first octant - you are looking at the axes from positive x, y, and z. The plot on the right is of the same two lines, but now you are looking from positive x, negative y, and positive z (the fourth octant). While it is not clear from the plot on the left that the lines do not intersect, the plot on the right shows they are in fact skew.



Example 4: Determine whether the following lines are parallel, skew(†), or intersecting. If they intersect, find the point of intersection.

a) $L_1: \quad x = -6t,\ y = 1 + 9t,\ z = -3t, \qquad L_2: \quad x = 1 + 2s,\ y = 4 - 3s,\ z = s$

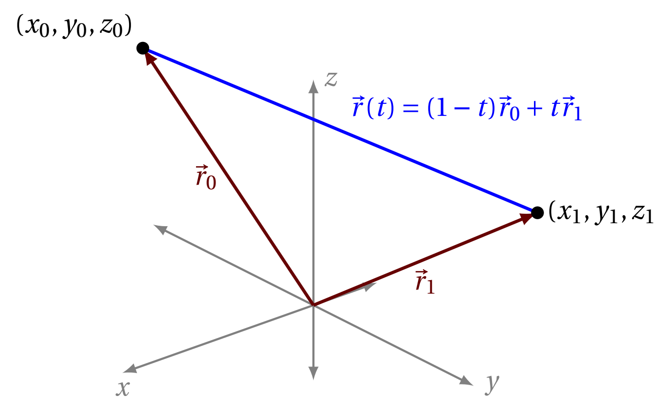
b) $L_1: \quad \dfrac{x}{1} = \dfrac{y-1}{2} = \dfrac{z-2}{3}, \qquad L_2: \quad \dfrac{x-3}{-4} = \dfrac{y-2}{-3} = \dfrac{z-1}{2}$

c) $L_1: \quad \vec{r}(t) = \langle 1-t, 2+t, -2-4t\rangle, \qquad L_2: \quad \vec{r}(s) = \langle -2s, 2s+3, -8s-6\rangle$

To parametrize a **line segment** between two points (x_0, y_0, z_0) and (x_1, y_1, z_1), one can simply take a **convex combination** of the position vectors $\vec{r}_0$ and $\vec{r}_1$ of the two points:

$$\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1, \qquad 0 \le t \le 1.$$

This just amounts to taking a weighted average of the two position vectors, with the extremes being $\vec{r}_0$ when $t = 0$ and $\vec{r}_1$ when $t = 1$.

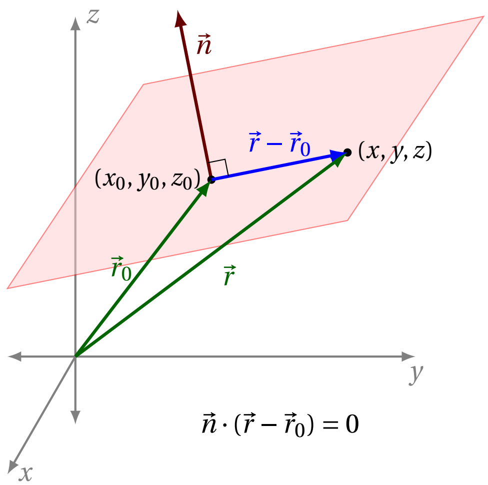


1.6.3 Planes

The "direction" of a plane in $\mathbb{R}^3$ is a little more difficult to describe. In fact, at first glance, it seems like there are *two* directions for a plane. Thus, the idea of using a point in the plane and a single vector that lies in the plane is insufficient for describing it completely. However, since a plane really spans two directions, and there are only three *independent* directions in $\mathbb{R}^3$, a single vector can be used to describe a plane.

Definition: A vector $\vec{n}$ is a **normal vector** to a plane if it is orthogonal to all vectors that lie in the plane.

Concept Check: How many different normal vectors does a plane have?



Examining the figure on the left, we see that a normal vector $\vec{n}$ must satisfy the relationship $\vec{n}\cdot(\vec{r}-\vec{r}_0)=0$ where $\vec{r}_0$ is the position vector of a given point (x_0,y_0,z_0) on the plane and $\vec{r}=\langle x,y,z\rangle$ is the position vector of any other point (x,y,z) on the plane. If we write $\vec{n}=\langle a,b,c\rangle$, then we have

$$\begin{aligned}\vec{n}\cdot(\vec{r}-\vec{r}_0) &= \langle a,b,c\rangle\cdot\big(\langle x,y,z\rangle-\langle x_0,y_0,z_0\rangle\big)\\ &= a(x-x_0)+b(y-y_0)+c(z-z_0)=0.\end{aligned}$$

This gives us two ways of representing the plane:

Vector Equation: $\vec{n}\cdot(\vec{r}-\vec{r}_0)=0$

Scalar Equation: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

The scalar equation is also known as the **point-normal form** of the equation of a plane. The scalar equation is what we most commonly use to represent a plane. In fact, any single linear equation in x, y, and z, such as $3x-y+2z=4$, represents a plane in $\mathbb{R}^3$. If you are given a scalar equation, it is very easy to spot the normal vector, as all you need to do is look at the coefficients of the variables. Also note that if $\vec{n}$ is normal to a plane, then so is $-\vec{n}$, or any other nonzero scalar multiple of $\vec{n}$.

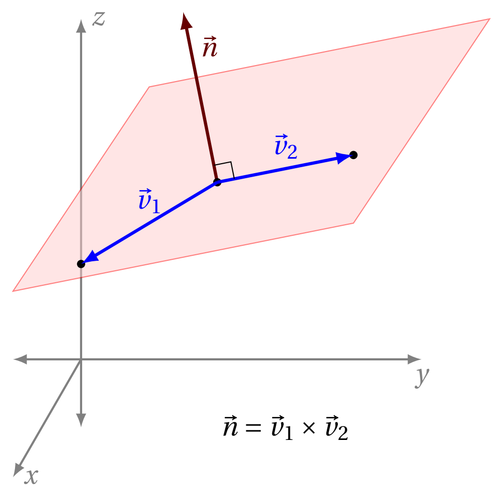
Example 5: Find the scalar equation of the plane passing through the point $(3,-1,7)$ with normal vector $\vec{n} = \langle 4,2,-5\rangle$.

Example 6: Determine if the planes $3x-4y+5z=0$ and $-6x+8y-10z=4$ are parallel.

Concept Check: (†)Can a plane contain skew lines?

Now, sometimes you are not given a point on the plane and its normal vector. Nevertheless, with enough information you can find a point and normal and find the equation of the plane.

- If you are given a point in the plane and two vectors $\vec{v}_1$ and $\vec{v}_2$ that lie in the plane (are parallel to the plane), then a normal vector can be found by computing $\vec{v}_1 \times \vec{v}_2$.
- If you are given two points P_1 and P_2 and a single vector $\vec{v}$ in the plane, then you can find another vector in the plane by finding the vector from P_1 to P_2, provided it is not parallel to $\vec{v}$, and then find the normal using the cross product.
- If you are given three *noncollinear* points in the plane, you can find two nonparallel vectors in the plane, and then find the normal using the cross product.

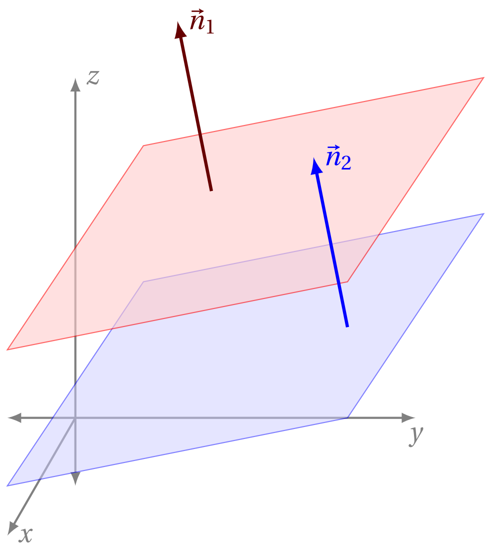


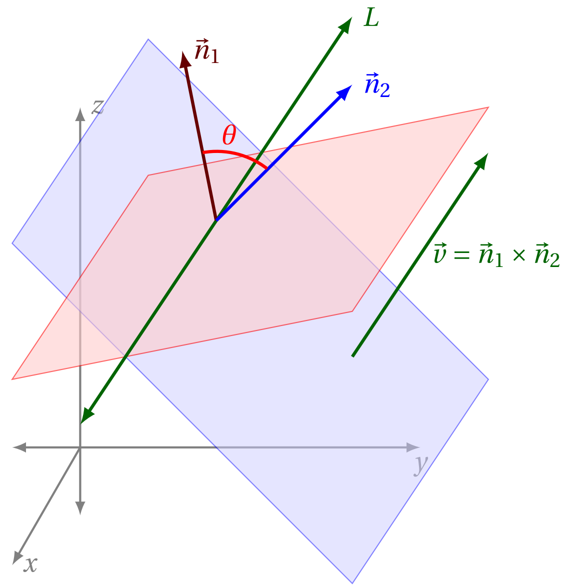
$\vec{n} = \vec{v}_1 \times \vec{v}_2$

Example 7: Find an equation of the plane that contains the point $(2,0,3)$ and the line $x=-1+t$, $y=t$, $z=-4+2t$.

Example 8: Find an equation of the plane that contains the line $x=-2+3t$, $y=4+2t$, $z=3-t$ and is perpendicular to the plane $x-2y+z=5$.

Concept Check: Is it possible to find the equation of a plane if you know the symmetric equations of two intersecting lines that lie in the plane? How would you find the plane equation?





Two planes in $\mathbb{R}^3$ are either parallel, or they intersect in a line. If they are parallel, then their normal vectors are also parallel, so the equations of the plans can be written as $ax + by + cz = d_1$ and $ax + by + cz = d_2$. If they intersect, then their normal vectors (call them $\vec{n}_1$ and $\vec{n}_2$) will have an angle θ between them where $0 \le \theta \le \pi/2$. Then, since the angle between $\vec{n}_1$ and $\vec{n}_2$ is the same as the angle between the two planes, we have that the *acute* angle between two planes satisfies

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}.$$

But what about the line of intersection L? How can we find it? Recall that we need a point on the line and a direction vector $\vec{v}$. One of the things you can see in the figure on the above right is that the direction vector $\vec{v}$ of the line must lie in both planes. So we know that $\vec{v}$ must be orthogonal to $\vec{n}_1$ and $\vec{n}_2$. To find a point on the line L, we know that the point must be on both planes. We also know that the line must pass through at least one of the coordinate planes $x = 0$, $y = 0$, or $z = 0$.

A direction vector of the line of intersection of the planes with normal vectors $\vec{n}_1$ and $\vec{n}_2$ is given by $\vec{v} = \vec{n}_1 \times \vec{n}_2$. A point on the line can be found by setting one of the variables in the plane equations to 0 and solving both equations for the remaining variables.

Example 9: Find the line of intersection of the planes $2x - 4y + 4z = 6$ and $6x + 2y - 3z = 4$.

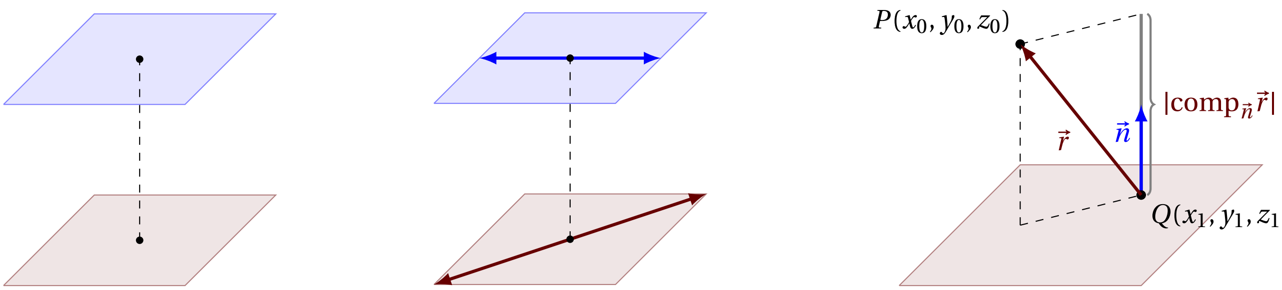
Concept Check: In what ways can three planes intersect?

1.6.4 (†, ‡) Distances

With a working knowledge of planes, several distance problems can be addressed:

- Find the distance between a point and a plane.
- Find the distance between two parallel planes.
- (†)Find the distance between two skew lines.

All of these problems actually reduce to the first one - finding the distance from a point to a plane. For example, to find the distance between two parallel planes, you can just take a point in one plane and find how far that point is from the other plane. To find the distance between skew lines, find parallel planes that contain the two lines (if they are skew, then this is possible).



Looking at the figure on the right, if $Q(x_1, y_1, z_1)$ is any point in the plane, and $\vec{r}$ is the vector $\overrightarrow{QP}$, then the distance from the point P to the plane is the (absolute value of the) scalar projection of $\vec{r}$ onto the normal vector $\vec{n}$. Now we know that $\vec{n} = \langle a, b, c\rangle$ and $\vec{r} = \langle x_0 - x_1, y_0 - y_1, z_0 - z_1\rangle$, and that

$$|\text{comp}_{\vec{n}}\vec{r}| = \left|\frac{\vec{r}\cdot\vec{n}}{|\vec{n}|}\right| = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|\big(ax_0 + by_0 + cy_0\big) - \big(ax_1 + by_1 + cz_1\big)|}{\sqrt{a^2 + b^2 + c^2}}.$$

The distance D from the point $P(x_0, y_0, z_0)$ to the plane $ax + by + cx + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Concept Check: (†)If two lines are skew, how can you find two parallel planes containing those lines?

Example 10: Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

Example 11: (†) Find the distance between the skew lines

$$L_1: \quad \frac{x-1}{1} = \frac{y+2}{3} = \frac{z-4}{-1}, \qquad L_2: \quad \frac{x}{2} = \frac{y-3}{1} = \frac{z+3}{4}.$$

1.6 Review of Concepts

- Terms to know: line, direction vector, symmetric equations, vector equation, parameter, parametric equations, (†)skew lines, line segment, convex combination, normal vector, point-normal form.
- Given two points, or single point and a direction, know how to find the symmetric, vector, and parametric equations of a line in $\mathbb{R}^2$ or $\mathbb{R}^3$.
- Know how to determine if two lines are parallel, (†) skew, or intersecting, and how to find their point of intersection.
- Know how to find the equation of a plane given enough information about the plane (point and normal vector, point and two vectors in the plane, three points in the plane, etc.).
- Know how to determine if two planes are parallel or intersecting, and how to find the equation of the line of intersection.
- Know how to apply the formula for the distance between two planes to other situations, such as finding the distance between two skew lines.

1.6 Practice Problems

1. What are the coordinates of the point where the line passing through the points $(2,4,-3)$ and $(3,-1,1)$ pass through the xy-plane?

2. Determine if the lines given by the equations $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ and $\frac{x-3}{-4} = \frac{y-2}{-3} = \frac{z-1}{2}$ are parallel, (†)skew, or intersecting. If they intersect, find their point of intersection. If they are parallel or (†)skew, find the distance between them.

3. Find a parametrization of the line segment from the point $(-1,3,5)$ to the point $(-2,2,4)$.

4. Find the scalar equation of the plane that passes through the point $(6,0,-2)$ and contains the line $x = 4-2t$, $y = 3+5t$, $z = 7+4t$.

5. Find parametric equations for the line through the point $(0,1,2)$ that is parallel to the plane $x + y + z = 2$ and perpendicular to the line $x = 1 + t$, $y = 1 - t$, $z = 2t$.

6. Find the equation of the plane through the points $(0,1,1)$, $(1,1,0)$, and $(1,0,1)$.

1.6 Exercises

1. Give a parametrization of the line segment from the point $(2,4,8)$ and $(7,5,3)$.

2. Find an equation of the line through the points $(-3,4,6)$ and $(5,-1,0)$.

3. Find an equation of the line through $(-3,4,2)$ that is perpendicular to both $\vec{u} = \langle 1,1,-5\rangle$ and $\vec{v} = \langle 0,4,0\rangle$.

4. Is the line through $(4,1,-1)$ and $(2,5,3)$ perpendicular to the line through $(-3,2,0)$ and $(5,1,4)$?

5. Determine whether the line $x = 3+8t$, $y = 4+5t$, $z = -3-t$ is parallel to the plane $x-3y+5z = 12$.

6. Find an equation of the line of intersection of the planes $x-y-2z=1$ and $x+y+z=-1$.

7. Find the point at which the line $x=1+2t$, $y=4t$, $z=2-3t$ intersects the plane $x+2y-z+1=0$.

8. Find an equation of the plane containing $(3,0,-2)$ that is parallel to both $\langle 1,-3,1\rangle$ and $\langle 4,2,0\rangle$.

9. Find an equation of the plane containing the points $(2,-1,4)$, $(1,1,-1)$, and $(-4,1,1)$.

10. Determine if the statements below are true or false, giving a brief explanation or counterexample.

a) The line $\vec{r}(t)=\langle 3,-1,4\rangle+t\langle 6,-2,8\rangle$ passes through the origin.

b) The plane containing the point $(1,1,1)$ with normal vector $\vec{n}=\langle 1,2,-3\rangle$ is the same as the plane containing the point $(3,0,1)$ with normal vector $\vec{n}=\langle -2,-4,6\rangle$.

c) If plane p_1 is orthogonal to plane p_2, and plane p_2 is orthogonal to plane p_3, then planes p_1 and p_3 are parallel.

d) Any two distinct lines in $\mathbb{R}^3$ determine a plane.

e) Given any plane p, there is exactly one plane that is orthogonal to p.

f) Given a plane p and a point P_0 not in the plane, there is exactly one plane that is orthogonal to p and contains P_0.

g) Three noncollinear points give enough information to determine a plane.

h) Two nonparallel vectors give enough information to determine a plane.

i) A line and a point not on the line give enough information to determine a plane.

11. Suppose you were given the equations of three planes

$$\begin{aligned} p_1: \quad a_1x+b_1y+c_1z &= d_1\\ p_2: \quad a_2x+b_2y+c_2z &= d_2\\ p_3: \quad a_3x+b_3y+c_3z &= d_3 \end{aligned}$$

and you were told that there is a single point where all three planes intersect. Describe, in words, the process you would use to find this point of intersection.

1.6 Answers to Practice Problems

1. The direction vector of the line is $\vec{v}=\langle 1,-5,4\rangle$. The line is given by $\vec{r}(t)=\langle 2+t,4-5t,-3+4t\rangle$ and it passes through the xy-plane when $z=0$, so when $-3+4t=0$, or $t=3/4$. Thus the line contains the point $(11/4,1/4,0)$.

2. Note that the lines are not parallel as their direction vectors $\vec{v}_1=\langle 1,2,3\rangle$ and $\vec{v}_2=\langle -4,-3,2\rangle$ are not parallel. To determine if the lines intersect or not, we must see if the system of equations

$$\begin{aligned} t &= 3-4s\\ 1+2t &= 2-3s\\ 2+3t &= 1+2s \end{aligned}$$

has a solution. Substituting the first equation into the second gives $1+2(3-4s) = 2-3s$, or $s = 5$ (and therefore $t = 3-20 = -17$), but these values do not satisfy the third equation. Thus these lines are skew. To find the distance between them, we need to find the plane containing one of the lines (parallel to the other line) and then find the distance from a point on the other line to that plane. The normal vector of the plane can be found by taking the cross product of the direction vectors of the lines (as it must be orthogonal to both). Thus

$$\vec{n} = \langle 1,2,3\rangle \times \langle -4,-3,2\rangle = \langle 13,-14,5\rangle,$$

and a point on the first line is $(0,1,2)$. Therefore a plane containing the first line is $13(x-0)-14(y-1)+5(z-2) = 0$, or $13x-14y+5z+4 = 0$. The distance between the lines is found by using a point on the second line (let's use $(3,2,1)$) and the distance formula

$$D = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}} = \frac{|13\cdot 3-14\cdot 2+5\cdot 1+4|}{\sqrt{13^2+14^2+5^2}} = \frac{20}{\sqrt{390}}.$$

3. $\vec{r}(t) = \langle -(1-t)-2t, 3(1-t)+2t, 5(1-t)+4t\rangle = \langle -t-1, 3-t, 5-t\rangle, \quad 0 \le t \le 1.$

4. We need two vectors in the plane. One is given by the direction of the line $\vec{v}_1 = \langle -2,5,4\rangle$ and the other can be found by finding the vector from a point on the line to the point $(6,0,-2)$. We know $(4,3,7)$ is on the line so let $\vec{v}_2 = \langle -2,3,9\rangle$. Thus a normal vector for the plane is $\vec{n} = \vec{v}_1 \times \vec{v}_2 = \langle 33,10,4\rangle$ so an equation of the plane is $33(x-6)+10y+4(z+2) = 0$.

5. A direction vector for the line will be orthogonal to the normal vector for the plane and the direction vector of the other line: $\langle 1,1,1\rangle \times \langle 1,-1,2\rangle = \langle 3,-1,-2\rangle$. Thus the parametric equations are $x = 3t$, $y = 1-t$, $z = 2-2t$.

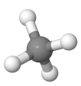
6. We find two vectors between the points and then take their cross product to get the normal vector. Thus

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \langle 1,-1,0\rangle \times \langle 1,0,-1\rangle = \langle 1,1,1\rangle.$$

The equation of the plane is then $(x-1)+(y-1)+(z-0) = 0$, or $x+y+z = 2$.

Chapter 2

Matrices and Linear Algebra



2.1 Systems of Linear Equations

Objectives and Concepts:

- Linear systems of equations can have no solution, a unique solution, or infinitely many solutions.
- In $\mathbb{R}^3$, three linear equations with a unique solution can be viewed as three planes with a single point of intersection.
- Solution sets of linear equations can be described with parameters.
- Chemical reaction equations can be balanced using systems of linear equations.

References: FCLA Chapter SLE, *Systems of Linear Equations*, Section SSLE, *Solving Systems of Linear Equations*

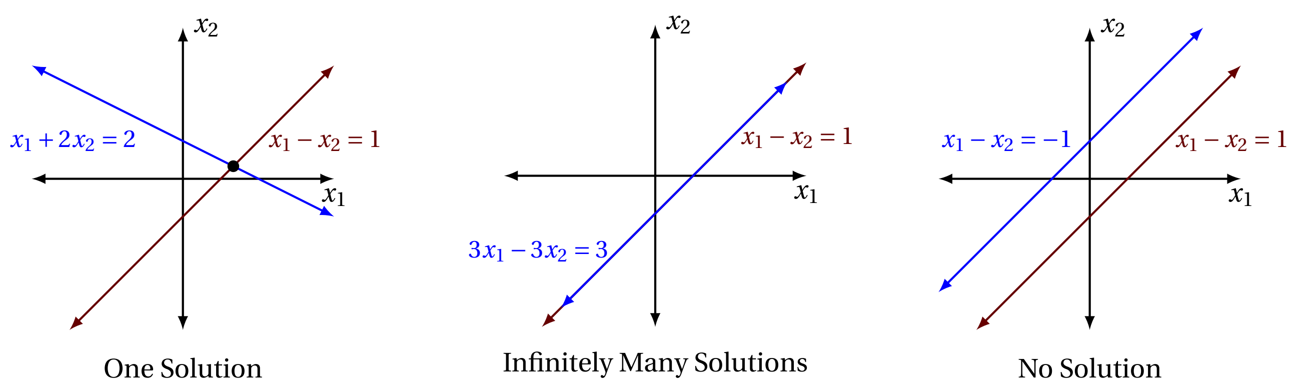
2.1.1 Linear Equations

Definition: A **linear equation** in the variables $x_1, x_2, \ldots, x_n$, where n is a positive integer, is an equation that can be written in the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b,$$

where the **coefficients** $a_1, \ldots, a_n$ and the right hand side b are real or complex numbers. A **system of linear equations** or **linear system** is a collection of one of more linear equations in the same variables. A **solution** to a linear system is a list of numbers $s_1, s_2, \ldots, s_n$ such that all equations in the system are satisfied when $x_i = s_i$ for all $1 \le i \le n$. The set of all possible solutions is called the **solution set** of the linear system, and two systems are **equivalent** if they have the same solution set.

There are three possibilities for a system of linear equations: the system has no solution, the system has one solution, or the system has infinitely many solutions. Below are examples of each case for linear equations in the two variables x_1 and x_2. Each linear equation in these two variables gives a line in the x_1x_2-plane. A solution is an (x_1, x_2) pair, i.e., a point in the x_1x_2-plane that belongs to both lines.



From the plots of the lines, it is clear that if the two lines are parallel but are not the same line, then there is no solution. However, if both equations represent the same line, there are infinitely many solutions. In this case the solution set can be described in terms of a **free variable**. For example, the solution set to the system

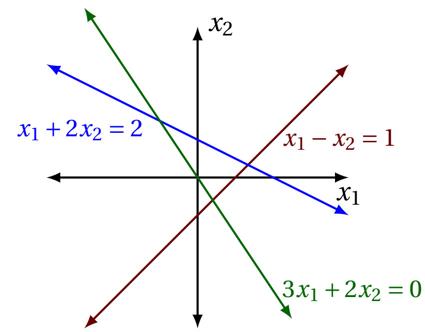
$$x_1 - x_2 = 1, \qquad 3x_1 - 3x_2 = 3,$$

can be described as $x_2 = x_1 - 1$ where x_1 is free (can be any real number). Another way to think about this is that $x_1 = t, x_2 = t - 1$ describes all possible solutions where t is a **parameter**.

Definition: A linear system is **consistent** if it has at least one solution, and it is **inconsistent** if it has no solution.

In each case previously considered, there were exactly as many equations as unknowns. However, this is not always the case. If there are more equations than unknowns, then the system is **overdetermined**. If there are more unknowns than equations, the system is **underdetermined**. A single equation in x_1 and x_2 is underdetermined. An example of an overdetermined system is given by the three equations

$$\begin{aligned} x_1 - x_2 &= 1, \\ x_1 + 2x_2 &= 2, \\ 3x_1 + 2x_2 &= 0. \end{aligned}$$



Overdetermined, No Solution

Even though none of the lines are parallel, this system is inconsistent as there is no point (x_1, x_2) that lies on all three lines.

Concept Check: Is it possible for an underdetermined system to have a unique solution?

Concept Check: Is it possible for an overdetermined system to have infinitely many solutions?

2.1.2 Solution by Substitution

When a linear system has a solution, it can be found by (possibly repeated) algebraic substitution. To do this, you simply use the information from one equation in another.

Example 1: Use substitution to find all solutions to the system $\quad 2x_1 - 4x_2 = 2, \quad x_1 + x_2 = 4.$

Example 2: Find all solutions to the system $x_1 - 4x_2 = 1$, $2x_1 - x_2 = -3$, and $x_1 + 3x_2 = 0$.

2.1.3 Solution Sets

When a linear system has infinitely many solutions, at least one free variable or parameter will be used to describe the solution set. For example, if we wanted to describe the solution set for the single equation $3x_1 - x_2 + 4x_3 = 1$ using the parameters s and t, we can do this in any of the following ways:

- $x_1 = s$, $x_2 = t$, $x_3 = \frac{1}{4}(1 - 3s + t)$
- $x_2 = s$, $x_3 = t$, $x_1 = \frac{1}{3}(1 + s - 4t)$
- $x_1 = s$, $x_3 = t$, $x_2 = 3s + 4t - 1$

Note that in each case, we are really describing a plane in $\mathbb{R}^3$.

Concept Check: What kind of geometric situation in $\mathbb{R}^3$ represents the solution set of the equations $2x_1 - x_2 = 1$, $x_2 + x_3 = 4$?

Example 3: Find all solutions to the system $x_1 + 6x_2 + 3x_4 = 0$, $x_3 - 4x_4 = 5$.

Concept Check: Is the system of equations that represents three planes, two of which are parallel but not the same plane, consistent?

2.1.4 Balancing Chemical Equations

Chemical equations that represent reactions are written with reactants on the left side and products on the right side, separated by an arrow. For example, to represent the reaction of sodium hydroxide with sulfuric acid to form sodium sulfate and water, we would write:

$$\mathrm{NaOH} + \mathrm{H_2SO_4} \longrightarrow \mathrm{Na_2SO_4} + \mathrm{H_2O}.$$

However, this equation is not *balanced* as the number of atoms of each element on the left is not the same as the number of atoms of each element on the right. To properly balance the equation, we would look for coefficients x_1, x_2, x_3, x_4 such that

$$x_1\mathrm{NaOH} + x_2\mathrm{H_2SO_4} \longrightarrow x_3\mathrm{Na_2SO_4} + x_4\mathrm{H_2O}.$$

Since the number of hydrogen atoms on the left must be the same as the number on the right, we know that the coefficients must satisfy the equation $x_1 + 2x_2 = 2x_4$. Looking at the other atoms, we arrive at the following system of four equations:

$$\begin{aligned} x_1 + 2x_2 \qquad\quad - 2x_4 &= 0 \\ x_1 + 4x_2 - 4x_3 - \ \ x_4 &= 0 \\ x_1 \qquad - 2x_3 \qquad\quad &= 0 \\ x_2 - \ \ x_3 \qquad\quad &= 0 \end{aligned}$$

This system has infinitely many solutions, one convenient integer solution is $x_1 = 2$, $x_2 = 1$, $x_3 = 1$, $x_4 = 2$.

Example 4: Balance the equation $\mathrm{C_6H_5COOH} + \mathrm{O_2} \longrightarrow \mathrm{CO_2} + \mathrm{H_2O}$.

Example 5: Balance the equation $\mathrm{Ba_3N_2} + \mathrm{H_2O} \longrightarrow \mathrm{Ba(OH)_2} + \mathrm{NH_3}$.

2.1 Review of Concepts

- Terms to know: solution set, equivalent systems, consistent, inconsistent, overdetermined, underdetermined, free variable, parameter.
- Know how to solve small linear systems using substitution.
- Know how to describe solution sets using free variables or parameters.
- Know how to set up linear systems of equations that balance a chemical equation.

2.1 Practice Problems

1. Find all solutions of the linear system

$$\begin{aligned} x_1 \quad\quad\; - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$$

2. Suppose the system below is consistent for all possible values of a and b. What can you say about the coefficients c and d?

$$\begin{aligned} x_1 + 3x_2 &= a \\ cx_1 + dx_2 &= b \end{aligned}$$

3. Balance the equation $K_4Fe(CN)_6 + H_2SO_4 + H_2O \longrightarrow K_2SO_4 + FeSO_4 + (NH_4)_2SO_4 + CO$.

2.1 Exercises

1. Determine if the statements below are true or false, giving a brief explanation or counterexample.

a) A system of 3 linear equations in 4 unknowns can have a unique solution.

b) A system of 3 linear equations in 2 unknowns can have a unique solution.

c) A system of 3 linear equations in 4 unknowns can have infinitely many solutions.

d) A system of 3 linear equations in 2 unknowns can have infinitely many solutions.

e) A system of 2 linear equations in 3 unknowns can have no solution.

2. Find the solution set (if it exists) of the linear system

$$\begin{aligned} x_1 + 3x_2 + 4x_3 &= 7 \\ 3x_1 + 9x_2 + 7x_3 &= 6 \end{aligned}$$

3. Find the solution set (if it exists) of the linear system

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 5 \\ 2x_1 + 4x_2 + 5x_3 &= 4 \\ 4x_1 + 5x_2 + 4x_3 &= 2 \end{aligned}$$

4. Find the solution set (if it exists) of the linear system

$$\begin{aligned} 2x_1 \quad\quad\quad\quad\quad\quad - 4x_4 &= -10 \\ 3x_2 + 3x_3 \quad\quad &= 0 \\ x_3 + 4x_4 &= -1 \\ -3x_1 + 2x_2 + 3x_3 + x_4 &= 5 \end{aligned}$$

5. Find an equation involving a, b, and c such that the following linear system is consistent:

$$\begin{aligned} x_1 - 4x_2 + 7x_3 &= a \\ -3x_2 + 5x_3 &= b \\ -2x_1 + 5x_2 - 9x_3 &= c \end{aligned}$$

6. Balance the equation $Cr_2O_7{}^{-2} + H^+ \xrightarrow{+6e} Cr^{+3} + H_2O$.

7. Balance the equation $PhCH_3 + KMnO_4 + H_2SO_4 \longrightarrow PhCOOH + K_2SO_4 + MnSO_4 + H_2O$.

8. How many of each coin and how much money in total do you have if all of the following are true:

- You have only nickels, dimes, and quarters.
- You have twice as many dimes as you have quarters.
- You have 13 coins that are nickels or quarters.
- You have \$1.25 in nickels and dimes.

2.1 Answers to Practice Problems

1. To find all solutions of the linear system

$$\begin{aligned} x_1 \qquad\quad - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$$

we can use the first and third equation to find x_1 and x_2 in terms of x_3:

$$x_1 = 8 + 3x_3, \qquad x_2 = -2 - 5x_3,$$

and substitute these into the second equation:

$$7 = 2(8 + 3x_3) + 2(-2 - 5x_3) + 9x_3 = 16 + 6x_3 - 4 - 10x_3 + 9x_3 = 12 + 5x_3.$$

This implies $5x_3 = 7 - 12 = -5$ so $x_3 = -1$. Then $x_1 = 8 - 3 = 5$ and $x_2 = -2 + 5 = 3$.

2. In order for the system

$$\begin{aligned} x_1 + 3x_2 &= a \\ cx_1 + dx_2 &= b \end{aligned}$$

to be consistent for all values of a and b, we need to ensure that the two equations do not contradict each other. The equations would be contradictory if the left hand sides were the same but the right hand sides were different. For example, if $c = 1$ and $d = 3$, the system is not consistent unless $a = b$. A similar situation occurs when $c = t$ and $d = 3t$ for any nonzero value of t. Thus, in order to ensure the system is consistent for ALL choices of a and b, we must have that $d - 3c \neq 0$, otherwise there are choices for a and b that would lead to an inconsistent system.

3. To balance $K_4Fe(CN)_6 + H_2SO_4 + H_2O \longrightarrow K_2SO_4 + FeSO_4 + (NH_4)_2SO_4 + CO$, we let variables represent the coefficient of each molecule:

$$x_1 K_4Fe(CN)_6 + x_2 H_2SO_4 + x_3 H_2O \longrightarrow x_4 K_2SO_4 + x_5 FeSO_4 + x_6 (NH_4)_2SO_4 + x_7 CO$$

Then we need to look at the equations that are formed when atoms of each type are balanced:

$$\begin{aligned} 4x_1 &= 2x_4 && \text{(K)} \\ x_1 &= x_5 && \text{(Fe)} \\ 6x_1 &= x_7 && \text{(C)} \\ 6x_1 &= 2x_6 && \text{(N)} \\ 2x_2 + 2x_3 &= 8x_6 && \text{(H)} \\ x_2 &= x_4 + x_5 + x_6 && \text{(S)} \\ 4x_2 + x_3 &= 4x_4 + 4x_5 + 4x_6 + x_7 && \text{(O)} \end{aligned}$$

To find an integer solution to these equations, we can just start substituting given relationships into others. Note that the first four equations all give $x_4, \dots x_7$ in terms of x_1, so trying to get everything in terms of x_1 might be useful. Then the (O) equation minus 2 times the (H) equation is

$$4x_2 + x_3 - 4x_2 - 4x_3 = 4x_4 + 4x_5 + 4x_6 + x_7 - 16x_6,$$

or

$$-3x_3 = 4x_4 + 4x_5 - 12x_6 + x_7 = 8x_1 + 4x_1 - 36x_1 + 6x_1 = -18x_1,$$

so $x_3 = 6x_1$. Also, the (S) equation gives

$$x_2 = x_4 + x_5 + x_6 = 2x_1 + x_1 + 3x_1 = 6x_1.$$

So now everything is given in terms of x_1. Let $x_1 = 1$ and see if all of the variables are integers: $x_2 = 6, x_3 = 6, x_4 = 2, x_5 = 1, x_6 = 3, x_7 = 6$. Thus the balanced equation is

$$K_4Fe(CN)_6 + 6H_2SO_4 + 6H_2O \longrightarrow 2K_2SO_4 + FeSO_4 + 3(NH_4)_2SO_4 + 6CO.$$

2.2 Matrices and Matrix Operations

Objectives and Concepts:

- A matrix is a rectangular array of numbers. An $m \times n$ matrix consist of m rows and n columns. Each row and column can be thought of as vectors.
- Matrix operations of addition and multiplication can be performed if the matrices are the correct size.
- A linear system has a matrix-vector equation representation.

References: TCMB §§18.1–3, 19.6; FCLA Chapter M, *Matrices*, Sections MO, *Matrix Operations* and MM, *Matrix Multiplication*

2.2.1 Matrices

Definition: A **matrix** is a rectangular array of data placed into **rows** and **columns**, and we often use capital letters to denote matrices. When $\boldsymbol{A}$ has m rows and n columns, we say $\boldsymbol{A}$ is of **size** $m \times n$.

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The matrix $\boldsymbol{A} = (a_{ij})$ is completely determined by its **entries** a_{ij} where the i and j are **indices** that represent the ith row and jth column of $\boldsymbol{A}$. Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are **equal** (written $\boldsymbol{A} = \boldsymbol{B}$) if and only if they are the same size and their corresponding entries are equal.

Example 1: The matrix

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 2 & -1 \\ 5 & -1 & 7 & 0 \\ 4 & 2 & 2 & -1 \end{bmatrix}$$

has 3 rows and 4 columns, so is of size 3×4. We have

$$a_{21} = 5, \quad a_{32} = 2, \quad a_{11} = 3, \quad a_{14} = -1.$$

Example 2: Find w, x, y, and z such that

$$\begin{bmatrix} x+y & y \\ w+z & w-z \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 4 & 6 \end{bmatrix}$$

Notation and Terminology: If $\boldsymbol{A}$ is an $m \times n$ matrix with real entries, we write $\boldsymbol{A} \in \mathbb{R}^{m\times n}$. The ith row of $\boldsymbol{A}$ is a **row vector** and is an element of $\mathbb{R}^{1\times n}$. The jth column of $\boldsymbol{A}$ is a **column vector** and we may write $\vec{a}_1 \in \mathbb{R}^{m\times 1}$. We can write $\boldsymbol{A}$ in terms of its columns: $\boldsymbol{A} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$.

Often when we are talking about row or column vectors, we may drop the "$\times 1$" and just say that the vectors are in $\mathbb{R}^n$ or $\mathbb{R}^m$.

2.2.2 Matrix Operations

Definition: Let $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ be $m \times n$ matrices. Then the **sum** of $\boldsymbol{A}$ and $\boldsymbol{B}$ is the matrix $\boldsymbol{A}+\boldsymbol{B} = (a_{ij}+b_{ij})$, i.e., each entry of $\boldsymbol{A}+\boldsymbol{B}$ is the sum of the corresponding entries in $\boldsymbol{A}$ and $\boldsymbol{B}$. The **scalar product** of a number c and the matrix $\boldsymbol{A}$ is $c\boldsymbol{A} = (ca_{ij})$. We define $-\boldsymbol{A} = (-1)\boldsymbol{A}$ and $\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + (-\boldsymbol{B})$.

Note: Addition is only defined for matrices of the same size!

Example 3: Let $\boldsymbol{A} = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} -2 & 0 & 7 \\ 4 & 5 & -3 \end{bmatrix}$. Find $\boldsymbol{A}+\boldsymbol{B}$, and $2\boldsymbol{A}-3\boldsymbol{B}$.

Definition: Let $\boldsymbol{A} = (a_{ij})$ be an $m \times n$ matrix and let $\boldsymbol{B} = (b_{ij})$ be a $n \times p$ matrix. Then the **product** of $\boldsymbol{A}$ and $\boldsymbol{B}$ is the $m \times p$ matrix

$$\boldsymbol{AB} = (c_{ik}) \quad \text{where} \quad c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}.$$

Note: Matrix multiplication is only defined when the matrix on the left has the same number of columns as the number of rows of the matrix on the right.

Concept Check: If $\boldsymbol{A}$ and $\boldsymbol{B}$ are the same size, when will $\boldsymbol{AB}$ be defined?

To form the ikth entry of $\boldsymbol{AB}$, you multiply the corresponding entries in the ith row of $\boldsymbol{A}$ with those in the kth column of $\boldsymbol{B}$ and add them up. **This is essentially taking the dot product of the vector that forms the ith row of $\boldsymbol{A}$ and the kth column of $\boldsymbol{B}$.**

$$\boxed{\boldsymbol{A} : m \text{ rows } n \text{ columns}} \qquad \boxed{\boldsymbol{B} : n \text{ rows } p \text{ columns}} \qquad \boxed{\boldsymbol{C} = \boldsymbol{AB} : m \text{ rows } p \text{ columns}}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ c_{31} & c_{32} & \cdots & c_{3p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{bmatrix}$$

$$c_{32} = a_{31}b_{12} + a_{32}b_{22} + \cdots + a_{3n}b_{n2}$$

Example 4: Let $\boldsymbol{A} = \begin{bmatrix} 1 & 6 \\ 4 & 2 \\ 3 & 1 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 7 & 0 \end{bmatrix}$. Compute $\boldsymbol{AB}$. Can we compute $\boldsymbol{BA}$?

Definition: A matrix $\boldsymbol{A}$ is **square** if it has the same number of rows as columns. Let $\boldsymbol{A}$ be an $n \times n$ matrix and let m be a positive integer. Then the mth power of $\boldsymbol{A}$ is the matrix

$$\boldsymbol{A}^m = \underbrace{\boldsymbol{A} \cdot \boldsymbol{A} \cdots \boldsymbol{A}}_{m \text{ times}}.$$

Example 5: Let A, B, C, and D be matrices. If A is 3×4, B is 2×2, C is 4×2, and D is 2×3, which of the following expressions are meaningful?

$$\boldsymbol{A}^2, \quad \boldsymbol{AB}, \quad \boldsymbol{BA}, \quad \boldsymbol{AC}, \quad \boldsymbol{CA}, \quad \boldsymbol{ADC}, \quad \boldsymbol{CB}^2, \quad \boldsymbol{DA}, \quad \boldsymbol{ACD}, \quad \boldsymbol{ACBD}, \quad \boldsymbol{BDC}, \quad \boldsymbol{CBDA}$$

Definition: Let $\boldsymbol{A} = (a_{ij})$ be an $m \times n$ matrix. The **transpose** of $\boldsymbol{A}$ is the $n \times m$ matrix $\boldsymbol{A}^T = (a_{ji})$.

Example 6: If $\boldsymbol{A} = \begin{bmatrix} 1 & 6 \\ 4 & 2 \\ 3 & 1 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 7 & 0 \end{bmatrix}$, compute $\boldsymbol{B}^T\boldsymbol{A}^T$ and $\boldsymbol{A}^T\boldsymbol{B}^T$.

Concept Check: What does the above example mean, in words, about the product of transposes?

Example 7: Compute $\vec{u}^T\vec{v}$ and $\vec{u}\vec{v}^T$ if $\vec{u} = \begin{bmatrix} -1 \\ 3 \\ -6 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

Concept Check: The dot product of the column vectors $\vec{u}$ and $\vec{v}$ is:

Concept Check: Let $\boldsymbol{A}$ be a 3×3 matrix with row vectors $\vec{a}_i^T$, i.e., $\boldsymbol{A}^T = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}$ and let $\boldsymbol{B} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix}$ also be a 3×3 matrix. Write $\boldsymbol{AB}$ in terms of dot products.

Definition: Let $\vec{u}$ and $\vec{v}$ be n-dimensional column vectors (i.e., $\vec{u}, \vec{v} \in \mathbb{R}^{n\times 1}$). The **inner product** of $\vec{u}$ and $\vec{v}$ is the scalar $\vec{u}^T\vec{v}$. (To be pedantic, this is a 1×1 matrix, not a scalar, but it is usually safe and convenient to blur this difference!) The **outer product** (or **tensor product**) of $\vec{u}$ and $\vec{v}$ is the $n\times n$ matrix $\vec{u}\vec{v}^T$.

Thus the inner product is another way to describe the scalar product of two vectors.

2.2.3 Some Special Matrices

Definition: The matrix $\mathbf{0} \in \mathbb{R}^{m\times n}$ with all entries equal to zero is the $m\times n$ **zero matrix**.

Concept Check: Is it possible for two nonzero matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ to have a product of the zero matrix? Can you come up with an example?

Definition: A matrix $\boldsymbol{D} = (d_{ij}) \in \mathbb{R}^{m\times n}$ is a **diagonal matrix** if $d_{ij} = 0$ for all $i \neq j$. The **identity matrix** $\boldsymbol{I} \in \mathbb{R}^{n\times n}$ is the $n\times n$ diagonal matrix with 1s along the diagonal, i.e.

$$\boldsymbol{I} = (I_{ij}) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

We often use $\boldsymbol{I}_n$ to represent the $n\times n$ identity matrix.

Matrix multiplication with diagonal matrices is fairly easy.

Example 8: Let $\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} -1 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}$. Find $\boldsymbol{AB}$ and $\boldsymbol{BA}$.

The identity matrix $\boldsymbol{I}_n$ is special in that it is the **multiplicative identity** for all $n\times n$ matrices. What we mean by this is that, for any $n\times n$ matrix $\boldsymbol{A}$, we have

$$\boldsymbol{A}\boldsymbol{I}_n = \boldsymbol{I}_n\boldsymbol{A} = \boldsymbol{A}.$$

Definition: The $n\times n$ matrix $\boldsymbol{A}$ is **symmetric** if $\boldsymbol{A}^T = \boldsymbol{A}$. The $n\times n$ matrix $\boldsymbol{A}$ is **skew-symmetric** if $\boldsymbol{A}^T = -\boldsymbol{A}$.

Concept Check: Is the product of symmetric matrices symmetric? Is the product of skew-symmetric matrices also skew?

Concept Check: Are $\boldsymbol{A}^T\boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{A}^T$ always defined? If they are, do they have any special properties?

Concept Check: Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $m \times n$ matrices. Is $(\boldsymbol{A}+\boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$?

2.2.4 Complex Matrices and Vectors

Complex Matrices

If the $m \times n$ matrix $\boldsymbol{A}$ contains complex entries, we write $\boldsymbol{A} \in \mathbb{C}^{m\times n}$. In this case there are matrices $\boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}^{m\times n}$ such that $\boldsymbol{A} = \boldsymbol{B} + i\boldsymbol{C}$.

Definition: The **complex conjugate** of the complex matrix $\boldsymbol{A} = \boldsymbol{B} + i\boldsymbol{C}$ is the matrix $\boldsymbol{A}^* = \boldsymbol{B} - i\boldsymbol{C}$. The **Hermitian conjugate** or **conjugate transpose** of $\boldsymbol{A}$ is the transpose of its complex conjugate:

$$\boldsymbol{A}^\dagger = (\boldsymbol{A}^*)^T = (\boldsymbol{A}^T)^*.$$

(Another notation for this is $\boldsymbol{A}^H$.)

Definition: A matrix $\boldsymbol{A}$ is **Hermitian** if $\boldsymbol{A} = \boldsymbol{A}^\dagger$.

For real matrices, Hermitian corresponds to symmetric. For complex matrices, being Hermitian is far more interesting than being symmetric—we will see an important manifestation of this in Section 2.6 on *Eigenvalues and Eigenvectors.*

Example 9: Find the Hermitian conjugate of $\boldsymbol{A} = \begin{bmatrix} 1 & i & 1-i \\ 2i & 0 & -1 \\ 1+i & -1-i & 3 \end{bmatrix}$ and compute $\boldsymbol{A}^\dagger \boldsymbol{A}$.

Concept Check: For any complex matrix $\boldsymbol{A}$, is the matrix $\boldsymbol{B} = \boldsymbol{A}^\dagger \boldsymbol{A}$ Hermitian?

Complex Vectors

If the n-component vector $\vec{z}$ contains complex entries, we write $\vec{z} \in \mathbb{C}^n$. In this case there are vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ such that $\vec{z} = \vec{u} + i\vec{v}$.

The only significant change from real vectors is the way that we define the norm and scalar product. For the case of $\vec{z} = [z_1] \in \mathbb{C}^1$ (which is a convoluted way of talking about complex numbers) the norm should match the absolute value: $\|\vec{z}\|^2 = |z_1|^2 = \overline{z_1} z_1$. This leads to the natural definition for a complex vector $\vec{z} \in \mathbb{C}^n$:

$$\|\vec{z}\|^2 = \sum_{i=1}^{n} |z_i|^2 = \sum_{i=1}^{n} \overline{z_i} z_i$$

It is also useful to redefine the scalar (dot) product to be compatible with this: for $\vec{z}, \vec{w} \in \mathbb{C}^n$,

$$\vec{z} \cdot \vec{w} = \sum_{i=1}^{n} \overline{z_i} w_i$$

so that it is still true that the norm comes from the scalar product: $\|\vec{z}\|^2 = \vec{z} \cdot \vec{z}$.

Recall that the scalar product of real vectors $\vec{u}$ and $\vec{v}$ is also given in terms of matrix multiplication as $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$. With complex vectors, the first factor in the scalar product is conjugated, so that for complex vectors $\vec{z}$ and $\vec{z}$ we now have

$$\vec{z} \cdot \vec{w} = (\vec{z}^T)^* \vec{w} = \vec{z}^\dagger \vec{w}$$

This is part of a general pattern: in almost everything done with complex matrices and vectors, we replace the transpose A^T by the Hermitian conjugate $A^\dagger$.

Note that when the elements of the vectors are all real, these new definitions match the previous ones, so there is no ambiguity or contradiction.

2.2.5 (†) Partitioned and Block Matrices

A **partition** of an $m \times n$ matrix $\boldsymbol{A}$ is a subdivision of the entries of $\boldsymbol{A}$ into rectangular **blocks**. Partitioned or block matrices can be very useful when$\boldsymbol{A}$ has a natural structure that might arise from a mathematical model of a physical process, or for handling a very large $\boldsymbol{A}$.

Example 10: The matrix $\boldsymbol{A} = \left[\begin{array}{rrr|rr|r} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ \hline 8 & -6 & 3 & 1 & 7 & -4 \end{array}\right]$ can be written as the matrix $\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \boldsymbol{A}_{13} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \boldsymbol{A}_{23} \end{bmatrix}$ Write down the six matrices $\boldsymbol{A}_{ij}$.

Multiplication of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ into the product $\boldsymbol{AB}$ via blocks is possible when the partitions of $\boldsymbol{A}$ and $\boldsymbol{B}$ are **conformable**, i.e., when the column partition of $\boldsymbol{A}$ matches the row partition of $\boldsymbol{B}$.

Example 11: Let

$$\boldsymbol{A} = \left[\begin{array}{rrr|rr} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & -4 & -2 & 7 & -1 \end{array}\right] = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix}, \qquad \boldsymbol{B} = \left[\begin{array}{rr} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hline -1 & 3 \\ 5 & 2 \end{array}\right] = \begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{bmatrix}.$$

Note that

$$\boldsymbol{AB} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{11}\boldsymbol{B}_1 + \boldsymbol{A}_{12}\boldsymbol{B}_2 \\ \boldsymbol{A}_{21}\boldsymbol{B}_1 + \boldsymbol{A}_{22}\boldsymbol{B}_2 \end{bmatrix}$$

where

$$\boldsymbol{A}_{11}\boldsymbol{B}_1 = \begin{bmatrix} 12+6-3 & 8-3+7 \\ 6-10+6 & 4+5-14 \end{bmatrix} = \begin{bmatrix} 15 & 12 \\ 2 & -5 \end{bmatrix}, \qquad \boldsymbol{A}_{12}\boldsymbol{B}_2 = \begin{bmatrix} 0-20 & 0-8 \\ -3-5 & 9-2 \end{bmatrix} = \begin{bmatrix} -20 & -8 \\ -8 & 7 \end{bmatrix},$$

$$\boldsymbol{A}_{21}\boldsymbol{B}_1 = \begin{bmatrix} 0+8+6 & 0-4-14 \end{bmatrix} = \begin{bmatrix} 14 & -18 \end{bmatrix}, \qquad \boldsymbol{A}_{22}\boldsymbol{B}_2 = \begin{bmatrix} -7-5 & 21-2 \end{bmatrix} = \begin{bmatrix} -12 & 19 \end{bmatrix}.$$

So

$$\boldsymbol{AB} = \begin{bmatrix} \boldsymbol{A}_{11}\boldsymbol{B}_1 + \boldsymbol{A}_{12}\boldsymbol{B}_2 \\ \boldsymbol{A}_{21}\boldsymbol{B}_1 + \boldsymbol{A}_{22}\boldsymbol{B}_2 \end{bmatrix} = \left[\begin{array}{rr} -5 & 4 \\ -6 & 2 \\ \hline 2 & 1 \end{array}\right].$$

2.2.6 Representation of Linear Systems of Equations

Given a linear system of m equations in n variables of the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

we can use a more compact notation to represent this system:

$$\boldsymbol{A}\vec{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \vec{b}.$$

$\boldsymbol{A}\vec{x} = \vec{b}$ is the **matrix-vector representation** of the linear system of equations. Another structure that is even more compact is the **augmented matrix**:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

This is called the augmented matrix as the coefficient matrix $\boldsymbol{A}$ has been augmented with the right hand side vector $\vec{b}$. It is understood that the first n columns represent the coefficients of the x_i in the system of equations.

Example 12: Write the augmented matrix of the system $\begin{aligned} -2x_1 + 2x_2 - 2x_3 &= 1 \\ x_1 + 4x_2 - 4x_3 &= -2 \\ 3x_1 \qquad - 2x_3 &= 5 \end{aligned}$.

Augmented matrices are useful for solving larger systems of equations via *Gaussian elimination* (a solution technique).

2.2 Review of Concepts

- Terms to know: matrix, index, entry, transpose, inner product, outer product, tensor product, zero matrix, diagonal matrix, identity matrix, multiplicative identity, symmetric matrix, skew-symmetric matrix, conjugate transpose, Hermitian conjugate, Hermitian matrix, (†) partitioned matrix, (†) block matrix, conformable, matrix-vector representation, augmented matrix.
- Know how to compute products and sums of matrices, and how to determine the size of the resulting matrix.
- (†) Know how to compute products of matrices using partitions and blocks.
- Know how to represent linear systems with a matrix-vector equation.

2.2 Practice Problems

1. Let

$$\boldsymbol{A} = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix}, \qquad \boldsymbol{C} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \qquad \boldsymbol{D} = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}.$$

Compute the following expressions, or explain why they are not defined.

a) $\boldsymbol{A}+2\boldsymbol{B}$ b) $3(\boldsymbol{C}-\boldsymbol{I}_2)\boldsymbol{B}$ c) $\boldsymbol{AD}+\boldsymbol{C}$ d) $\boldsymbol{C}^T\boldsymbol{AB}^T$

2. Compute $\boldsymbol{AB}$ where $\boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$.

3. Let $\vec{u}, \vec{v} \in \mathbb{R}^{n\times 1}$ and let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$. What kind of object is $\vec{u}^T\boldsymbol{A}\vec{v}$?

4. Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric $n \times n$ matrices. Is $\boldsymbol{A}-2\boldsymbol{B}$ symmetric?

5. The **symmetric part** $\boldsymbol{A}_{sym}$ and **skew part** $\boldsymbol{A}_{skew}$ of a matrix $\boldsymbol{A}$ are given by

$$\boldsymbol{A}_{sym} = \frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^T), \qquad \boldsymbol{A}_{skew} = \frac{1}{2}(\boldsymbol{A}-\boldsymbol{A}^T),$$

and $\boldsymbol{A}$ can be written as $\boldsymbol{A} = \boldsymbol{A}_{sym} + \boldsymbol{A}_{skew}$. Find the symmetric part and the skew part of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 4 \\ 2 & -2 & 1 \\ 0 & 5 & 3 \end{bmatrix}.$$

2.2 Exercises

1. Let $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ be arbitrary matrices for which the given sums and products are defined. Determine if the following statements are true or false, giving a brief explanation or counterexample in each case.

a) $\boldsymbol{AB}+\boldsymbol{AC} = \boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C})$.

b) If $\boldsymbol{A} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix}$ are both 2×2, then $\boldsymbol{AB} = \begin{bmatrix} \vec{a}_1\vec{b}_1 & \vec{a}_2\vec{b}_2 \end{bmatrix}$.

c) $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{CB})$.

d) If $\boldsymbol{A}$ and $\boldsymbol{B} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix}$ are both 3×3, then $\boldsymbol{AB} = \begin{bmatrix} \boldsymbol{A}\vec{b}_1 + \boldsymbol{A}\vec{b}_2 + \boldsymbol{A}\vec{b}_3 \end{bmatrix}$.

2. Suppose the second column of $\boldsymbol{B}$ is all zeros. What can you say about the second column of $\boldsymbol{AB}$ (if $\boldsymbol{AB}$ is defined)?

3. Let $\boldsymbol{A} = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$. Construct a matrix $\boldsymbol{B}$ (with two different nonzero columns) such that $\boldsymbol{AB}$ is the zero matrix.

4. If $\vec{u}$ and $\vec{v}$ are in $\mathbb{R}^n$ (column vectors), how are $\vec{u}^T\vec{v}$ and $\vec{v}^T\vec{u}$ related? How are $\vec{u}\vec{v}^T$ and $\vec{v}\vec{u}^T$ related?

5. (†) Assume that the matrices are partitioned conformably for block multiplication. Compute the following products.

a) $\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{D} & \boldsymbol{0} \\ \boldsymbol{E} & \vec{F} \end{bmatrix}$

b) $\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{0} & -\boldsymbol{D} \\ \boldsymbol{D} & \boldsymbol{0} \end{bmatrix}$

6. Find the symmetric and skew parts of the matrix $\boldsymbol{A} = \begin{bmatrix} 4 & 0 & -1 \\ 2 & -2 & 1 \\ -3 & 2 & -6 \end{bmatrix}$. (See practice problems.)

7. Let $\boldsymbol{A} = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$. Compute $\boldsymbol{AB}$. Then write each column of $\boldsymbol{AB}$ as a linear combination of the columns of $\boldsymbol{A}$.

8. Suppose that the matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E}$ all have the following sizes:

$$\boldsymbol{A}: 4 \times 5 \qquad \boldsymbol{B}: 4 \times 5 \qquad \boldsymbol{C}: 5 \times 2 \qquad \boldsymbol{D}: 4 \times 2 \qquad \boldsymbol{E}: 5 \times 4$$

Determine which of the following matrix expressions are defined. For those that are defined, give the size of the resulting matrix, and for those that are not defined, indicate why.

1. $\boldsymbol{BA}$
2. $\boldsymbol{AC} + \boldsymbol{D}$
3. $\boldsymbol{AE} + \boldsymbol{B}$
4. $\boldsymbol{AB} + \boldsymbol{B}$
5. $\boldsymbol{E}(\boldsymbol{A} + \boldsymbol{B})$
6. $\boldsymbol{E}(\boldsymbol{AC})$
7. $\boldsymbol{E}^T\boldsymbol{A}$
8. $(\boldsymbol{A}^T + \boldsymbol{E})\boldsymbol{D}$

2.2 Answers to Practice Problems

1. For

$$\boldsymbol{A}=\begin{bmatrix}2&0&-1\\4&-5&2\end{bmatrix},\qquad \boldsymbol{B}=\begin{bmatrix}7&-5&1\\1&-4&-3\end{bmatrix},\qquad \boldsymbol{C}=\begin{bmatrix}1&2\\-2&1\end{bmatrix},\qquad \boldsymbol{D}=\begin{bmatrix}3&5\\-1&4\end{bmatrix},$$

a) $\boldsymbol{A}+2\boldsymbol{B}=\begin{bmatrix}2&0&-1\\4&-5&2\end{bmatrix}+2\begin{bmatrix}7&-5&1\\1&-4&-3\end{bmatrix}=\begin{bmatrix}16&-10&1\\6&-13&-4\end{bmatrix}$ b)

$$3(\boldsymbol{C}-\boldsymbol{I}_2)\boldsymbol{B}=3\left(\boldsymbol{C}=\begin{bmatrix}1&2\\-2&1\end{bmatrix}-\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\begin{bmatrix}7&-5&1\\1&-4&-3\end{bmatrix}=3\begin{bmatrix}0&2\\-2&0\end{bmatrix}\begin{bmatrix}7&-5&1\\1&-4&-3\end{bmatrix}=\begin{bmatrix}6&-24&-18\\-42&30&-6\end{bmatrix}$$

c) $\boldsymbol{AD}+\boldsymbol{C}$ is not defined as $\boldsymbol{AD}$ is 2×3 and $\boldsymbol{C}$ is 2×2.

d)

$$\boldsymbol{C}^T\boldsymbol{AB}^T=\left(\begin{bmatrix}1&2\\-2&1\end{bmatrix}\right)^T\begin{bmatrix}2&0&-1\\4&-5&2\end{bmatrix}\left(\begin{bmatrix}7&-5&1\\1&-4&-3\end{bmatrix}\right)^T$$
$$=\begin{bmatrix}1&-2\\2&1\end{bmatrix}\begin{bmatrix}2&0&-1\\4&-5&2\end{bmatrix}\begin{bmatrix}7&1\\-5&-4\\1&-3\end{bmatrix}=\begin{bmatrix}-6&10&-5\\8&-5&0\end{bmatrix}\begin{bmatrix}7&1\\-5&-4\\1&-3\end{bmatrix}$$
$$=\begin{bmatrix}-42-50-5&-6-40+15\\56+25+0&8+20+0\end{bmatrix}=\begin{bmatrix}-97&-31\\81&28\end{bmatrix}$$

2. $\boldsymbol{A}=\begin{bmatrix}2&3\\1&-5\end{bmatrix}$, $\boldsymbol{B}=\begin{bmatrix}4&3&6\\1&-2&3\end{bmatrix}$

$$\boldsymbol{AB}=\begin{bmatrix}2&3\\1&-5\end{bmatrix}\begin{bmatrix}4&3&6\\1&-2&3\end{bmatrix}=\begin{bmatrix}8+3&6-6&12+9\\4-5&3+10&6-15\end{bmatrix}=\begin{bmatrix}11&0&21\\-1&13&-9\end{bmatrix}$$

3. $\vec{u}^T\boldsymbol{A}\vec{v}$ is a scalar as $\vec{u}$ is $1\times n$, $\boldsymbol{A}$ is $n\times n$, and $\vec{v}$ is $n\times 1$, and the resulting matrix size is 1×1.

4. $(\boldsymbol{A}-2\boldsymbol{B})^T=\boldsymbol{A}^T-2\boldsymbol{B}^T$ because the transpose of a sum is the sum of the transposes (see Concept Check earlier in section). Now $\boldsymbol{A}^T-2\boldsymbol{B}^T=\boldsymbol{A}-\boldsymbol{B}^T2^T=\boldsymbol{A}-2\boldsymbol{B}$ since $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric and the transpose of a scalar is itself. Thus we have $(\boldsymbol{A}-2\boldsymbol{B})^T=\boldsymbol{A}-2\boldsymbol{B}$ so yes, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric, then so is $\boldsymbol{A}-2\boldsymbol{B}$.

5. To compute $\boldsymbol{A}_{sym}$ we have

$$\boldsymbol{A}_{sym}=\frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^T)=\frac{1}{2}\left(\begin{bmatrix}1&-1&4\\2&-2&1\\0&5&3\end{bmatrix}+\begin{bmatrix}1&2&0\\-1&-2&5\\4&1&3\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}2&1&4\\1&-4&6\\4&6&6\end{bmatrix}=\begin{bmatrix}1&1/2&2\\1/2&-2&3\\2&3&3\end{bmatrix}$$

$$\boldsymbol{A}_{skew}=\frac{1}{2}(\boldsymbol{A}-\boldsymbol{A}^T)=\frac{1}{2}\left(\begin{bmatrix}1&-1&4\\2&-2&1\\0&5&3\end{bmatrix}-\begin{bmatrix}1&2&0\\-1&-2&5\\4&1&3\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}0&-3&4\\3&0&-4\\-4&4&0\end{bmatrix}=\begin{bmatrix}0&-3/2&2\\3/2&0&-2\\-2&2&0\end{bmatrix}$$

2.3 Determinants and Inverses

Objectives and Concepts:

- The determinant of a square matrix is a scalar value that is computed from the entries in the matrix. A determinant of zero means that one of the rows (or columns) of the matrix can be written as a linear combination of the others. A nonzero determinant means that the rows and columns of the matrix are linearly independent, and that a unique solution of $\boldsymbol{A}\vec{x} = \vec{b}$ exists for any $\vec{b} \in \mathbb{R}^n$.
- When a matrix $\boldsymbol{A}$ has a nonzero determinant, there is a matrix $\boldsymbol{A}^{-1}$ such that $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$.
- The inverse matrix and Cramer's Rule can both be used to solve small systems of linear equations.

References: TCMB Chapter 17, §§1–5 and §18.4; FCLA Chapter D, *Determinants* and Sections MISLE, *Matrix Inverses and Systems of Linear Equations* and MISLE, *Matrix Inverses and Nonsingular Matrices*

2.3.1 Determinants of Matrices

Definition: For $n \geq 2$, the **determinant** of an $n \times n$ matrix $\boldsymbol{A} = (a_{ij})$ is the sum of n terms of the form $\pm a_{1j} \det \boldsymbol{A}_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of $\boldsymbol{A}$. In symbols,

$$\det \boldsymbol{A} = a_{11} \det \boldsymbol{A}_{11} - a_{12} \det \boldsymbol{A}_{12} + \cdots + (-1)^{1+n} a_{1n} \det \boldsymbol{A}_{1n} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det \boldsymbol{A}_{1j}.$$

Here $\boldsymbol{A}_{ij}$ is the submatrix of $\boldsymbol{A}$ formed by removing the ith row and jth column of $\boldsymbol{A}$.

For the 2×2 matrix $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\det \boldsymbol{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$.

Example 1: Compute the determinants of both

$$\boldsymbol{A} = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{B} = \begin{bmatrix} 1 & -5 \\ 3 & 3 \end{bmatrix}.$$

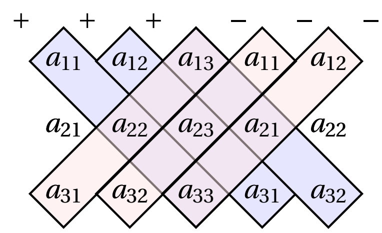
For a 3×3 matrix, we can compute the determinant as follows:

$$\det \boldsymbol{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= (a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{33}) - (a_{13}a_{22}a_{31} + a_{23}a_{32}a_{11} + a_{33}a_{12}a_{31})$$

A mnemonic device can be found by placing a copy of the first two columns after the third column and using diagonals, just like we did for cross products. In fact, the cross product of $\vec{u}$ and $\vec{v}$ is really a determinant:



$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{\boldsymbol{i}} & \vec{\boldsymbol{j}} & \vec{\boldsymbol{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Example 2: Compute the determinant of the matrix $A = \begin{bmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{bmatrix}$.

2.3.2 Cofactor Expansion

Definition: Given an $n \times n$ matrix $\boldsymbol{A} = [a_{ij}]$, the (i, j)-**cofactor** of $\boldsymbol{A}$ is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det \boldsymbol{A}_{ij}$$

where $\boldsymbol{A}_{ij}$ is the **minor** of $\boldsymbol{A}$ formed by removing the ith row and jth column. The determinant of an $n \times n$ matrix $\boldsymbol{A}$ can be computed by a cofactor expansion across any row or down any column of $\boldsymbol{A}$. The expansion across the ith row using cofactors is

$$\det \boldsymbol{A} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

and the cofactor expansion down the jth column is

$$\det \boldsymbol{A} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

This means that we can actually compute a determinant using any row or column! This is incredibly useful as some rows or columns are more "convenient" than others. All we need to do is ensure that we are computing the sign of the corresponding cofactors correctly.

Example 3: Compute the determinant of the matrix $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

Example 4: Compute the determinant of the matrix

$$A = \begin{bmatrix} 0 & 1 & 5 & 0 \\ 0 & 4 & -1 & 3 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -5 & 0 \end{bmatrix}$$

Concept Check: What does this say about the determinant of a **triangular** matrix? What about a diagonal matrix?

2.3.3 Properties of Determinants

Theorem: Let $\boldsymbol{A}$ be an $n \times n$ matrix.

a. If a multiple of one row of $\boldsymbol{A}$ is added to another row to produce a matrix $\boldsymbol{B}$, then $\det \boldsymbol{A} = \det \boldsymbol{B}$.
b. If two rows of $\boldsymbol{A}$ are interchanged to produce B, then $\det \boldsymbol{B} = -\det \boldsymbol{A}$.
c. If one row of $\boldsymbol{A}$ is multiplied by k to produce B, then $\det \boldsymbol{B} = k \cdot \det \boldsymbol{A}$.

This is useful when performing **row reduction** on a matrix, as you can transform a matrix into a triangular matrix via a sequence of successive **row operations** (adding a multiple of a row to another, interchanging two rows, and scaling rows).

Theorem: If $\boldsymbol{A}$ is an $n \times n$ matrix, then $\det \boldsymbol{A} = \det \boldsymbol{A}^T$.

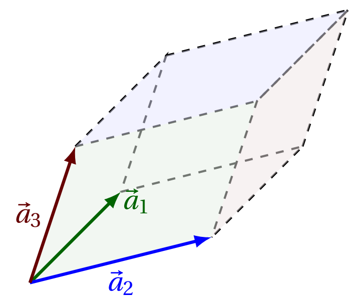
Theorem: If $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n \times n$ matrices, then $\det(\boldsymbol{AB}) = \det \boldsymbol{A} \det \boldsymbol{B}$.

Concept Check: If $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n \times n$ and $\det(\boldsymbol{A}) = 2$ and $\det(\boldsymbol{B}) = -3$, what is the determinant of the matrix $\boldsymbol{A}^2\boldsymbol{B}^T(\boldsymbol{A}^T\boldsymbol{B})$?

Example 5: Find $\det(\boldsymbol{AB})$ where

$$\boldsymbol{A} = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix}.$$

Let $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$ be the columns of a 3×3 matrix $\boldsymbol{A}$. Then the volume of the parallelepiped determined by $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$ is given by $|\vec{a}_1\cdot(\vec{a}_2\times\vec{a}_3)|$, the absolute value of the scalar triple product of those vectors. Note that



$$\vec{a}_1\cdot(\vec{a}_2\times\vec{a}_3) = \vec{a}_1\cdot\begin{bmatrix} a_{22}a_{33}-a_{23}a_{32} \\ a_{32}a_{13}-a_{12}a_{13} \\ a_{12}a_{23}-a_{22}a_{13}\end{bmatrix}$$

$$= a_{11}(a_{22}a_{33}-a_{23}a_{32}) + a_{21}(a_{32}a_{13}-a_{12}a_{33}) + a_{31}(a_{12}a_{23}-a_{22}a_{13}) = \det\boldsymbol{A}.$$

> If $\boldsymbol{A}$ is a 2×2 matrix, then the area of the parallelogram formed by the columns of $\boldsymbol{A}$ is $|\det\boldsymbol{A}|$. If $\boldsymbol{A}$ is a 3×3 matrix, then the volume of the parallelepiped formed by the columns of $\boldsymbol{A}$ is $|\det\boldsymbol{A}|$.

2.3.4 Inverses

Determinants play a crucial role in systems of n linear equations in n unknowns. For the moment, let's examine the linear system

$$\begin{aligned} ax_1+bx_2 &= y_1, \\ cx_1+dx_2 &= y_2. \end{aligned}$$

Note that the coefficient matrix of this linear system is $\boldsymbol{A}=\begin{bmatrix} a & b \\ c & d\end{bmatrix}$. Now, this system of equations will have unique solution for *any* values of y_1 and y_2 when the lines described by the equations are not parallel. When will this be the case? Well, if the lines are parallel, then their slopes b/a and d/c in the x_1x_2-plane must be the same. So we have

$$\frac{b}{a}=\frac{d}{c} \quad\Longrightarrow\quad bc=ad \quad\Longrightarrow\quad ad-bc=0.$$

Thus, the lines will not be parallel and therefore the system of equations will have a solution when $ad-bc\neq 0$, or in other words, when $\det\boldsymbol{A}\neq 0$. In this case, we can actually find the solution of the system of equations to be

$$x_1=\frac{1}{ad-bc}(dy_1-by_2), \quad x_2=\frac{1}{ad-bc}(ay_2-cy_1).$$

A way to represent this in matrix form is

$$\vec{x}=\boldsymbol{I}\vec{x}=\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix} x_! \\ x_2\end{bmatrix}=\begin{bmatrix} \dfrac{dy_1-by_2}{ad-bc} \\[2ex] \dfrac{-cy_1+ay_2}{ad-bc}\end{bmatrix}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a\end{bmatrix}\begin{bmatrix} y_1 \\ y_2\end{bmatrix}=\boldsymbol{M}\vec{y}$$

So the matrix-vector equation $\boldsymbol{A}\vec{x}=\vec{y}$ is equivalent to the equation $\boldsymbol{I}\vec{x}=\boldsymbol{M}\vec{y}$. What we really did was multiply $\boldsymbol{A}\vec{x}=\vec{y}$ by this special matrix $\boldsymbol{M}$ that satisfied $\boldsymbol{M}\boldsymbol{A}=\boldsymbol{I}$.

Definition: An $n \times n$ matrix $\boldsymbol{A}$ is **invertible** if there exists a matrix $\boldsymbol{A}^{-1}$ such that

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}.$$

In this case $\boldsymbol{A}^{-1}$ is the **inverse** of the matrix $\boldsymbol{A}$. An invertible matrix $\boldsymbol{A}$ is also said to be a **nonsingular** matrix. If no such $\boldsymbol{A}^{-1}$ exists, then $\boldsymbol{A}$ is said to be **singular**.

So the inverse of the matrix $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ exists when $ad - bc \neq 0$ and is given by

$$\boldsymbol{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Thus the unique solution to $\boldsymbol{A}\vec{x} = \vec{y}$ is $\vec{x} = \boldsymbol{A}^{-1}\vec{y}$. Although we will not prove this here, it is true that the determinant determines invertibility.

Theorem: An $n \times n$ matrix $\boldsymbol{A}$ is nonsingular if and only if $\det \boldsymbol{A} \neq 0$.

Concept Check: If the matrix $\boldsymbol{A}$ is invertible, is $\boldsymbol{A}^T$ also invertible?

Example 6: Solve the system of equations $\begin{aligned} x_1 - 2x_2 &= -1 \\ 4x_1 - 6x_2 &= 3 \end{aligned}$ by finding the inverse of the coefficient matrix.

Note that the rows of $\boldsymbol{A}^{-1}$ are made from the transposes of cross products of the columns of $\boldsymbol{A}$.

Some properties of invertible matrices:

Theorem:

a) If $\boldsymbol{A}$ is an invertible $n \times n$ matrix, then $\boldsymbol{A}^{-1}$ is invertible and $\left(\boldsymbol{A}^{-1}\right)^{-1} = \boldsymbol{A}$.
b) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible $n \times n$ matrices, then so is $\boldsymbol{AB}$, and $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.
c) If $\boldsymbol{A}$ is an invertible $n \times n$ matrix, then $\boldsymbol{A}^T$ is invertible and $\left(\boldsymbol{A}^T\right)^{-1} = \left(\boldsymbol{A}^{-1}\right)^T$.

Example 7: (†) Suppose $\boldsymbol{A}_{11}$ is an invertible matrix. Find matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that the product below has the form indicated, and compute $\boldsymbol{B}_{22}$.

$$\begin{bmatrix} \boldsymbol{I} & 0 & 0 \\ \boldsymbol{X} & \boldsymbol{I} & 0 \\ \boldsymbol{Y} & 0 & \boldsymbol{I} \end{bmatrix}\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \\ \boldsymbol{A}_{31} & \boldsymbol{A}_{32} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ 0 & \boldsymbol{B}_{22} \\ 0 & \boldsymbol{B}_{32} \end{bmatrix}$$

2.3.5 (†) Cramer's Rule

This is an explicit formula for solving systems of equations, using determinants:

Cramer's Rule: Let $\boldsymbol{A}$ be an $n \times n$ invertible matrix. For any $\vec{b} \in \mathbb{R}^n$, the solution $\vec{x}$ of $\boldsymbol{A}\vec{x} = \vec{b}$ is given by

$$x_i = \frac{\det \boldsymbol{A}_i(\vec{b})}{\det \boldsymbol{A}}, \qquad 1 \le i \le n,$$

where the matrix $\boldsymbol{A}_i(\vec{b})$ is the matrix obtained by replacing the ith column of $\boldsymbol{A}$ with the vector $\vec{b}$.

This can be useful for solving very small systems of equations, such as 2×2 or 3×3 systems. However it is impractical for larger systems, because the number of mathematical operations required to compute a determinant increases dramatically with the size of the matrix. For example, computing the determinant of a 30×30 matrix by cofactor expansion involves about 31! arithmetic operations while the fastest computer as of 2020 can perform about 5×10^{17} arithmetic operations per second, so it would take about half a million years to do this. Instead, using the method of substitution or Gaussian elimination, the processor on a mobile phone can solve such a system in a small fraction of a second.

Even the fastest algorithm for computing the determinant involves almost as much computation as solving the corresponding equations, so computing the $n+1$ determinants in Cramer's Rule makes the computational cost about n times greater.

Example 8: (†). Solve the system of equations below using Cramer's Rule.

$$\begin{bmatrix} 2 & 4 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 21 \\ 4 \\ 10 \end{bmatrix}$$

2.3 Review of Concepts

- Terms to know: determinant, cofactor, minor, triangular matrix, inverse matrix, invertible, singular, nonsingular, Cramer's rule.
- Know how to compute determinants of matrices by cofactor expansion.
- Know how to find the inverse of a 2×2 matrix.
- Know how to manipulate partitioned matrices and their inverses.

2.3 Practice Problems

1. Find the determinants of the following matrices:

a) $\boldsymbol{A} = \begin{bmatrix} 4 & -3 \\ 3 & -4 \end{bmatrix}$ b) $\boldsymbol{B} = \begin{bmatrix} 2 & -3 & 4 \\ 0 & 1 & -3 \\ 1 & 2 & 2 \end{bmatrix}$ c) $\boldsymbol{C} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -2 & 0 \\ -1 & 1 & 0 \end{bmatrix}$ d) $\boldsymbol{D} = \begin{bmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{bmatrix}$

2. Compute the determinant of $\boldsymbol{A}^4$ where $\boldsymbol{A} = \begin{bmatrix} 1 & 1/2 & 0 \\ 2 & -1 & 1 \\ 4 & 3 & 0 \end{bmatrix}$.

3. Find the solution of the linear system

$$\begin{aligned} 2x_1 - 3x_2 + 4x_3 &= 8 \\ x_2 - 3x_3 &= -7 \\ x_1 + 2x_2 + 2x_3 &= 11 \end{aligned}.$$

4. (†) Find formulas for the matrices $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$ in terms of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$. You may need to make assumptions about the size and invertibility of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$.

$$\begin{bmatrix} \boldsymbol{X} & 0 \\ \boldsymbol{Y} & \boldsymbol{Z} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & 0 \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{I} \end{bmatrix}.$$

2.3 Exercises

1. Determine if the statements below are true or false, giving a brief explanation or counterexample.

a) For an $n \times n$ matrix $\boldsymbol{A}$, $\det(\boldsymbol{A}^2) = (\det(\boldsymbol{A}))^2$.

b) For $n \times n$ matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, $\det(\boldsymbol{ABC}) = \det(\boldsymbol{C})\det(\boldsymbol{A})\det(\boldsymbol{B})$.

c) For $n \times n$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, $\det(\boldsymbol{A}+\boldsymbol{B}) = \det(\boldsymbol{A}) + \det(\boldsymbol{B})$.

d) The product of invertible matrices is always invertible.

e) The product of singular matrices is always singular.

f) For invertible $n \times n$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, $\left((\boldsymbol{AB})^T\right)^{-1} = \left(\boldsymbol{A}^{-1}\right)^T \left(\boldsymbol{B}^{-1}\right)^T$.

2. Find the determinants of the following matrices:

a) $\boldsymbol{A} = \begin{bmatrix} 0 & 3 & 2 \\ 2 & 0 & 1 \\ 2 & 6 & 0 \end{bmatrix}$ b) $\boldsymbol{B} = \begin{bmatrix} 3 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 4 & 2 \end{bmatrix}$ c) $\boldsymbol{C} = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$

3. (†) Find the inverse of the matrix $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ 0 & \boldsymbol{A}_{22} \end{bmatrix}$ where $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22}$ are $m \times m$ and $n \times n$ invertible matrices, respectively. You should check that your inverse works on both the left and the right of $\boldsymbol{A}$.

4. How is $\det \boldsymbol{A}^{-1}$ related to $\det \boldsymbol{A}$? Explain.

5. Explain why the matrix $\boldsymbol{A} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is invertible for any value of θ. Give the inverse matrix (hint: one approach is to use blocks).

6. Let $\boldsymbol{A} = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$. Compute $\boldsymbol{A}^3$, $\boldsymbol{A}^{-3}$, and $\boldsymbol{A}^2 - 2\boldsymbol{A} + \boldsymbol{I}$.

7. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Find the inverse of $\boldsymbol{A}$ by solving the matrix equation $\boldsymbol{AX} = \boldsymbol{I}$ for $\boldsymbol{X}$.

8. Find 2×2 matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ such that $(\boldsymbol{A}+\boldsymbol{B})^2 \neq \boldsymbol{A}^2 + 2\boldsymbol{AB} + \boldsymbol{B}^2$. What relationship would $\boldsymbol{A}$ and $\boldsymbol{B}$ need to satisfy in order to ensure $(\boldsymbol{A}+\boldsymbol{B})^2 = \boldsymbol{A}^2 + 2\boldsymbol{AB} + \boldsymbol{B}^2$?

9. The **trace** of a square matrix $\boldsymbol{A}$, denoted $\mathrm{tr}(\boldsymbol{A})$, is the sum of the entries on the diagonal of $\boldsymbol{A}$:

$$\mathrm{tr}(\boldsymbol{A}) = \sum_{i=1}^{n} a_{ii}.$$

Explain why $\mathrm{tr}(\boldsymbol{AB}) = \mathrm{tr}(\boldsymbol{BA})$. Is it possible for any square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ to satisfy $\boldsymbol{AB} - \boldsymbol{BA} = \boldsymbol{I}$?

2.3 Answers to Practice Problems

1. a)

$$\det(\boldsymbol{A}) = \begin{vmatrix} 4 & -3 \\ 3 & -4 \end{vmatrix} = -16 - (-9) = -7.$$

b)

$$\det(\boldsymbol{B}) = \begin{vmatrix} 2 & -3 & 4 \\ 0 & 1 & -3 \\ 1 & 2 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & -3 \\ 2 & 2 \end{vmatrix} - 0 + 1\begin{vmatrix} -3 & 4 \\ 1 & -3 \end{vmatrix} = 2(2-(-6)) + (9-4) = 16 + 5 = 21.$$

c)

$$\det(\boldsymbol{C}) = \begin{vmatrix} 2 & 1 & 3 \\ 4 & -2 & 0 \\ -1 & 1 & 0 \end{vmatrix} = 3\begin{vmatrix} 4 & -2 \\ -1 & 1 \end{vmatrix} - 0 + 0 = 3(4-2) = 6.$$

d)

$$\det(\boldsymbol{D}) = \begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix} = (-1)^{2+3} \cdot 2 \begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix} = -2\left((-1)^{2+2} \cdot 3 \begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix} \right)$$

$$= -2\left(3\left(0 - (-1)\begin{vmatrix} 4 & -5 \\ 5 & -3 \end{vmatrix} + 2\begin{vmatrix} 4 & 3 \\ 5 & 2 \end{vmatrix}\right)\right) = -2\,(3\,((-12+25) + 2(8-15))) = -2 \cdot 3 \cdot (-1) = 6.$$

2. Because $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$, we have that $\det(\boldsymbol{A}^4) = (\det(\boldsymbol{A}))^4$. Now

$$\det(\boldsymbol{A}) = \begin{vmatrix} 1 & 1/2 & 0 \\ 2 & -1 & 1 \\ 4 & 3 & 0 \end{vmatrix} = (-1)^{2+3} \begin{vmatrix} 1 & 1/2 \\ 4 & 3 \end{vmatrix} = -1(3-2) = -1,$$

so $\det(\boldsymbol{A}^4) = (-1)^4 = 1$.

3. We want to solve

$$2x_1 - 3x_2 + 4x_3 = 8, \tag{2.1}$$

$$x_2 - 3x_3 = -7, \tag{2.2}$$

$$x_1 + 2x_2 + 2x_3 = 11. \tag{2.3}$$

Take (2.1) and subtract 2 times (2.3) to get

$$2x_1 - 3x_2 + 4x_3 - 2x_1 - 4x_2 - 4x_3 = 8 - 22$$

or

$$-7x_2 = -14.$$

Then $x_2 = 2$, and (2.2) then implies

$$2 - 3x_3 = -7$$

which implies $3x_3 = -7 - 2 = -9$ so $x_3 = 3$. Finally substituting these values for x_2 and x_3 into (2.1) we have

$$2x_1 - 3(2) + 4(3) = 8,$$

or $2x_1 = 2$, so $x_1 = 1$.

4. From the equation $\begin{bmatrix} \boldsymbol{X} & 0 \\ \boldsymbol{Y} & \boldsymbol{Z} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & 0 \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{I} \end{bmatrix}$ we have the following:

$$\begin{bmatrix} \boldsymbol{XA} + 0 & 0 \\ \boldsymbol{YA} + \boldsymbol{ZB} & \boldsymbol{ZC} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{I} \end{bmatrix}$$

which implies the three equations

$$\boldsymbol{XA} = \boldsymbol{I}, \qquad \boldsymbol{YA} + \boldsymbol{ZB} = 0, \qquad \boldsymbol{ZC} = \boldsymbol{I}.$$

We know that $\boldsymbol{A}$ must be invertible and that $\boldsymbol{X}$ is its inverse, i.e. multiplying the first equation on the right by $\boldsymbol{A}^{-1}$ gives

$$\boldsymbol{X} = \boldsymbol{XI} = \boldsymbol{X}(\boldsymbol{AA}^{-1}) = \boldsymbol{IA}^{-1} = \boldsymbol{A}^{-1}.$$

Also, we have that $\boldsymbol{C}$ must be invertible and therefore $\boldsymbol{Z} = \boldsymbol{C}^{-1}$. Finally, we have that $\boldsymbol{YA} + \boldsymbol{ZB} = 0$ implies

$$\boldsymbol{YA} = -\boldsymbol{ZB} \qquad \Longrightarrow \qquad \boldsymbol{Y} = -\boldsymbol{C}{-}1\boldsymbol{BA}^{-1}.$$

2.4 Linear Independence and Bases

Objectives and Concepts:

- A set of vectors in linearly dependent if one of the vectors can be written as a linear combination of the other vectors in the set, otherwise they are linearly independent.
- Determinants can be used to determine whether or not a set of n vectors in $\mathbb{R}^n$ are independent.
- Any independent set of n vectors in $\mathbb{R}^n$ forms a basis, and every vector in $\mathbb{R}^n$ can be written in terms of any basis for $\mathbb{R}^n$.

References: FCLA Sections *Linear Combinations*, *Spanning Sets*, *Linear Independence*, *Linear Dependence and Spans*, and Chapter *Vector Spaces* up to Section *Bases*

2.4.1 Linear Independence and Dependence

Recall in our discussion of the invertibility of the matrix $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that we determined the matrix is invertible if the lines $ax_1 + bx_2 = y_1$ and $cx_1 + dx_2 = y_2$ were not parallel - if the rows of $\boldsymbol{A}$ we not scalar multiples of each other. This also translates to the columns of $\boldsymbol{A}$ not being scalar multiples of each other. Another way to say this is that the vectors $\vec{a}_1$ and $\vec{a}_2$ that formed the columns of $\boldsymbol{A}$ were **linearly independent**.

Definition: Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be vectors in $\mathbb{R}^n$. We say that the set $\{\vec{v}_1, \ldots, \vec{v}_m\}$ is a **linearly independent** set if the vector equation

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_m \vec{v}_m = \vec{0}.$$

has only the **trivial** solution $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$. The set $\{\vec{v}_1, \ldots, \vec{v}_m\}$ is **linearly dependent** if there are choices for $\alpha_1, \alpha_2, \ldots, \alpha_m$, not all zero, such that

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_m \vec{v}_m = \vec{0}.$$

In this case The above equation is called a **linear dependence relation** among $\vec{v}_1, \ldots, \vec{v}_m$.

If $\boldsymbol{A} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{bmatrix}$, the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are linearly independent if the system of equations $\boldsymbol{A}\vec{x} = \vec{0}$ has only $\vec{x} = \vec{0}$ as a solution. Also, if the $\vec{v}_i$ are dependent, there will be a **nontrivial** solution to $\boldsymbol{A}\vec{x} = \vec{0}$. **This means that at least one column of $\boldsymbol{A}$ can be written as a linear combination of the other columns of $\boldsymbol{A}$.**

Now, any single nonzero vector in $\mathbb{R}^n$ is linearly independent. Also, any two vectors in $\mathbb{R}^n$ $(n \geq 2)$ are independent as long as they are not scalar multiples of each other. For more than two vectors, it is not always clear. To determine if a set of vectors is independent or dependent, you must determine if the system $\boldsymbol{A}\vec{x} = \vec{0}$, where the vectors form the columns of $\boldsymbol{A}$, has only the trivial solution.

Example 1: Determine if the set of vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

is linearly independent or dependent. If they are dependent, find a dependence relation.

An interesting case that can be considered immediately is when $m > n$:

Theorem: Any set of more than n vectors in $\mathbb{R}^n$ is linearly dependent.

This is true because any consistent system of n equations with at least $n+1$ unknowns has at least one free variable. When you have exactly n vectors in $\mathbb{R}^n$, you can determine dependence or independence using determinants.

Theorem: The columns of $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ are linearly independent if and only if $\det \boldsymbol{A} \neq 0$.

Example 2: Find the value(s) of h such that the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}$ are linearly dependent.

2.4.2 Vector Spaces

A **vector space** is a nonempty set V of objects, called vectors, on which the operations of addition and scalar multiplication are defined. A formal definition is now given.

Definition: A **vector space** (over $\mathbb{R}$) consists of a set V along with the operations of vector addition and scalar multiplication such that for all vectors $\vec{u}, \vec{v}, \vec{w} \in V$, and all scalars $r, s \in \mathbb{R}$:

1. the set V is closed under vector addition, that is, $\vec{v} + \vec{w} \in V$
2. vector addition is commutative, $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
3. vector addition is associative, $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$
4. there is a zero vector $\vec{0} \in V$ such that $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v} \in V$
5. each $\vec{v} \in V$ has an additive inverse $\vec{w} \in V$ such that $\vec{w} + \vec{v} = \vec{0}$
6. the set V is closed under scalar multiplication, that is, $r\vec{v} \in V$
7. addition of scalars distributes over scalar multiplication, $(r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$
8. scalar multiplication distributes over vector addition, $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$
9. ordinary multiplication of scalars associates with scalar multiplication, $(rs) \cdot \vec{v} = r \cdot (s \cdot \vec{v})$
10. multiplication by the scalar 1 is the identity operation, $1 \cdot \vec{v} = \vec{v}$.

Example 3: The set of all polynomials $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ where n is a nonnegative integer and the $a_i \in \mathbb{R}$ for all i is a vector space. (Discuss why this set satisfies each part of the definition.)

Example 4: For positive integer m and n, the set of all complex matrices of size $m \times n$ is a vector space.

Example 5: For a real interval $[a, b]$, the set of all continuous functions with domain $[a, b]$ is a vector space.

While there are many different types of vector spaces, including spaces of functions, matrices, sequences, polynomials, and many other objects, here we are primarily concerned with the vector space $\mathbb{R}^n$, in which the objects are exactly the vectors we have been talking about.

When the standard rules of arithmetic apply (vector addition and scalar multiplication are associative, commutative, and distributive laws hold) the fundamental characterization of a vector space is that it is **closed** under addition and scalar multiplication - this means that any linear combination of vectors from the space results in a vector that is also in the space.

Definition: Let $\{\vec{v}_1, \ldots, \vec{v}_m\}$ be a set of m vectors in $\mathbb{R}^n$. The **span** of the set of vectors is the set of all possible linear combinations of the $\vec{v}_i$:

$$\text{span}\{\vec{v}_1, \ldots, \vec{v}_m\} = \left\{\vec{w} \in \mathbb{R}^n \mid \vec{w} = c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m \text{ for some } c_i \in \mathbb{R}, 1 \le i \le n\right\}$$

The span of a set of vectors from $\mathbb{R}^n$ forms another set of vectors that is known as a **subspace** of $\mathbb{R}^n$. For example, when the vectors are considered to be position vectors of points in space, the span of any single vector in $\mathbb{R}^3$ is a line through the origin, and the span of any two independent vectors in $\mathbb{R}^3$ forms a plane through the origin. This is a subspace because any vector in the plane can be written as a linear combination of the two independent vectors.

Concept Check: Describe the span of the two vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$.

Concept Check: Must the span of any nonempty set of vectors in $\mathbb{R}^n$ contain the zero vector?

Example 6: (a) For what value(s) of h is $\vec{v}_3$ in span$\{\vec{v}_1, \vec{v}_2\}$, and (b) for what value(s) of h are $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly independent?

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -2 \\ 9 \\ 15 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 2 \\ -5 \\ h \end{bmatrix}$$

2.4.3 Bases of $\mathbb{R}^n$

Recall that the vectors $\vec{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{\jmath} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ formed the **standard basis** for $\mathbb{R}^3$. We now explain what we mean by this name.

Definition: Let V be a subspace of $\mathbb{R}^n$. The linearly independent set of vectors $\mathscr{B} = \{\vec{v}_1, \ldots, \vec{v}_m\}$ $(m \le n)$ is a **basis** for V if span$\{\vec{v}_1, \ldots, \vec{v}_m\} = V$, that is, if any vector $\vec{v} \in V$ can be written as $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$ for some choice of the constants $c_1, \ldots, c_m$.

So while the set $\left\{\vec{\imath}, \vec{\jmath}, \vec{k}\right\}$ is a basis for $\mathbb{R}^3$, it is just one of infinitely many bases. For example, any vector $\vec{v} \in \mathbb{R}^3$ can also be written in terms of the basis vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

When we write a vector as a linear combination of vectors in a basis $\mathscr{B}$ (different from the standard basis) we are really performing a **change of basis**. The actual values of the constants in the linear combination are the **coordinates** of the vector in the new basis $\mathscr{B}$.

Example 7: What are the values of c_1 and c_2 if $\begin{bmatrix} 4 \\ -7 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$?

Much like we used a subscript to denote the same point in different coordinate systems, we can use a subscript to denote the coordinate vector of a vector in a basis $\mathscr{B}$.

Example 8: The coordinate vector of $\vec{x} = \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}$ in the basis $\mathscr{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} \right\}$ is the vector $\begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}_{\mathscr{B}}$.

What is the coordinate vector $\vec{y}$ of $\vec{x}$ in the basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$?

Note that we really are solving a linear system to find the coordinate vector $\vec{y}$! So, if the matrix $\boldsymbol{B}$ has the columns $\vec{v}_1, \vec{v}_2, \vec{v}_3$, then finding the coordinate vector is equivalent to solving the matrix-vector equation $\boldsymbol{B}\vec{y} = \vec{x}$ for $\vec{y}$. In this situation we refer to $\boldsymbol{B}$ as the **change of coordinates matrix** of $\mathscr{B}$. Since the columns of $\boldsymbol{B}$ form a linearly independent set, we have $\det \boldsymbol{B} \neq 0$, and therefore to convert from the standard basis to the $\mathscr{B}$-coordinate vector, we multiply $\boldsymbol{B}^{-1}\vec{x}$ to get $\vec{y}$. **Every vector in $\mathbb{R}^3$ has a unique coordinate representation in any basis of $\mathbb{R}^3$.**

2.4 Review of Concepts

- Terms to know: linearly independent, linearly dependent, linear dependence relation, trivial, nontrivial, vector space, subspace, closed, span, basis, coordinate vector, change of basis matrix.
- Know how to determine if a set of vectors is linearly independent or dependent, and how to find a linear dependence relation if the set is dependent.
- Know how to find coordinates of a vector in different bases.

2.4 Practice Problems

1. Determine which sets of vectors in $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$. If the set is not a basis, find a linear dependence relation among the vectors.

a) $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}$ b) $\begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$ c) $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ 4 \end{bmatrix}$

2. Find a basis $\mathscr{B}$ in $\mathbb{R}^3$ such that the coordinate vector of $\vec{x} = \begin{bmatrix} 3 \\ 1 \\ -10 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}_{\mathscr{B}}$.

2.4 Exercises

1. Determine if the following statements are true or false, giving a brief explanation or counterexample in each case.

a) The columns of a matrix $\boldsymbol{A}$ are linearly independent if and only if the equation $\boldsymbol{A}\vec{x} = \vec{0}$ has the trivial solution.

b) If S is a linearly dependent set of vectors, then each vector in S is a linear combination of the other vectors in S.

c) If the vectors $\vec{x}$ and $\vec{y}$ are independent and $\vec{z}$ is in span$\{\vec{x}, \vec{y}\}$, then the set $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly independent.

d) If a set of vectors contains fewer vectors than there are entries in the vectors, then the set is linearly independent.

e) The columns of a 4×5 matrix must be linearly dependent.

2. The vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ -8 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$ span $\mathbb{R}^2$ but do not form a basis. Find two different ways to express $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

3. Find the coordinates of $\vec{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ in the basis $\mathscr{B} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \end{bmatrix} \right\}$.

4. Find the values of λ for which the following system of equations has a **nontrivial** (not all zeros) solution, and find the solution for these values of λ.

$$\begin{aligned} 2x_1 + x_2 &= \lambda x_1 \\ x_1 + 2x_2 &= \lambda x_2 \end{aligned}$$

2.4 Answers to Practice Problems

1. a) These vectors are dependent. To see why, compute the determinant

$$\begin{vmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ -2 & -4 & 1 \end{vmatrix} = (2-20) - 2(-15+6) = -18 + 18 = 0.$$

To find a linear dependence relation, we must find a nontrivial solution to $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$. Thus we want to solve the system

$$\begin{aligned} x_1 + 3x_2 - 3x_3 &= 0 \\ 2x_2 - 5x_3 &= 0 \\ -2x_1 - 4x_2 + x_3 &= 0 \end{aligned}$$

This system has the solution

$$x_1 = -\frac{9}{2}t, \qquad x_2 = \frac{5}{2}t, \qquad x_3 = t,$$

for any real number t. Thus a linear dependence relation can be obtained (set $t = 2$) as

$$9\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - 5\begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} - 2\begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

b) These vectors are independent as

$$\begin{vmatrix} 0 & 3 & 0 \\ 3 & -5 & 2 \\ -1 & 4 & -2 \end{vmatrix} = -3(-6+2) = 12.$$

c) These vectors are independent as

$$\begin{vmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{vmatrix} = 2(-12-10) - 1(-8-5) - 7(-4+3) = -44 + 13 + 7.$$

2. To find a basis $\mathscr{B}$ in $\mathbb{R}^3$ such that the coordinate vector of $\vec{x} = \begin{bmatrix} 3 \\ 1 \\ -10 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}_{\mathscr{B}}$, we simply need to find three linearly independent vectors $\vec{u}, \vec{v}, \vec{w}$ such that

$$\vec{u} - 4\vec{v} + 2\vec{w} = \begin{bmatrix} 3 \\ 1 \\ -10 \end{bmatrix}.$$

You can find such a basis by simply choosing one of the basis vectors and then finding a choice of the others that will work (and then check that they are independent). For example, let $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Then we must have that

$\vec{v}$ and $\vec{w}$ satisfy

$$4\vec{v}+2\vec{w}=\begin{bmatrix}3\\1\\-10\end{bmatrix}-\vec{u}=\begin{bmatrix}2\\0\\-10\end{bmatrix}.$$

Another way to say this is we need a solution to the system

$$-4v_1+2w_1=2, \qquad -4v_2+2w_2=0, \qquad -4v_3+2w_3=-10.$$

Let $\vec{v}=\begin{bmatrix}0\\0\\2\end{bmatrix}$. Then $\vec{w}=\begin{bmatrix}1\\0\\-1\end{bmatrix}$ satisfies the other equations. Thus one basis is

$$\mathscr{B}=\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\2\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}.$$

There are many more, for example

$$\mathscr{B}=\left\{\begin{bmatrix}3\\1\\-4\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}2\\2\\-1\end{bmatrix}\right\}$$

is another basis such that the coordinate vector of $\vec{x}=\begin{bmatrix}3\\1\\-10\end{bmatrix}$ is $\begin{bmatrix}1\\-4\\2\end{bmatrix}_{\mathscr{B}}$.

2.5 (‡) Linear Transformations and Operators

Objectives and Concepts:

- A transformation or operator maps a vector in a vector space to another vector in the same or different vector space.
- A linear transformation is a transformation with additional properties that describe linearity.
- Linear transformations in $\mathbb{R}^2$ or $\mathbb{R}^3$ can be determined by their action on the standard basis vectors and can be represented by matrices.

References: TCMB §§18.1,5,6, 19.6; FCLA Chapter LT, *Linear Transformations*,

2.5.1 Transformations and Operators

We have previously described the matrix-vector product $\boldsymbol{A}\vec{x}$ (where $\boldsymbol{A}$ is $m \times n$ and $\vec{x}$ is $n \times 1$) as a linear combination (with weights $x_1, \ldots, x_n$) of the vectors that form the columns of $\boldsymbol{A}$. However, another way to think about matrix-vector multiplication is that $\boldsymbol{A}$ is a **transformation** (or **operator**) acting on the vector $\vec{x}$. In this sense, an $m \times n$ matrix $\boldsymbol{A}$ is really a function that takes a vector from $\mathbb{R}^n$ and maps it to a vector in $\mathbb{R}^m$. We can also extend this idea to mapping points in $\mathbb{R}^n$ to points in $\mathbb{R}^m$ by considering the vectors to be position vectors of the points.

Definition: A **transformation** or **operator** T from a vector space V to another vector space W is a rule that assigned to each vector v in V a vector w in W. The set V is is called the **domain** of T, and W is called the **codomain** of T. The notation $T : V \to W$ indicates the domain of T is V and the codomain is W. For v in v, the vector $w = T(v)$ in W is the **image** of v under T. The set of all images $T(v)$ is called the **range** of T. The transformation $T : V \to W$ is said to be **onto** W if each w in W is the image of *at least one* v in V. The transformation $T : V \to W$ is said to be **one-to-one** if each w in W is the image of *at most one* v in V.

Many of these concepts coincide with ideas we are familiar with regarding functions defined on the set of real numbers. Indeed, a function on $\mathbb{R}$ is also a transformation or operator, however transformations and operators can be much more general in the sense of the objects (vectors) that they act upon and map to.

Many of the transformations we will encounter are **matrix transformations**. In this case, we usually write $T : \mathbb{R}^n \to \mathbb{R}^m$ and T maps a vector $\vec{x}$ in $\mathbb{R}^n$ to a vector $T(\vec{x}) = \boldsymbol{A}\vec{x}$ in $\mathbb{R}^m$, where $\boldsymbol{A}$ is the **matrix of the transformation** T.

Example 1: Let $\boldsymbol{A} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\boldsymbol{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, and define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\vec{x}) = \boldsymbol{A}\vec{x}$, so that

$$T(\vec{x}) = \boldsymbol{A}\vec{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

a) Find $T(\vec{u})$, the image of $\vec{u}$ under T.

b) Find an $\vec{x}$ in $\mathbb{R}^2$ whose image under T is $\boldsymbol{b}$.

c) Is there more than one $\vec{x}$ whose image under T is $\boldsymbol{b}$?

d) Determine if $\boldsymbol{c}$ is in the range of the transformation T.

e) Is T onto $\mathbb{R}^3$?

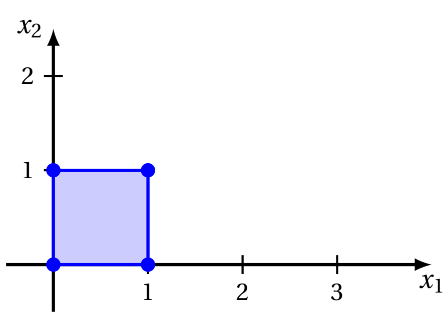
Example 2: If $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then the transformation $\vec{x} \mapsto \boldsymbol{A}\vec{x}$ *projects* vectors (points) in $\mathbb{R}^3$ onto the x_1x_2-plane because

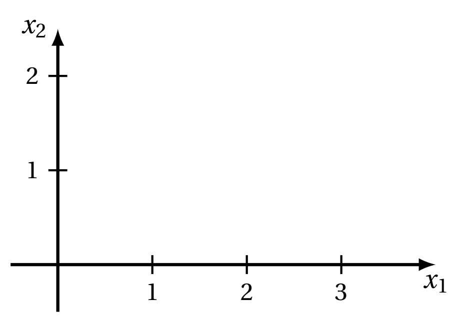
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}.$$

Is T one-to-one? Is T onto $\mathbb{R}^3$?

Example 3: Let $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{x}) = \boldsymbol{A}\vec{x}$ is called a **shear transformation**. To see why, examine the action of T on the points that define the unit square in the first quadrant:

$$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \qquad T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \qquad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \qquad T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) =$$





Of course, transformations and operators are not limited to simply matrices acting on vectors.

Example 4: Let P be the set of all polynomials in the indeterminate x: that is, let

$$P = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid n \text{ is a positive integer}, a_i \in \mathbb{R}, 0 \le i \le n\}.$$

(It is true that P is a vector space). Let the mapping T be defined by

$$T(p) = \frac{d}{dx} p,$$

i.e., $T(p)$ is the derivative of p with respect to x. For example,

$$T(3 - 4x + 2x^2 + 6x^5) = -4 + 4x + 30x^4.$$

Then T is a transformation, it is often referred to as the *differentiation operator*.

Is T one-to-one? Is T onto P?

Example 5: Let F be the set of all functions from the real numbers to the real numbers and let T be the operator defined by $T(f) = f^2$. For example, $T(\sin x) = \sin^2 x$.

Is T one-to-one? Is T onto F?

2.5.2 Linear Transformations

Definition: A transformation (or mapping) T is **linear** if

i. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in the domain of T;

ii. $T(c\vec{u}) = cT(\vec{u})$ for all scalars c and all $\vec{u}$ in the domain of T.

Often the two conditions for linearity are checked at the same time by determining if

$$T(\alpha\vec{x} + \beta\vec{y}) = \alpha T(\vec{x}) + \beta T(\vec{y}).$$

It is fairly easy to see that every matrix transformation is a linear transformation. Let $T(\vec{x}) = \boldsymbol{A}\vec{x}$. Then

$$T(\alpha\vec{x} + \beta\vec{y}) = \boldsymbol{A}(\alpha\vec{x} + \beta\vec{y}) = \boldsymbol{A}(\alpha\vec{x}) + \boldsymbol{A}(\alpha\vec{y}) = \alpha\boldsymbol{A}\vec{x} + \beta\boldsymbol{A}\vec{y} = \alpha T(\vec{x}) + \beta T(\vec{y}).$$

Concept Check: Is the differentiation operator on the set of polynomials a linear operator?

Concept Check: Is the operator defined in Example 5 a linear operator?

Example 6: Determine if the transformation $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = 3x - 2$ is a linear transformation.

If T is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$.

2.5.3 Matrix Transformations

Recall the standard basis vectors $\vec{\imath}, \vec{\jmath}, \vec{k}$. We will use a slightly different notation for these that extends well when the dimensions of the vector space $\mathbb{R}^3$ are indexed.

> **Definition:** Let $\vec{e}_j$, $1 \le j \le n$ represent the jth column of the $n \times n$ identity matrix. We say that $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ is the **standard basis** (or **elementary basis**) for $\mathbb{R}^n$.

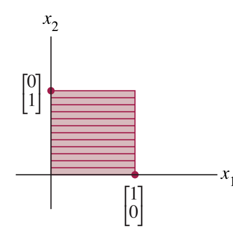
> **Theorem:** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix $\boldsymbol{A}$ such that $T(\vec{x}) = \boldsymbol{A}\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^n$. In fact, $\boldsymbol{A}$ is the $m \times n$ matrix whose jth column is the vector $T(\boldsymbol{e}_j)$, where $\boldsymbol{e}_j$ is the jth column of I_n. We have
>
> $$\boldsymbol{A} = \begin{bmatrix} T(\boldsymbol{e}_1) & T(\boldsymbol{e}_2) & \cdots & T(\boldsymbol{e}_n) \end{bmatrix}.$$

Thus the action of an $n \times n$ matrix transformation T is completely determined by the action of T on the standard basis. The columns of the matrix $\boldsymbol{A}$ of the transformation T are given by $T(\vec{e}_j)$. The matrix $\boldsymbol{A}$ above is called the **standard matrix for the linear transformation** T.

Example 7: Find the standard matrix $\boldsymbol{A}$ for the dilation transformation $T(\vec{x}) = 3\vec{x}$, for $\vec{x}$ in $\mathbb{R}^2$.

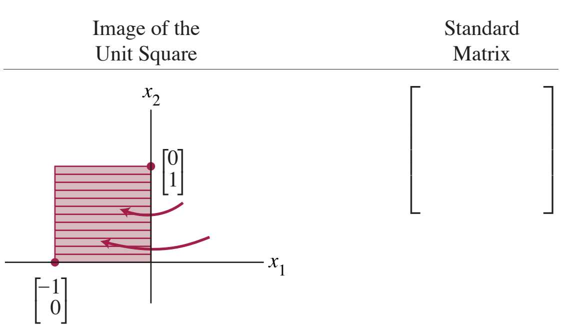
For any particular action on the unit square in $\mathbb{R}^2$, we can find the matrix $\boldsymbol{A}$ of T by determining the action of T on the basis vectors

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

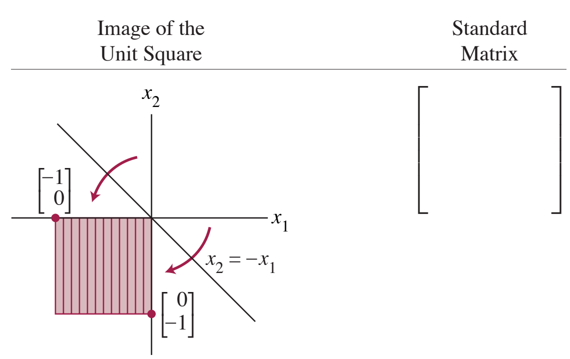


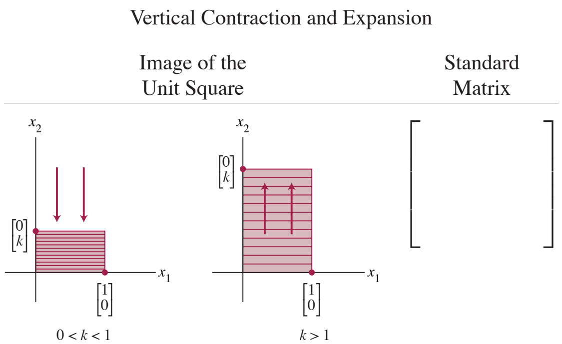
Example 8: Find the standard matrices of the given linear transformations.

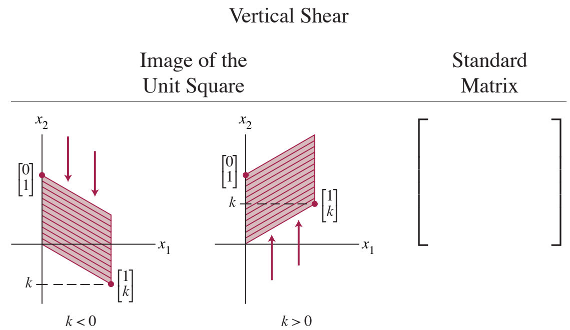
Reflection in the x_2-axis



Reflection in the line $x_2 = -x_1$







Example 9: Find the standard matrices of the transformations of reflection through the line $x_1 = x_2$ and reflection through the origin.

Concept Check: What is the standard matrix of the transformation that represents reflection of a vector in $\mathbb{R}^3$ across the x_1x_2-plane? What about through the x_2x_3-plane?

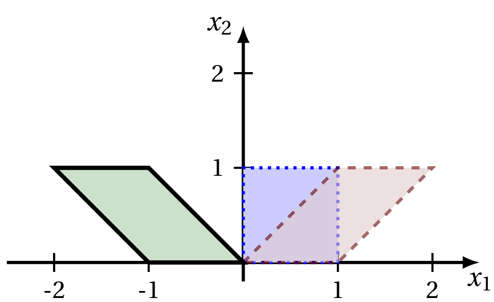
If we want to apply several transformations to a vector, we can do so through left multiplication. For example, if we first want to reflect a vector $\vec{x}$ across the x_2 axis (T_1) and then dilate the x_1 coordinate by 2 (T_2), we would perform the composite transformation $(T_2 \circ T_1)(\vec{x}) = T_2(T_1(\vec{x})) = \boldsymbol{A}_2\boldsymbol{A}_1\vec{x}$. Performing these transformations in this order would really just be the matrix multiplication of

$$\boldsymbol{A}_2\boldsymbol{A}_1\vec{x} = \begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-2x_1\\x_2\end{bmatrix}.$$

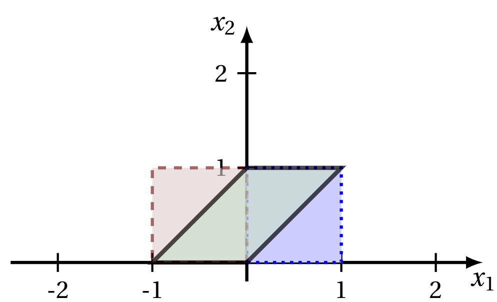
Note that we would end up with the same result had we performed T_2 first and then T_1:

$$\boldsymbol{A}_1\boldsymbol{A}_2\vec{x} = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-2x_1\\x_2\end{bmatrix}.$$

However, it is usually **not** the case that the order of transformations does not matter in their composition. For example, if you perform a horizontal shear of 1 unit to the right and then reflect through across the x_2-axis, then you do not get the same result as if you perform those operations in reverse order.



$$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-x_1-x_2\\x_2\end{bmatrix}$$



$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_2-x_1\\x_2\end{bmatrix}$$

In these cases, the matrices of the transformations do not commute.

Definition: The **commutator** of the operators $\boldsymbol{A}$ and $\boldsymbol{B}$ is the operator $[\boldsymbol{A},\boldsymbol{B}] = \boldsymbol{AB} - \boldsymbol{BA}$.

Example 10: Compute the commutator of $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

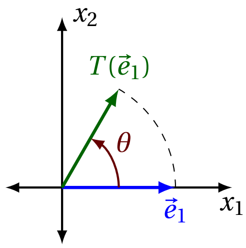
Example 11: Let V be the vector space of functions of a single variable x, let A represent the linear operator of differentiation with respect to x, i.e., if $f \in V$, then $A(f) = \dfrac{d}{dx} f(x)$. Let B represent the operation of multiplication by x, i.e., if $f \in V$, then $B(f) = x \cdot f(x)$.

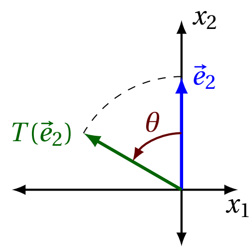
Compute $[A, B]$.

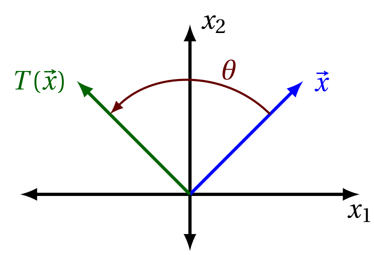
2.5.4 Rotation Matrices

Of particular importance to chemists are linear transformations that represent rotations.

Example 12: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle θ, with counterclockwise rotation for a positive angle. Find the standard matrix for this transformation. (It may be helpful to recall that $\sin(\theta + \pi/2) = \cos(\theta)$ and $\cos(\theta + \pi/2) = -\sin(\theta)$.)







$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \quad \end{bmatrix}$ $\qquad$ $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \quad \end{bmatrix}$ $\qquad$ $T = \begin{bmatrix} \qquad \end{bmatrix}$

Example 13: Give the 2×2 matrix $\boldsymbol{A}$ that rotates a vector in $\mathbb{R}^2$ an angle of $7\pi/6$.

The rotation of a vector in $\mathbb{R}^3$ is accomplished by a 3×3 rotation matrix. Rotations about the coordinate axes are straightforward extensions of the 2×2 rotation matrix found above. Recall that we use the right-hand rule to denote the direction of rotation - your thumb points in the direction of the positive axis of rotation.

Rotation about the positive x_3-axis (rotation in the x_1x_2-plane)	Rotation about the positive x_2-axis (rotation in the x_1x_3-plane)	Rotation about the positive x_1-axis (rotation in the x_2x_3-plane)
$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

Example 14: Give the 3×3 matrix $\boldsymbol{A}$ that rotates a vector in $\mathbb{R}^3$ an angle of $2\pi/3$ about the x_2 axis.

Example 15: Give the 3×3 matrix $\boldsymbol{A}$ that first reflects a vector across the x_1x_3-plane and then rotates the vector an angle of $\pi/6$ about the x_1 axis.

Example 16: Find the matrix of the transformation that represents the rotation of $2\pi/3$ about the x_3-axis, followed by the rotation of $2\pi/3$ about the x_2 axis, followed by reflection across the x_1x_2-plane.

Concept Check: Are rotation transformations also linear transformations? Are they one-to-one and onto?

> **Theorem:** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:
>
> a) T is onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$;
>
> b) T is one-to-one if and only if the columns of A are linearly independent.

2.5.5 Orthogonal and Unitary Matrices

> **Definition:** Let $\vec{v}_1 \dots, \vec{v}_n$ be a set of vectors in $\mathbb{R}^m$. We say that the set $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an **orthogonal set** if $\vec{v}_i \cdot \vec{v}_j = 0$ for all $1 \le i, j \le n$ with $i \ne j$.

Example 17: Determine if the vectors $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$ form an orthogonal set.

The rotation matrices described above have a special property. Note that each column, as a vector in $\mathbb{R}^n$, is orthogonal to every other column, and each column has a magnitude of 1.

> **Definition:** A set of n vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ in $\mathbb{R}^n$ is an **orthonormal basis** for $\mathbb{R}^n$ if $|\vec{v}_i| = 1$ for $1 \le i \le n$ and $\vec{v}_i \cdot \vec{v}_j = 0$ for all $1 \le i, j \le n$ with $i \ne j$. An $n \times n$ matrix $\boldsymbol{A}$ is **orthogonal** if each column of the matrix (as a vector in $\mathbb{R}^n$) has length 1 and is orthogonal to each other column in the matrix.

The columns of an orthogonal matrix form an orthonormal basis for $\mathbb{R}^n$. For this reason, orthogonal matrices have a very special property:

> **Theorem:** If $\boldsymbol{A}$ is an orthogonal matrix, then $\boldsymbol{A}^{-1} = \boldsymbol{A}^T$.

Example 18: The matrix $\boldsymbol{A} = \begin{bmatrix} 3/\sqrt{11} & 1/\sqrt{11} & 1/\sqrt{11} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{66} & -4/\sqrt{66} & 7/\sqrt{66} \end{bmatrix}$ is an orthogonal matrix because $A^T A = \boldsymbol{I}$. Can you construct an orthogonal matrix from the vectors in Example 17?

> **Definition:** An $n \times n$ complex matrix $\boldsymbol{A}$ is **unitary** if $\boldsymbol{A}^{-1} = \boldsymbol{A}^\dagger$.

Example 19: Verify that the matrix $\boldsymbol{A} = \begin{bmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ -i/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ is unitary.

Concept Check: Is the product of two orthogonal matrices also orthogonal? Is the product of two unitary matrices also unitary?

2.5 Review of Concepts

- Terms to know: transformation, operator, domain, codomain, range, one-to-one, onto, matrix of the transformation, shear transformation, reflection, rotation transformation, linear transformation, matrix transformation, commutator, orthogonal set, orthonormal set, orthogonal matrix, unitary matrix.
- Know how to construct the matrix of a linear transformation in $\mathbb{R}^2$ and $\mathbb{R}^3$ by analyzing the action of the transformation on the elementary basis.
- Know how to determine if a set of vectors is orthogonal, and make an orthonormal set from those vectors. Know how to determine if a matrix is orthogonal or unitary.

2.5 Practice Problems

1. Let the transformation T be defined by $T(\vec{x}) = \boldsymbol{A}\vec{x}$. For the given matrix $\boldsymbol{A}$ and vector $\vec{b}$, find a vector $\vec{x}$ whose image under T is $\vec{b}$, and determine whether $\vec{x}$ is unique.

a) $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & 5 \end{bmatrix}, \vec{b} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$

b) $\boldsymbol{A} = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 9 \\ 3 \\ -6 \end{bmatrix}$

2. Show that the transformation T defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 3x_2 \\ x_1 + 4 \\ 5x_2 \end{bmatrix}$ is not linear.

3. Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, and $\vec{v}_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\vec{x}$ to $x_1\vec{v}_1 + x_2\vec{v}_2$. Find a matrix $\boldsymbol{A}$ such that $T(\vec{x}) = \boldsymbol{A}\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^2$.

4. Find the standard matrix of the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that maps $\vec{e}_1$ to $\vec{e}_1 - 2\vec{e}_2$ and leaves $\vec{e}_2$ unchanged.

5. Find the (single) standard matrix of the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that first reflects points across the x_2-axis and then rotates points $\pi/6$ radians.

2.5 Exercises

1. Determine if the following statements are true or false, giving a brief explanation or counterexample in each case.

a) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is completely determined by its effect on the columns of the $n \times n$ identity matrix.

b) When two linear transformations are performed one after another, the combined effect may not always be a linear transformation.

c) If $\boldsymbol{A}$ is a 3×2 matrix, then the transformation $T(\vec{x}) = \boldsymbol{A}\vec{x}$ cannot be one-to-one.

d) If $\boldsymbol{A}$ is a 3×2 matrix, then the transformation $T(\vec{x}) = \boldsymbol{A}\vec{x}$ cannot be onto.

e) A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each vector in $\mathbb{R}^n$ maps onto a unique vector in $\mathbb{R}^m$.

f) Let V be the vector space of all twice-differentiable functions of the variable x. The transformation A defined by, if $f \in V$, then $A(f) = \dfrac{d^2}{dx^2} f(x) + f(x)$ is a linear transformation.

2. Find the (single) standard matrix of the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that first reflects points across the x_1-axis and then reflects points across the line $x_2 = x_1$. Show that this is a rotation transformation by finding the angle of rotation.

3. Find the (single) standard matrix of the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that first rotates points an angle of θ about the x_1-axis, then reflects across the x_1x_2-plane, and finally rotates an angle of ϕ about the x_3-axis. Then give the matrix for $\theta = \pi/3$ and $\phi = -\pi/6$.

4. The spin matrices for a nucleus with spin quantum number 1 are

$$\boldsymbol{I}_x = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \boldsymbol{I}_y = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \qquad \boldsymbol{I}_z = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Find the commutators $[\boldsymbol{I}_x, \boldsymbol{I}_y]$, $[\boldsymbol{I}_y, \boldsymbol{I}_z]$, and $[\boldsymbol{I}_z, \boldsymbol{I}_x]$.

5. Describe (in words) the geometric action of the following matrices on a vector $\vec{v} \in \mathbb{R}^2$.

a) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ c) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

6. It is true that the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ represents a particular rotation transformation on a vector in $\mathbb{R}^2$. Can you find two non-rotational transformations that, when subsequently applied, will result in the same transformation matrix? In what order should they be applied? (In essence you are finding a **factorization** of the above rotation matrix.)

2.5 Answers to Practice Problems

1. a) For $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & 5 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$, to find a $\vec{x}$ whose image under T is $\vec{b}$, we need to see if the system

$$\begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$$

has a solution. Note that $\det(\boldsymbol{A}) = 15$ so this system should have a solution, and it will be unique as there are as many equations as there are unknowns. Using substitution, the solution is found to be $\vec{x} = \begin{bmatrix} 1/3 \\ 11/3 \\ 2/3 \end{bmatrix}$.

b) For $\boldsymbol{A} = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 9 \\ 3 \\ -6 \end{bmatrix}$, note that this system is overdetermined, as there are more equations than unknowns in

$$\boldsymbol{A} = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix} \vec{b} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{b} = \begin{bmatrix} 1 \\ 9 \\ 3 \\ -6 \end{bmatrix}.$$

Using substitution, a description of the solution set is found to be $x_1 = 7-3t, x_2 = 3-t, x_3 = t$, so there are infinitely many solutions, one of which is $\vec{x} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}$.

2. That the transformation T defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 3x_2 \\ x_1 + 4 \\ 5x_2 \end{bmatrix}$ is not linear can be shown in multiple ways. One think to note immediately is that $T(\mathbf{0}) = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} \neq \mathbf{0}$, so this violates a theorem in the section. Also note that

$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 5 \\ 5 \end{bmatrix}$ but $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 5 \end{bmatrix} \neq T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

3. If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, and $\vec{v}_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$, then the matrix that maps $\vec{x}$ to $x_1\vec{v}_1 + x_2\vec{v}_2$ must satisfy

$$\boldsymbol{A}\vec{x} = \boldsymbol{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 = \begin{bmatrix} -2x_1 + 7x_2 \\ 5x_1 - 3x_2 \end{bmatrix}.$$

This matrix is $\boldsymbol{A} = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}$.

4. The standard matrix of the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that maps $\vec{e}_1$ to $\vec{e}_1 - 2\vec{e}_2$ and leaves $\vec{e}_2$ unchanged is

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

5. The matrix $\boldsymbol{A}_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ reflects points across the x_2-axis, and the matrix $\boldsymbol{A}_2 = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$ rotates points $\pi/6$ radians, so the (single) standard matrix of the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that first reflects points across the x_2-axis and then rotates points $\pi/6$ radians is given by

$$\boldsymbol{A} = \boldsymbol{A}_2\boldsymbol{A}_1 = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}.$$

2.6 Eigenvalues and Eigenvectors

Objectives and Concepts:

- An eigenvector of an $n \times n$ matrix $\boldsymbol{A}$ is a nonzero vector $\vec{x}$ such that $\boldsymbol{A}\vec{x} = \lambda\vec{x}$ for some scalar λ. A scalar λ is called an eigenvalue of $\boldsymbol{A}$ if there is a nontrivial solution $\vec{x}$ of $\boldsymbol{A}\vec{x} = \lambda\vec{x}$.
- The eigenspace of $\boldsymbol{A}$ corresponding to λ is the set of all solutions to the equation $(\boldsymbol{A} - \lambda\boldsymbol{I})\vec{x} = \vec{0}$.
- The eigenvalues of a triangular matrix are the entries on the diagonal.
- Eigenvectors corresponding to distinct eigenvalues are linearly independent.
- The algebraic multiplicity of an eigenvalue is its multiplicity as a root of the characteristic polynomial.

References: TCMB §§19.1–3,6; FCLA Chapter E, *Eigenvalues*,

2.6.1 Eigenvalues

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ whose action is described by $\vec{x} \mapsto \boldsymbol{A}\vec{x}$ may move vectors in a variety of directions. However, there exist special vectors on which the action of $\boldsymbol{A}$ is just a scaling - in other words, there are special vectors $\vec{x}$ where the action of $\boldsymbol{A}$ on $\vec{x}$ results in a vector with the same direction as $\vec{x}$.

Example 1: Let $\boldsymbol{A} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Compute the action of $\boldsymbol{A}$ on $\vec{u}$ and $\vec{v}$.

> **Definition:** An **eigenvector** of an $n \times n$ matrix $\boldsymbol{A}$ is a nonzero vector $\vec{x}$ such that $\boldsymbol{A}\vec{x} = \lambda\vec{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of $\boldsymbol{A}$ if there is a nontrivial solution $\vec{x}$ of $\boldsymbol{A}\vec{x} = \lambda\vec{x}$; such an $\vec{x}$ is called an **eigenvector corresponding to** λ.

Example 2: Let $\boldsymbol{A} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are $\vec{u}$ and $\vec{v}$ eigenvectors of A?

Example 3: Show that 7 is an eigenvalue of the matrix in the above example, and find an eigenvector of this eigenvalue. (Find a solution to $\boldsymbol{A}\vec{x} = 7\vec{x}$.)

Concept Check: Why must we stipulate that an eigenvector must be a *nonzero* vector?

> **Definition:** The **eigenspace** of $\boldsymbol{A}$ corresponding to λ is the set of all solutions to the equation $(\boldsymbol{A} - \lambda\boldsymbol{I})\vec{x} = \vec{0}$.

To find the eigenspace of a particular eigenvalue λ of a matrix $\boldsymbol{A}$, we simply find the set of all solutions to $(\boldsymbol{A}-\lambda\boldsymbol{I})\vec{x}=\vec{0}$. We can construct a **basis** for the eigenspace by choosing appropriate values for any free variables that arise in the solution set. The number of basis vectors will be the same as the number of free variables in the description of the solution set.

Concept Check: Why do we know that there will always be infinitely many solutions to $(\boldsymbol{A}-\lambda\boldsymbol{I})\vec{x}=\vec{0}$ if λ is an eigenvalue?

Example 4: Let $\boldsymbol{A}=\begin{bmatrix}3 & 0\\ 8 & -1\end{bmatrix}$. An eigenvalue of $\boldsymbol{A}$ is 3. Find a basis for the corresponding eigenspace.

Example 5: Let $\boldsymbol{A}=\begin{bmatrix}4 & -1 & 6\\ 2 & 1 & 6\\ 2 & -1 & 8\end{bmatrix}$. An eigenvalue of $\boldsymbol{A}$ is 2. Find a basis for the corresponding eigenspace.

2.6.2 The Characteristic Polynomial (or Characteristic Equation)

Now λ is an eigenvalue of $\boldsymbol{A}$ whenever $\boldsymbol{A}\vec{x}=\lambda\vec{x}$ has a nontrivial solution, or when the linear system $(\boldsymbol{A}-\lambda\boldsymbol{I})\vec{x}=\vec{0}$ has a nontrivial solution. This will only be true when the matrix $\boldsymbol{A}-\lambda\boldsymbol{I}$ is a singular matrix, which means its determinant is 0. This gives us a way to find eigenvalues.

> **Definition:** For an $n\times n$ matrix $\boldsymbol{A}$, the scalar equation $\det(\boldsymbol{A}-\lambda\boldsymbol{I})=0$ is called the **characteristic equation** of $\boldsymbol{A}$, and $\det(\boldsymbol{A}-\lambda\boldsymbol{I})$ is a polynomial of degree n in λ known as the **characteristic polynomial** of $\boldsymbol{A}$.

> **Theorem:** λ is an eigenvalue of $\boldsymbol{A}$ if and only if it is a solution of the equation $\det(\boldsymbol{A}-\lambda\boldsymbol{I})=0$.

This gives us a way to find eigenvalues of any matrix: first compute the determinant of the matrix $\boldsymbol{A}-\lambda\boldsymbol{I}$ to find the characteristic polynomial, then find all values of λ that are roots of the characteristic polynomial.

Example 6: Find the eigenvalues of $\boldsymbol{A}=\begin{bmatrix}2 & 3\\ 3 & -6\end{bmatrix}$ and the corresponding eigenvectors.

Example 7: Find the eigenvalues of $\boldsymbol{A}=\begin{bmatrix}5 & -2 & 6 & -1\\ 0 & 3 & -8 & 0\\ 0 & 0 & 5 & 4\\ 0 & 0 & 0 & 1\end{bmatrix}$. (Use what you know about determinants.)

Example 8: Find the eigenvalues of $\boldsymbol{A}=\begin{bmatrix}a & -b\\ b & a\end{bmatrix}$ where a and b are real numbers.

Concept Check: What does this say about the eigenvalues of rotation matrices?

Example 9: Find the eigenvalues of $\boldsymbol{A} = \begin{bmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{bmatrix}$.

Example 10: Find the eigenvalues and the corresponding eigenvectors of $\boldsymbol{A} = \begin{bmatrix} -2 & 1 & 1 \\ -11 & 4 & 5 \\ -1 & 1 & 0 \end{bmatrix}$.

2.6.3 Properties of Eigenvalues and Eigenvectors

Definition: Let λ be an eigenvalue of the $n \times n$ matrix $\boldsymbol{A}$. The **algebraic multiplicity** of λ is the number of times λ appears as a root of the characteristic polynomial. The **geometric multiplicity** of λ is the number of linearly independent eigenvectors associated with λ.

Example 11: The eigenvalues of $\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ are $\lambda = 1, -2, -2$. Thus the algebraic multiplicity of $\lambda = -2$ is 2. Find the eigenvectors for each eigenvalue. What is the geometric multiplicity of $\lambda = -2$?

Concept Check: What is the relationship between the algebraic multiplicity and the geometric multiplicity of an eigenvalue?

Theorem: If $\vec{x}$ is an eigenvector of $\boldsymbol{A}$, then so is $c\vec{x}$ for any nonzero scalar c.

Theorem: If $\boldsymbol{A}$ is a symmetric $n \times n$ matrix with real entries

1. All the eigenvalues of $\boldsymbol{A}$ are real.
2. All the eigenvectors of $\boldsymbol{A}$ are real valued.
3. The geometric multiplicity of each eigenvalue is the same as its algebraic multiplicity—and thus the eigenvectors form a basis for all of $\mathbb{R}^n$.
4. Eigenvectors corresponding to distinct eigenvalues are orthogonal.

For a Hermitian matrix, this is again all true except the bit about eigenvectors being real-valued. (And of course they form a basis for $\mathbb{C}^n$ instead of $\mathbb{R}^n$.)

Theorem: If $\boldsymbol{A}$ is an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, then $\det \boldsymbol{A} = \lambda_1 \lambda_2 \cdots \lambda_n$.

Concept Check: If $\boldsymbol{A}$ is a singular matrix, then what can we say about its eigenvalues?

2.6.4 Eigenfunctions

Transformations and operators (other than matrix transformations) also have eigenvalues. The associated elements of the vector space on which the transformation operates may be vectors, or functions, or some other objects. In order to be an eigenvalue/"eigenobject" pair for an operator T acting on a vector space V, the relationship $T(v) = \lambda v$ for some scalar λ and some nonzero $v \in V$ must be satisfied. One common situation is to consider the vector space V to be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ that are differentiable. In this case, linear operators have **eigenfunctions** associated with their eigenvalues.

Example 12: Let V be the set of all differentiable functions defined on the real line. What are the eigenvalues and associated eigenfunctions of the differentiation operator $D : V \to V$ defined by $D(f) = \dfrac{df}{dx}$ for $f(x)$ in V?

2.6 Review of Concepts

- Terms to know: eigenvalue, eigenvector, eigenspace, characteristic polynomial, characteristic equation, algebraic multiplicity, geometric multiplicity, eigenfunction.
- Know how to find eigenvalues and eigenvectors of 2×2 and 3×3 matrices, and a basis for the eigenspace of an eigenvalue.

2.6 Practice Problems

1. Find the eigenvalues and associated eigenvectors of the following matrices:

a) $\boldsymbol{A} = \begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}$ b) $\boldsymbol{A} = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ c) $\boldsymbol{A} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & 0 & 3 \end{bmatrix}$

2. Find the eigenvalues and associated eigenvectors of the matrix $\boldsymbol{A} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

3. Find the eigenvalues and associated eigenvectors of the matrix $\boldsymbol{A} = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$.

4. Find the eigenvalues and associated eigenvectors of the matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 2 \\ -2 & 0 & 3 \end{bmatrix}$.

2.6 Exercises

1. Determine if the following statements are true or false, giving a brief explanation or counterexample in each case.

a) $\boldsymbol{A}$ and $\boldsymbol{A}^T$ have the same eigenvalues.

b) The eigenvalues of $\boldsymbol{M} = \boldsymbol{AB}$ are the products of the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$.

c) If $\boldsymbol{A}$ is singular, then 0 is an eigenvalue of $\boldsymbol{A}$.

d) The sum of two eigenvectors of $\boldsymbol{A}$ is also an eigenvector of $\boldsymbol{A}$.

2. Find the eigenvalues and eigenvectors of the matrix $\boldsymbol{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Given your knowledge of the action of this matrix as a linear transformation, give a geometric interpretation of the eigenvectors.

3. Find a basis for the eigenspace of each eigenvalue of the matrix $\boldsymbol{A} = \begin{bmatrix} -4 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & 3 & -2 \end{bmatrix}$.

4. Find the eigenvalues and eigenvectors of the matrices $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$. Use these results to make a conjecture about the eigenvalues and eigenvectors of $\begin{bmatrix} a & a+1 \\ 1 & 0 \end{bmatrix}$ where a is any real number.

5. Find the eigenvalues and associated eigenvectors of the following matrices:

a) $\boldsymbol{A} = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ b) $\boldsymbol{A} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ c) $\boldsymbol{A} = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}$

6. Suppose you knew the eigenvalues of the product $\boldsymbol{AB}$. What would you be able to say about the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$?

7. The Hamiltonian operator for a one-dimensional harmonic oscillator moving in the x direction is

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{kx^2}{2}.$$

Find the value of the coefficient a such that the function e^{-ax^2} is an eigenfunction of the Hamiltonian operator. The quantity k is the force constant, m is the mass of the oscillating particle, and $\hbar$ is Planck's constant divided by 2π.

2.6 Answers to Practice Problems

1. a) To find the eigenvalues, we have

$$\det(\boldsymbol{A}-\lambda\boldsymbol{I}) = \begin{vmatrix} 4-\lambda & 1 \\ 3 & 6-\lambda \end{vmatrix} = (4-\lambda)(6-\lambda)-3 = \lambda^2-10\lambda+21 = (\lambda-3)\lambda-7),$$

so the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 7$. For λ_1 we have

$$\boldsymbol{A}-\lambda_1\boldsymbol{I} = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

and the solutions of $(\boldsymbol{A}-\lambda_1\boldsymbol{I})\vec{x} = \vec{0}$ are of the form $x_1 = -x_2$, so $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector. For λ_2 we have

$$\boldsymbol{A}-\lambda_1\boldsymbol{I} = \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$$

and the solutions of $(\boldsymbol{A}-\lambda_2\boldsymbol{I})\vec{x} = \vec{0}$ are of the form $3x_1 = x_2$, so $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector.

b) To find the eigenvalues, we have

$$\det(\boldsymbol{A}-\lambda\boldsymbol{I}) = \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = \lambda^3-6\lambda^2+11\lambda-6.$$

By observation, we see that one eigenvalue is $\lambda_1 = 1$, so we can find the others via polynomial long division.

$$\begin{array}{r|rrrr}
 & \lambda^2 & -5\lambda & +6 & \\ \hline
\lambda-1 & \lambda^3-6\lambda^2 & +11\lambda & -6 & \\
 & -\lambda^3+\lambda^2 & & & \\ \hline
 & -5\lambda^2 & +11\lambda & & \\
 & 5\lambda^2 & -5\lambda & & \\ \hline
 & & 6\lambda & -6 & \\
 & & -6\lambda & +6 & \\ \hline
 & & & 0 &
\end{array}$$

Thus the other eigenvalues are found by factoring:

$$\lambda^2-5\lambda+6 = (\lambda-2)(\lambda-3),$$

so the remaining eigenvalues are $\lambda_2 = 2$ and $\lambda_3 = 3$. For λ_1 we have

$$\boldsymbol{A}-\lambda_1\boldsymbol{I} = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

and the solution set of $(\boldsymbol{A}-\lambda_1\boldsymbol{I})\vec{x}=\vec{0}$ is given by $x_1=x_3=0$ with x_2 free. Thus an eigenvector is $\vec{v}_1=\begin{bmatrix}0\\1\\0\end{bmatrix}$. For λ_2, we have

$$\boldsymbol{A}-\lambda_2\boldsymbol{I}=\begin{bmatrix}2 & 0 & 1\\-2 & -1 & 0\\-2 & 0 & -1\end{bmatrix}$$

and the solution set of $(\boldsymbol{A}-\lambda_2\boldsymbol{I})\vec{x}=\vec{0}$ is given by $x_2=-2x_1, x_3=-2x_1$ with x_1 free. Thus an eigenvector is $\vec{v}_2=\begin{bmatrix}-1\\2\\2\end{bmatrix}$. For λ_3, we have

$$\boldsymbol{A}-\lambda_3\boldsymbol{I}=\begin{bmatrix}1 & 0 & 1\\-2 & -2 & 0\\-2 & 0 & -2\end{bmatrix}$$

and the solution set of $(\boldsymbol{A}-\lambda_3\boldsymbol{I})\vec{x}=\vec{0}$ is given by $x_2=-x_1, x_3=-x_1$ with x_1 free. Thus an eigenvector is $\vec{v}_3=\begin{bmatrix}-1\\1\\1\end{bmatrix}$.

c) To find the eigenvalues, we have

$$\det(\boldsymbol{A}-\lambda\boldsymbol{I})=\begin{vmatrix}1-\lambda & -2 & 2\\-2 & 1-\lambda & 2\\-2 & 0 & 3-\lambda\end{vmatrix}=\lambda^3-5\lambda^2+7\lambda-3.$$

By observation, we see that one eigenvalue is $\lambda_1=1$, so we can find the others via polynomial long division.

$$\begin{array}{r l}
 & \lambda^2-4\lambda+3\\ \cline{2-2}
\lambda-1 \big) & \lambda^3-5\lambda^2+7\lambda-3\\
 & -\lambda^3+\lambda^2\\ \cline{2-2}
 & -4\lambda^2+7\lambda\\
 & 4\lambda^2-4\lambda\\ \cline{2-2}
 & 3\lambda-3\\
 & -3\lambda+3\\ \cline{2-2}
 & 0
\end{array}$$

Thus the other eigenvalues are found by factoring: $\lambda^2-4\lambda+3=(\lambda-1)(\lambda-3)$. Thus the eigenvalue $\lambda_1=1$ has an algebraic multiplicity of 2, and the other eigenvalue is $\lambda_2=3$. For λ_1 we have

$$\boldsymbol{A}-\lambda_1\boldsymbol{I}=\begin{bmatrix}0 & -2 & 2\\-2 & 0 & 2\\-2 & 0 & 2\end{bmatrix}$$

and the solution set of $(\boldsymbol{A}-\lambda_1\boldsymbol{I})\vec{x}=\vec{0}$ is given by $x_1=x_2=x_3$ with x_1 free. Thus an eigenvector is $\vec{v}_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$, which means λ_1 has a geometric multiplicity of only 1. For λ_2, we have

$$\boldsymbol{A}-\lambda_2\boldsymbol{I}=\begin{bmatrix}-2 & -2 & 2\\ -2 & -2 & 2\\ -2 & 0 & 0\end{bmatrix}$$

and the solution set of $(\boldsymbol{A}-\lambda_2\boldsymbol{I})\vec{x}=\vec{0}$ is given by $x_1=0$ and $x_2=x_3$ with x_2 free. Thus an eigenvector is $\vec{v}_2=\begin{bmatrix}0\\1\\1\end{bmatrix}$.

2. The characteristic polynomial of $\boldsymbol{A}=\begin{bmatrix}5 & 4\\ 1 & 2\end{bmatrix}$ is $\lambda^2-7\lambda+6$, so $\boldsymbol{A}$ has eigenvalues $\lambda_1=6$ and $\lambda_2=1$. An eigenvector for $\lambda_1=6$ is $\vec{v}_1=\begin{bmatrix}4\\1\end{bmatrix}$ and an eigenvector for $\lambda_2=1$ is $\vec{v}_2=\begin{bmatrix}1\\-1\end{bmatrix}$.

3. The characteristic polynomial of $\boldsymbol{A}=\begin{bmatrix}1 & -2\\ 2 & -3\end{bmatrix}$ is $\lambda^2+2\lambda+1$, so the only eigenvalue is $\lambda=-1$. Now

$$\boldsymbol{A}-\lambda\boldsymbol{I}=\boldsymbol{A}+\boldsymbol{I}=\begin{bmatrix}2 & -2\\ 2 & -2\end{bmatrix},$$

which only has one eigenvector $\vec{v}=\begin{bmatrix}1\\1\end{bmatrix}$.

4. This matrix has the characteristic polynomial $\lambda^3-5\lambda^2+7\lambda-3$, which is the same as the matrix in practice problem 1c). Then the eigenvalues are $\lambda_1=1$ (with algebraic multiplicity 2) and $\lambda_2=3$. Now for λ_1, we get

$$\boldsymbol{A}-\lambda_1\boldsymbol{I}=\begin{bmatrix}0 & 0 & 0\\ -2 & 0 & 2\\ -2 & 0 & 2\end{bmatrix}$$

and the solution set of $(\boldsymbol{A}-\lambda_1\boldsymbol{I})\vec{x}=\vec{0}$ is given by $x_1=x_3$ with x_1 free, and x_2 is free as well. Thus two independent eigenvectors are $\vec{v}_1=\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\vec{v}_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$. For λ_2, we have

$$\boldsymbol{A}-\lambda_2\boldsymbol{I}=\begin{bmatrix}-2 & 0 & 0\\ -2 & -2 & 2\\ -2 & 0 & 0\end{bmatrix}$$

which implies $x_1=0$ and $x_2=x_3$. Thus an eigenvector is $\vec{v}_3=\begin{bmatrix}0\\1\\1\end{bmatrix}$.

2.7 (†) Matrix Groups and Symmetry

Objectives and Concepts:

- A group is an algebraic structure that satisfies conditions of closure, associativity, identity, and invertibility of elements in a set under a particular binary operation.
- Several different sets of matrices form groups under matrix multiplication.
- The point group of a molecule is the set of all symmetry operations on that molecule under the binary operation of composition.

2.7.1 Groups

A **binary operation**, such as addition or multiplication, is an operation applied to two members of a set of objects. When the set of objects and the binary operation have special properties, they form one of the most fundamental algebraic structures.

Definition: A **group** is a set G together with a binary operation $*$ that satisfies

1. **Closure:** If $a, b \in G$, then $a * b \in G$.
2. **Associativity:** If $a, b, c \in G$, then $(a * b) * c = a * (b * c)$.
3. **Identity:** There is an element $e \in G$ such that $a * e = e * a = a$.
4. **Inverse:** For every $a \in G$, there is an element $a' \in G$ such that $a * a' = a' * a = e$.

Example 1: The set of integers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ with the operation of addition $(+)$ form a group.

Example 2: The set of rational numbers $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$ under the operation of addition forms a group, but does **not** form a group under multiplication. However, the set of nonzero rational numbers $\mathbb{Q}^* = \{p/q \mid p, q \in \mathbb{Z}, p, q \neq 0\}$ under the operation of multiplication forms a group.

Concept Check: Do the nonzero integers $\mathbb{Z}^*$ form a group under multiplication?

Concept Check: Let G be the set of all real numbers of the form 10^k where $k \in \mathbb{Z}$. Does this set form a group under multiplication?

Note that it is not assumed that the group operation is **commutative**. In fact, many of the groups we will discuss are **non-commutative**, i.e., $a * b \neq b * a$ in general. (As an aside, a commutative group is also known as an **abelian group**.)

Example 3: Let G be the set of all matrices in $\mathbb{R}^{2\times 2}$ that are invertible. It is true that G forms a group under matrix multiplication, however the group is not commutative. Give an example of A, B in G such that $AB \neq BA$.

Many finite sets form groups as well (in fact, one school of thought considers finite groups to be far more interesting that infinite groups).

Example 4: The set $S = \{1, -1, i, -i\}$ under multiplication of complex numbers is a group. The identity element is 1, and the set is closed under multiplication. What is the inverse of each element?

Example 5: The set of positive and negative multiples of standard basis vectors $G = \{\pm\vec{\imath}, \pm\vec{\jmath}, \pm\vec{k}\}$ under the cross product operation do **not** form a group, as the set G does not have an identity element under $\times$. (If e is an identity element in a group, it must satisfy $e * e = e$. Is there any vector $\vec{v} \in \mathbb{R}^3$ that satisfies $\vec{v} \times \vec{v} = \vec{v}$?)

We often use **multiplicative notation** to describe properties of abstract groups. In this sense the group operation is represented by juxtaposition of elements, i.e., $a * b$ would be written as ab. This allows the use of **exponents** which represent repeated application of the group operation. For example, $a * a * a * a$ can be written as a^4. It is natural to write the inverse of a as a^{-1}. Even if the operation is additive in nature, multiplicative notation can be translated:

$$ab = a + b, \qquad a^n = \underbrace{a + a + \cdots + a}_{n \text{ times}}.$$

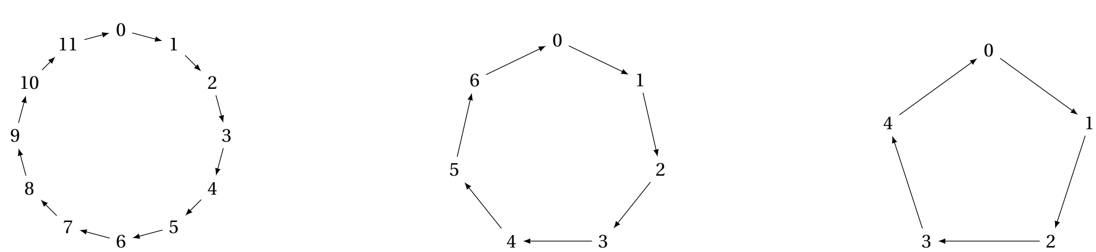
Definition: The **order** of a group is the number of elements in the group. The **order** of an element a in a group is the smallest number of copies of a required to become the identity under the group operation. In multiplicative notation, the order of a is the smallest positive integer m such that $a^m = e$. In additive notation, the order of a is the smallest positive integer m where

$$\underbrace{a + a + \cdots + a}_{m \text{ times}} = e.$$

If no such integer m exists, then the order of a is ∞.

Definition: Let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$. We define the operation of **addition modulo** n to be the remainder when the sum of two numbers is divided by n. In other words, we write $a + b = r \bmod n$ when $a + b = nq + r$ for some integer q and some r satisfying $0 \leq r \leq n$, i.e., $r \in \mathbb{Z}_n$. We also define **multiplication modulo** n to be the remainder when the product of two numbers is divided by n.

Modular arithmetic is a lot like the arithmetic we perform when we look at an analog clock. For example, six hours after 7:00 is 1:00, and four hours before 2:00 is 10:00. This is addition modulo 12.



Example 6: Verify that $\mathbb{Z}_7$ under the operation of addition modulo 7 is a group. What is the identity element? What is the result of $5+6$? What is the inverse of 3?

Example 7: Is $\mathbb{Z}_5$, the set of integers modulo 5, a group under multiplication modulo 5?

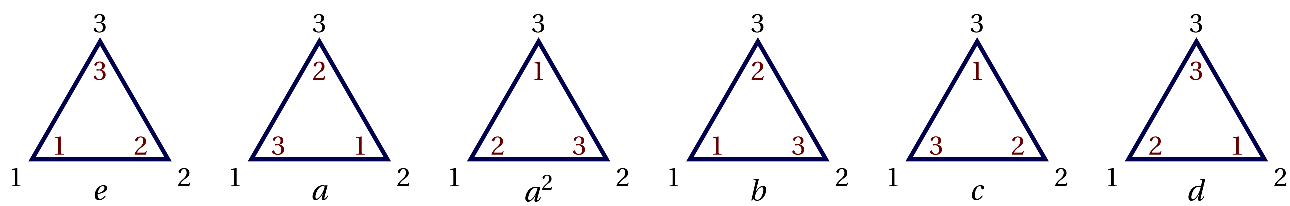
When the order of a group is small, it is often useful to write out the **group table** (or **multiplication table**) of the group. In a group table the result of the product $a * b$ is the entry in the a row and the b column of the table. Here are some examples of group tables:

	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

$\mathbb{Z}_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\mathbb{Z}_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Example 8: Consider an equilateral triangle with vertices labeled 1, 2, and 3 as shown in the figures below. A group can be performed by permutations of the vertices. Define the element a to be the resulting permutation when rotating counterclockwise 120°. Then a^2 is a rotation of 240°. Define b, c, and d to be the permutations obtained by flipping the triangle 180° when holding vertex 1, 2, and 3 fixed, respectively.

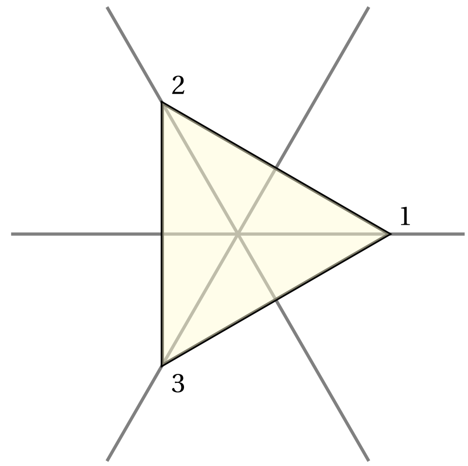


The group operation is *composition*, i.e., ab is the permutation that results from performing b first, then performing a (note the right-to-left convention). The group table is given to the right. Notice that the group table is not symmetric across the diagonal, i.e., the group is not abelian. This is known as the **dihedral group** D_3. It is the smallest non-abelian group! What is the order of each element of the group?

D_3	e	a	a^2	b	c	d
e	e	a	a^2	b	c	d
a	a	a^2	e	d	b	c
a^2	a^2	e	a	c	d	b
b	b	c	d	e	a	a^2
c	c	d	b	a^2	e	a
d	d	b	c	a	a^2	e

2.7.2 Matrix Groups

Another way to view the group D_3 is to consider the group members to be rotation transformations in the x_1x_2-plane of $2\pi/3$ (a) and $4\pi/3$ (a^2) about the origin and reflections across the x_1-axis (b), the line $x_2 = -\sqrt{3}x_1$ (c), and $x_2 = \sqrt{3}x_1$ (d). The identity transformation is e. We can associate a 2×2 matrix with each transformation.



$$\begin{array}{cccccc}
e & a & a^2 & b & c & d \\
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &
\begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} &
\begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix} &
\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} &
\begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} &
\begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \\
\boldsymbol{I} & \boldsymbol{A} & \boldsymbol{A}^2 & \boldsymbol{B} & \boldsymbol{C} & \boldsymbol{D}
\end{array}$$

The group table for D_3 in matrix representation is the same, with the exception that composition of operators is now represented by multiplication of matrices. Note that the matrices $\boldsymbol{I}$, $\boldsymbol{A}$, and $\boldsymbol{A}^2$ by themselves form a group - this is a **subgroup** of D_3. In fact, it is an abelian subgroup of a nonabelian group.

D_3	$\boldsymbol{I}$	$\boldsymbol{A}$	$\boldsymbol{A}^2$	$\boldsymbol{B}$	$\boldsymbol{C}$	$\boldsymbol{D}$
$\boldsymbol{I}$	$\boldsymbol{I}$	$\boldsymbol{A}$	$\boldsymbol{A}^2$	$\boldsymbol{B}$	$\boldsymbol{C}$	$\boldsymbol{D}$
$\boldsymbol{A}$	$\boldsymbol{A}$	$\boldsymbol{A}^2$	$\boldsymbol{I}$	$\boldsymbol{D}$	$\boldsymbol{B}$	$\boldsymbol{C}$
$\boldsymbol{A}^2$	$\boldsymbol{A}^2$	$\boldsymbol{I}$	$\boldsymbol{A}$	$\boldsymbol{C}$	$\boldsymbol{D}$	$\boldsymbol{B}$
$\boldsymbol{B}$	$\boldsymbol{B}$	$\boldsymbol{C}$	$\boldsymbol{D}$	$\boldsymbol{I}$	$\boldsymbol{A}$	$\boldsymbol{A}^2$
$\boldsymbol{C}$	$\boldsymbol{C}$	$\boldsymbol{D}$	$\boldsymbol{B}$	$\boldsymbol{A}^2$	$\boldsymbol{I}$	$\boldsymbol{A}$
$\boldsymbol{D}$	$\boldsymbol{D}$	$\boldsymbol{B}$	$\boldsymbol{C}$	$\boldsymbol{A}$	$\boldsymbol{A}^2$	$\boldsymbol{I}$

Several other sets of matrices also form groups. The group D_4, which represents the image of a square under rotation and reflection transformations, contains the following matrices.

$$\begin{array}{cccccccc}
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &
\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} &
\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} &
\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} &
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} &
\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} &
\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} &
\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\
\boldsymbol{I} & \boldsymbol{A} & \boldsymbol{A}^2 & \boldsymbol{A}^3 & \boldsymbol{B} & \boldsymbol{C} & \boldsymbol{D} & \vec{F}
\end{array}$$

The group table for D_4 is given below:

D_4	$\boldsymbol{I}$	$\boldsymbol{A}$	$\boldsymbol{A}^2$	$\boldsymbol{A}^3$	$\boldsymbol{B}$	$\boldsymbol{C}$	$\boldsymbol{D}$	$\vec{F}$
$\boldsymbol{I}$	$\boldsymbol{I}$	$\boldsymbol{A}$	$\boldsymbol{A}^2$	$\boldsymbol{A}^3$	$\boldsymbol{B}$	$\boldsymbol{C}$	$\boldsymbol{D}$	$\vec{F}$
$\boldsymbol{A}$	$\boldsymbol{A}$	$\boldsymbol{A}^2$	$\boldsymbol{A}^3$	$\boldsymbol{I}$	$\vec{F}$	$\boldsymbol{D}$	$\boldsymbol{B}$	$\boldsymbol{C}$
$\boldsymbol{A}^2$	$\boldsymbol{A}^2$	$\boldsymbol{A}^3$	$\boldsymbol{I}$	$\boldsymbol{A}$	$\boldsymbol{C}$	$\boldsymbol{B}$	$\vec{F}$	$\boldsymbol{D}$
$\boldsymbol{A}^3$	$\boldsymbol{A}^3$	$\boldsymbol{I}$	$\boldsymbol{A}$	$\boldsymbol{A}^2$	$\boldsymbol{D}$	$\vec{F}$	$\boldsymbol{C}$	$\boldsymbol{B}$
$\boldsymbol{B}$	$\boldsymbol{B}$	$\boldsymbol{D}$	$\boldsymbol{C}$	$\vec{F}$	$\boldsymbol{I}$	$\boldsymbol{A}^2$	$\boldsymbol{A}$	$\boldsymbol{A}^3$
$\boldsymbol{C}$	$\boldsymbol{C}$	$\vec{F}$	$\boldsymbol{B}$	$\boldsymbol{D}$	$\boldsymbol{A}^2$	$\boldsymbol{I}$	$\boldsymbol{A}^3$	$\boldsymbol{A}$
$\boldsymbol{D}$	$\boldsymbol{D}$	$\boldsymbol{C}$	$\vec{F}$	$\boldsymbol{B}$	$\boldsymbol{A}^3$	$\boldsymbol{A}$	$\boldsymbol{I}$	$\boldsymbol{A}^2$
$\vec{F}$	$\vec{F}$	$\boldsymbol{B}$	$\boldsymbol{D}$	$\boldsymbol{C}$	$\boldsymbol{A}$	$\boldsymbol{A}^3$	$\boldsymbol{A}^2$	$\boldsymbol{I}$

2.7.3 (†) Symmetry Point Groups and Molecules

The mathematical structure of groups has a wide variety of applications in physical chemistry. Symmetry groups are an important component of the analysis of the structure, bonding, and spectroscopy of molecules, as well as understanding some properties of wave-functions and normal mode vibrations.

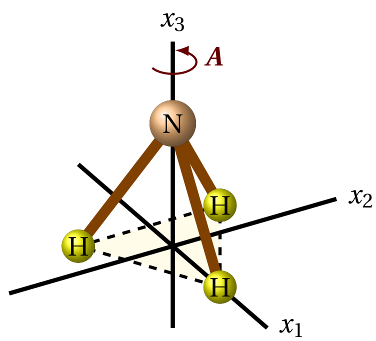
To begin understanding how groups can help understand molecular symmetry, we extend the matrix representation of the group D_3 to 3×3 matrices (that leave the x_3-coordinate unchanged). Now we have a complete specification of a particular set of **symmetry operators** on points in $\mathbb{R}^3$.

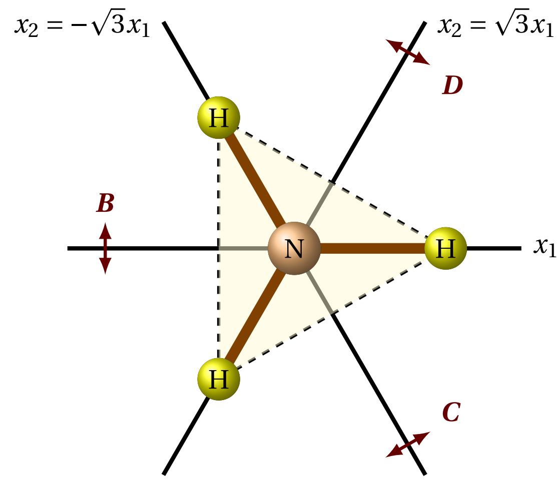
$$\underset{\boldsymbol{I}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \quad \underset{\boldsymbol{A}}{\begin{bmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \quad \underset{\boldsymbol{A}^2}{\begin{bmatrix} -1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\underset{\boldsymbol{B}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \quad \underset{\boldsymbol{C}}{\begin{bmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \quad \underset{\boldsymbol{D}}{\begin{bmatrix} -1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

So the operator $\boldsymbol{I}$ represents the action of leaving all points fixed, $\boldsymbol{A}$ represents the action of rotating all points an angle of $2\pi/3$ about the x_3-axis, $\boldsymbol{A}^2$ represents rotating all points an angle of $4\pi/3$ about the x_3-axis, $\boldsymbol{B}$ represents reflection in the x_2x_3-plane, $\boldsymbol{C}$ is reflection in the plane $\sqrt{3}x_1 + x_2 = 0$, and $\boldsymbol{D}$ is reflection in the plane $\sqrt{3}x_1 - x_2 = 0$.

Revisiting the ammonia molecule NH_3, note that if we apply any of these transformations to that molecule, the resulting molecule is the same. Thus these transformations form a group of **symmetries of the molecule**.





Indeed, the matrix group $\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}\}$ is also the group of symmetries of an ammonia molecule.

Definition: The **symmetry elements** of a molecule are geometric descriptions of the types of symmetry the molecule exhibits. The **symmetry transformations** or **symmetry operators** of a molecule are transformations that result in the interchange of identical nuclei. A **point group** is a group of symmetry transformations that leave at least one point fixed.

A symmetry element is a point of reference about which symmetry operations can take place. In particular, symmetry elements can be centers of inversion, axes of rotation and mirror planes.

Thus, the symmetry of a molecule is characterized by a set of symmetry elements, each of which are associated to one or more symmetry operations (linear transformations). For example, the symmetry elements of ammonia are characterized by rotation of $2\pi/3$ about the x_3-axis, and reflection in the x_1x_3-plane, the plane $\sqrt{3}x_1 + x_2 = 0$, and the plane $\sqrt{3}x_1 - x_2 = 0$. Associated with the rotation of $2\pi/3$ are the operations of rotation by $2\pi/3$ and $4\pi/3$, and there is one operation associated with each planar reflection.

Below is a list of standard symmetry elements with a brief description of each. The representation of each element in **Schoenflies** (or **Schönflies**) notation, commonly used in spectroscopy, is also given.

- **Identity:** the symmetry element that represents no change. Every molecule has this symmetry element. Notation: E (einheit).

- **Inversion:** the symmetry element of reflection through the origin. A molecule has inversion when, for any atom in the molecule, an identical atom exists diametrically opposite this center an equal distance from it. There may or may not be an atom at the center. Notation: i.

- **Planar Reflection (or mirror plane):** the symmetry element that represents a plane of reflection through which an identical copy of the original molecule is given. Standard notation for a symmetry plane is σ, with a subscript that indicates the direction of the plane. Examples include a vertical plane σ_v, a horizontal plane σ_h, a particular coordinate plane such as $\sigma_{x_1x_2}$ or σ_{xz}, or a dihedral plane σ_d (reflection in a plane through the origin and the axis with the 'highest' symmetry, but also bisecting the angle between the twofold axes perpendicular to the symmetry axis).

- **Proper Rotation (or symmetry axis):** the symmetry element that represents a counter-clockwise rotation of $2\pi/n$ radians about a given axis. A molecule can have multiple symmetry axes in different directions. A molecule can also have more than one symmetry axis in a given direction; the one with the highest n is called the *principal axis.* Notation: C_n or $C_n(x)$, where x represents the direction of the axis of rotation.
- **Improper Rotation (or rotation-reflection axis):** the symmetry element that represents a counter-clockwise rotation of $2\pi/n$ radians followed by a reflection through a plane perpendicular to the axis of rotation. This is a composition of a proper rotation (C_n) element and a planar reflection (σ) element. Notation: S_n or $S_n(x)$, where x represents the axis of rotation.

Often when there are multiple planes or axes of the same type, an apostrophe ($'$) or two may be added to distinguish them. An example of the Schoenfiles notation applied to the ammonia point group would be $\boldsymbol{I} = E$, $\boldsymbol{A} = C_3$, $\boldsymbol{A}_2 = C_3^2$, $\boldsymbol{B} = \sigma_v$, $\boldsymbol{C} = \sigma'_v$, and $\boldsymbol{D} = \sigma''_v$, resulting in the point group table below.

	E	C_3	C_3^2	σ_v	σ'_v	σ''_v
E	E	C_3	C_3^2	σ_v	σ'_v	σ''_v
C_3	C_3	C_3^2	E	σ''_v	σ_v	σ'_v
C_3^2	C_3^2	E	C_3	σ'_v	σ''_v	σ_v
σ_v	σ_v	σ'_v	σ''_v	E	C_3	C_3^2
σ'_v	σ'_v	σ''_v	σ_v	C_3^2	E	C_3
σ''_v	σ''_v	σ_v	σ'_v	C_3	C_3^2	E

The symmetry elements are the geometric objects (the mirror planes, rotation axes, inversion center, etc.). The symmetry operations are the actual linear transformations that performs the permutation of the nuclei of the molecule. The point group of a molecule is the set of all symmetry operations on that molecule under the binary operation of composition.

A very valuable online resource is the Symmetry at Otterbien University website:

`http://symmetry.otterbein.edu`

2.7 Review of Concepts

- Terms to know: binary operation, identity, closed, inverse, group, order of element, order of group, subgroup, dihedral group, matrix group, point group, symmetry element, symmetry operation.
- Know how to verify that a given set with a binary operation is a group.
- Know how to calculate the group table of groups of small order.
- Know how to find the order of an element in a finite group.
- Know how to find the matrix that represents a symmetry operation.

2.7 Practice Problems

1. Consider the set $\mathbb{Z}_2 = \{0,1\}$ and let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ (the Cartesian product). Let the binary operation $*$ be defined by, for $(a,b),(c,d) \in G$, $(a,b)*(c,d) = (a+c \bmod 2, b+d \bmod 2)$. Verify that G is a group under this operation and give the group table.

2.7 Exercises

1. Consider the set of positive integers $\mathbb{Z}_7^* = \{1,2,3,4,5,6\}$. Let the binary operation $*$ be defined by, for $a,b \in \mathbb{Z}_7^*$, $a*b = ab \bmod 7$, i.e., the remainder when the product of a and b is divided by 7. Verify that this is a group under this operation. What is the identity element? Give the group table for this group. Comment on whether or not this represents the same group as D_3 discussed in the text.

2. List the symmetry elements of the water molecule and give the table of the corresponding point group.

3. Find 3×3 matrices that represent the following symmetry operations.

a) $C_8(x)$ b) $S_4(z)$ c) σ_{xz} d) $C_6(y)$

2.7 Answers to Practice Problems

1. Closure is verified by noting that the result of $(a,b)*(c,d)$ is also an element of G as $a+c \bmod 2 \in \mathbb{Z}_2$ and $b+d \bmod 2 \in \mathbb{Z}_2$. The operation of G is associative because modular arithmetic is also associative. The identity element in G is $(0,0)$ as $(a,b)*(0,0) = (a,b) = (0,0)*(a,b)$. Finally, if $(a,b) \in G$, then the inverse of (a,b) is itself, as

$$(a,b)*(a,b) = (a+a \bmod 2, b+b \bmod 2) = (2a \bmod 2, 2b \bmod 2) = (0,0).$$

The group table is given below.

$*$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(1,0)$	$(1,0)$	$(0,0)$	$(1,1)$	$(0,1)$
$(0,1)$	$(0,1)$	$(1,1)$	$(0,0)$	$(1,0)$
$(1,1)$	$(1,1)$	$(0,1)$	$(1,0)$	$(0,0)$

Chapter 3

Single-Variable Calculus

3.1 Integration By Parts

Objectives and Concepts:

- Integration by parts is a technique that allows an integral to be rewritten as a product and another, possibly easier, integral.
- The choice of functions for u and dv should be motivated by ease of differentiation and integration, respectively.
- If integration by parts multiple times produces the same integral that was present initially, the equation can e solved for the unknown integral.

References: OSC-2 §3.1, CET §7.1, TCMB §6.4.

3.1.1 Integration By Parts

Substitution is an incredibly useful technique of integration, however there are several antiderivatives that cannot be found using it. For example, what kind of function do we have if we take its derivative and arrive at $x\cos x$? In other words, what is

$$\int x\cos x\,dx.$$

It's a simple matter to take the derivative of the integrand using the Product Rule, but there is no Product Rule for integrals. However, this section introduces *Integration by Parts*, a method of integration that is based on the Product Rule for derivatives. It will enable us to evaluate this integral.

The Product Rule says that if u and v are functions of x, then $(uv)' = u'v + uv'$. For simplicity, we've written u for $u(x)$ and v for $v(x)$. Suppose we integrate both sides with respect to x. This gives

$$\int (uv)'\,dx = \int (u'v + uv')\,dx.$$

By the Fundamental Theorem of Calculus, the left side integrates to uv. The right side can be broken up into two integrals, and we have

$$uv = \int u'v\,dx + \int uv'\,dx.$$

Solving for the second integral we have

$$\int uv'\,dx = uv - \int u'v\,dx.$$

Using differential notation, we can write $du = u'(x)dx$ and $dv = v'(x)dx$ and the expression above can be written as follows:

$$\int u\,dv = uv - \int v\,du.$$

This is the Integration by Parts formula. For reference purposes, we state this in a theorem.

Integration by Parts: Let u and v be differentiable functions of x on an interval I containing a and b. Then

$$\int u\;dv = uv - \int v\;du,$$

and

$$\int_{x=a}^{x=b} u\;dv = uv\Big|_a^b - \int_{x=a}^{x=b} v\;du.$$

The key to Integration by Parts is to identify part of the integrand as "u" and part as "dv." Once those choices are made, the other necessary components of the formula du and v are found by differentiation and integration, respectively. Regular practice will help one make good identifications, and later we will introduce some principles that help. In general, the motivating principle should be that **the choice of u is easy (or simplifies) when differentiated** and **the choice of dv should be easy to integrate**.

Example 1: Evaluate $\displaystyle\int x\cos x\;dx$.

For now, let $u = x$ and $dv = \cos x\;dx$. It is generally useful to make a small table of these values as done below. Right now we only know u and dv, so we need to identify du and v.

$u = x$	$dv = \cos x\;dx$
$du =$??	$v =$??

On the right side of the formula we can see that we need du and v. We get du by taking the derivative of u, and we get $du = (1)\,dx$, or simply $du = dx$. We get v by finding an antiderivative of dv. Here we get $v = \sin x$. (Choose the antiderivative with a constant of 0.)

Now substitute all of this into the Integration by Parts formula, giving

$$\int x\cos x = x\sin x - \int \sin x\,dx.$$

$$\begin{array}{ll} u = x & dv = \cos x\,dx \\ du = dx & v = \sin x \end{array}$$

We can then integrate $\sin x$ to get $-\cos x + C$ and thus the answer is

$$\int x\cos x\,dx = x\sin x + \cos x + C.$$

The example above demonstrates how integration by parts works in general. We try to identify u and dv in the integral we are given, and the key is that we usually want to choose u and dv so that du is simpler than u and v is hopefully not too much more complicated than dv. This will mean that the integral on the right side of the Integration by Parts formula, $\int v\,du$ will be simpler to integrate than the original integral $\int u\,dv$.

In the example above, we chose $u = x$ and $dv = \cos x\,dx$. Then $du = dx$ was simpler than u and $v = \sin x$ is no more complicated than dv. Therefore, instead of integrating $x\cos x\,dx$, we could integrate $\sin x\,dx$, which we know how to do.

A useful mnemonic for helping to determine the best choice u is “LIATE,” where

L = **L**ogarithmic, I = **I**nverse Trig., A = **A**lgebraic (polynomials),

T = **T**rigonometric, and E = **E**xponential.

If the integrand contains both a logarithmic and an algebraic term, in general letting u be the logarithmic term works best, as indicated by L coming before A in LIATE. However, sometimes you have to try several different choices for u before getting one that works.

Example 2: Evaluate $\int xe^x\,dx$.

Example 3: Evaluate $\int x^2\cos x\,dx$.

Example 4: Evaluate $\int e^x\cos x\,dx$.

This is a classic problem. Our mnemonic suggests letting u be the trigonometric function instead of the exponential. In this particular example, one can let u be either $\cos x$ or e^x. To demonstrate that we do not have to follow LIATE, we choose $u = e^x$ and hence $dv = \cos x\,dx$. Then $du = e^x\,dx$ and $v = \sin x$ as shown below.

$$\begin{array}{ll} u = e^x & v = \sin x \\ du = e^x\,dx & dv = \cos x\,dx \end{array}$$

Notice that du is no simpler than u, going against our general rule (but bear with us). The Integration by Parts formula yields

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

$u = e^x$	$v = -\cos x$
$du = e^x \, dx$	$dv = \sin x \, dx$

The integral on the right is not much different than the one we started with, so it seems like we have gotten nowhere. Let's stick keep working and apply Integration by Parts to the new integral, using $u = e^x$ and $dv = \sin x \, dx$.

The Integration by Parts formula then gives:

$$\begin{aligned}\int e^x \cos x \, dx &= e^x \sin x - \left(-e^x \cos x - \int -e^x \cos x \, dx\right)\\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.\end{aligned}$$

It seems we are back right where we started, as the right hand side contains $\int e^x \cos x \, dx$. But this actually a good thing.

Add $\int e^x \cos x \, dx$ to both sides. This gives

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x$$

Now divide both sides by 2:

$$\int e^x \cos x \, dx = \frac{1}{2}\left(e^x \sin x + e^x \cos x\right).$$

Simplifying a little and adding the constant of integration, our answer is thus

$$\int e^x \cos x \, dx = \frac{1}{2} e^x \left(\sin x + \cos x\right) + C.$$

How would the result change for $\int e^x \sin x \, dx$?

Example 5: Evaluate $\int \ln x \, dx$. (**Hint:** let $u = \ln x$.)

Example 6: Evaluate $\int \arctan x \, dx$.

Example 7: Evaluate $\int_1^2 x^2 \ln x \, dx$.

3.1.2 Using u-Substitution with Integration by Parts

When taking derivatives, it was common to employ multiple rules (such as, using both the Quotient and the Chain Rules). It should then come as no surprise that some integrals are best evaluated by combining integration techniques. In particular, here we illustrate making an "unusual" substitution first before using Integration by Parts.

Example 8: Evaluate $\int \cos(\ln x)\ dx$.

The integrand contains a composition of functions, leading us to think Substitution would be beneficial. Letting $u = \ln x$, we have $du = 1/x\ dx$. This seems problematic, as we do not have a $1/x$ in the integrand. But consider:

$$du = \frac{1}{x}\ dx \Rightarrow x \cdot du = dx.$$

Since $u = \ln x$, we can use inverse functions and conclude that $x = e^u$. Therefore we have that

$$\begin{aligned} dx &= x \cdot du \\ &= e^u\ du. \end{aligned}$$

We can thus replace $\ln x$ with u and dx with $e^u\ du$. Thus we rewrite our integral as

$$\int \cos(\ln x)\ dx = \int e^u \cos u\ du.$$

We evaluated this integral in a previous example. Using the result there, we have:

$$\begin{aligned} \int \cos(\ln x)\ dx = \int e^u \cos u\ du &= \frac{1}{2}e^u\big(\sin u + \cos u\big) + C \\ &= \frac{1}{2}e^{\ln x}\big(\sin(\ln x) + \cos(\ln x)\big) + C = \frac{1}{2}x\big(\sin(\ln x) + \cos(\ln x)\big) + C \end{aligned}$$

Example 9: Evaluate $\int \sin(\sqrt{x})\, dx$.

3.1 Review of Concepts

- Know how to evaluate definite and indefinite integrals using integration by parts.

3.1 Practice Problems

1. Evaluate the following integrals:

a) $\displaystyle\int e^{\sqrt{x}}\,dx$

b) $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$

c) $\displaystyle\int x\sin x\cos x\,dx$

d) $\displaystyle\int_1^e (\ln x)^2\,dx$

3.1 Exercises

1. Evaluate the following integrals:

a) $\displaystyle\int_0^1 \ln(x^2+1)\,dx$

b) $\displaystyle\int \arcsin(x)\,dx$

c) $\displaystyle\int 9t^2\sin(3t)\,dt$

d) $\displaystyle\int (\ln x)^3\,dx$

e) $\displaystyle\int (t+1)e^{5t}\,dt$

f) $\displaystyle\int_1^e \frac{\ln x^2}{x^2}\,dx$

g) $\displaystyle\int \theta\sec^2\theta\,d\theta$

h) $\displaystyle\int_0^{\sqrt[4]{3}} y\tan^{-1}y^2\,dy$

2. In the study of periodic (wave-like) functions, definite integrals of the form

$$I=\int_0^{\pi}\theta\cos(n\theta)\,d\theta$$

must often be evaluated for some positive integer n. Find a formula for I in terms of n.

3. The **hyperbolic sine** and **hyperbolic cosine** functions $\sinh x$ and $\cosh x$ are defined by the following formulas:

$$\sinh x=\frac{e^x-e^{-x}}{2},\qquad \cosh x=\frac{e^x+e^{-x}}{2}.$$

Using these definitions, find $\dfrac{d}{dx}(\sinh x)$ and $\dfrac{d}{dx}(\cosh x)$ in terms of each other. Use this information to evaluate the integral

$$\int \cosh x\sinh x\,dx.$$

3.1 Answers to Practice Problems

1. a) Let $u=\sqrt{x}$, then $du=\dfrac{1}{2\sqrt{x}}\,dx$, so $2u\,du=dx$. Then we have

$$\int e^{\sqrt{x}}\,dx=2\int ue^u\,du=2(ue^u-e^u)+C=2\sqrt{x}e^{\sqrt{x}}-2e^{\sqrt{x}}+C.$$

b) Let $u = x$ and let $dv = \dfrac{1}{\sqrt{x+1}}\,dx$. Then $du = dx$ and $v = 2\sqrt{x+1}$. Then integration by parts produces

$$\int \frac{x}{\sqrt{x+1}}\,dx = 2x\sqrt{x+1} - 2\int \sqrt{x+1}\,dx = 2x\sqrt{x+1} - 2\left(\frac{2}{3}(x+1)^{3/2}\right) + C = 2x\sqrt{x+1} - \frac{4}{3}(x+1)^{3/2} + C.$$

How does this compare to Example 16 of the Appendix B.2 (Introductory Calculus Review)?

c) Let $u = x$ and let $dv = \sin x \cos x\,dx$. Then $du = dx$, and we can find v by a simple substitution: let $t = \sin x$, then $dt = \cos x\,dx$, so

$$\int \sin x \cos x\,dx = \int t\,dt = \frac{t^2}{2} + C = \frac{1}{2}\sin^2 x + C.$$

So $v = (1/2)\sin^2 x$. Then we have

$$\int x \sin x \cos x\,dx = \frac{1}{2}x\sin^2 x - \frac{1}{2}\int \sin^2 x\,dx.$$

To evaluate the last integral, recall the double-angle identity:

$$\sin^2 x = \frac{1-\cos 2x}{2}.$$

This means that

$$\int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{1}{2}x - \frac{1}{2}\int \cos 2x\,dx.$$

This requires another substitution, let $s = 2x$, then $ds = 2\,dx$, so $ds/2 = dx$. Then we have

$$\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{2}\int \cos 2x\,dx = \frac{1}{2}x - \frac{1}{4}\int \cos s\,ds = \frac{1}{2}x + \frac{1}{4}\sin s + C = \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

Finally, putting it all together we have

$$\begin{aligned}\int x\sin x\cos x\,dx = \frac{1}{2}x\sin^2 x - \frac{1}{2}\int \sin^2 x\,dx &= \frac{1}{2}x\sin^2 x - \frac{1}{2}\left(\frac{1}{2}x - \frac{1}{2}\int \cos 2x\,dx\right)\\ &= \frac{1}{2}x\sin^2 x - \frac{1}{2}\left(\frac{1}{2}x + \frac{1}{4}\sin 2x\right) + C = \frac{1}{2}x\sin^2 x - \frac{x}{4} - \frac{1}{8}\sin 2x + C.\end{aligned}$$

d) Let $u = (\ln x)^2$ and $dv = dx$. Then $du = 2(\ln x)(1/x)\,dx$ and $v = x$. Then we have

$$\int_1^e (\ln x)^2\,dx = x(\ln x)^2\Big|_1^e - 2\int_1^e x\left(\frac{\ln x}{x}\right)\,dx = x(\ln x)^2\Big|_1^e - 2\int_1^e \ln x\,dx.$$

Now we evaluated the second integral in Example 5. Thus we have

$$\begin{aligned}\int_1^e (\ln x)^2\,dx = x(\ln x)^2\Big|_1^e - 2\int_1^e x\ln x\,dx &= x(\ln x)^2\Big|_1^e - 2\,(x\ln x - x)\Big|_1^e\\ &= e(\ln e)^2 - 1(\ln 1)^2 - 2\,[(e\ln e - e) - (1\ln 1 - 1)] = e - 2[(e-e) - (0-1)] = e - 2.\end{aligned}$$

3.2 Using Tables of Integrals and (†)Partial Fractions Decomposition

Objectives and Concepts:

- Several known antiderivatives are given in a table of integrals.
- In many cases, a substitution, integration by parts, or partial fractions will help rewrite the integral into a form present in the table.

References: OSC-2 §3.5 and CET §7.6 for Table of Integrals; OSC-2 §3.4 and CET §7.4 and TCMB §6.6 for (†) Partial Fractions Decomposition.

3.2.1 Tables of Integrals

Several standard antiderivatives are given in the **Table of Integrals** (following this section). The formulas can be applied to a wide variety of integrals directly, but it is often necessary to perform some simplification first as well as to use u-substitution.

Example 1: Evaluate $\displaystyle\int x^5 \cos(x^3)\, dx$.

Example 2: Evaluate $\displaystyle\int \frac{x^3}{x^2+9}\, dx$.

Example 3: Evaluate $\displaystyle\int \frac{7x}{\sqrt{4x-x^2}}\, dx$.

Example 4: Example $\displaystyle\int \sin^2 x \cos^2 x\, dx$.

Example 5: Example $\displaystyle\int \sin^2(2x)\cos^2 x\, dx$.

Example 6: Example $\displaystyle\int \frac{1}{\sqrt{e^{2x}-9}}\, dx$.

3.2.2 (†) Partial Fractions Decomposition

Many rational functions that will arise will not fit one of the forms given in the table of integrals. However, if the denominator contains more than one polynomial factor, a **partial fraction decomposition** of the rational function can be found. One way to think about partial fraction decomposition is to consider it as the "undoing" of the act of getting a common denominator. We set up a system of equations that will allow us to find the decomposition of the rational function into a sum of rational functions with smaller denominators. One important fact that is used in partial fraction decomposition is that two polynomials are equal if and only if the coefficients for each term in the polynomial are equal.

Partial Fraction Decomposition: Let $\dfrac{p(x)}{q(x)}$ be a rational function, where the degree of p is less than the degree of q. (If the degree of p is greater than or equal to the degree of q, then perform polynomial long division to reduce $\dfrac{p(x)}{q(x)}$ to the sum of a polynomial and a rational function.)

1. **Linear Terms:** Let $(x-a)$ divide $q(x)$, where $(x-a)^n$ is the highest power of $(x-a)$ that divides $q(x)$. Then the decomposition of $\dfrac{p(x)}{q(x)}$ will contain the sum
$$\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_n}{(x-a)^n}.$$
2. **Quadratic Terms:** Let x^2+bx+c divide $q(x)$, where $(x^2+bx+c)^n$ is the highest power of x^2+bx+c that divides $q(x)$. Then the decomposition of $\dfrac{p(x)}{q(x)}$ will contain the sum
$$\frac{B_1x+C_1}{x^2+bx+c}+\frac{B_2x+C_2}{(x^2+bx+c)^2}+\cdots+\frac{B_nx+C_n}{(x^2+bx+c)^n}.$$

To find the coefficients A_i, B_i and C_i:

1. Multiply all fractions by $q(x)$, clearing the denominators. Collect like terms.
2. Equate the resulting coefficients of the powers of x and solve the resulting system of linear equations.

Example 7: Evaluate $\displaystyle\int \frac{1}{x^2(x^2+4)}\,dx$ by first finding the partial fraction decomposition of $\dfrac{1}{x^2(x^2+4)}$.

3.2.3 Using Symmetry

In many cases, a definite integral can be simplified or even solved because of properties of the integrand.

Definition: A function $f(x)$ is **odd** if $f(-x)=-f(x)$ for all x. A function $f(x)$ is **even** if $f(-x)=f(x)$ for all x.

Theorem: If f is an odd function, then
$$\int_{-a}^{a} f(x)\,dx = 0.$$
If f is an even function, then
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

Example 8: Evaluate $\displaystyle\int_{-3\pi/2}^{3\pi/2} 4x^2 \sin(2x)\cos(4x)\,dx$

3.2 Review of Concepts

- Know how to evaluate definite and indefinite integrals using tables of integrals.
- Know how to rewrite a rational function using partial fraction decomposition.

3.2 Practice Problems

1. Evaluate the following integrals:

a) $\displaystyle\int e^{4x}\sqrt{1+e^{2x}}\,dx$

b) $\displaystyle\int \frac{1}{\sqrt{2x^3+3x^2}}\,dx$

c) (†) $\displaystyle\int \frac{2x^2-x+4}{x^3+4x}\,dx$

d) (†) $\displaystyle\int \frac{x^3}{x^2-x-2}\,dx$

3.2 Exercises

1. Evaluate the following integrals:

a) $\displaystyle\int e^{-3x}\cos(4x)\,dx$

b) $\displaystyle\int \frac{1}{x^2\sqrt{4x-9}}\,dx$

c) $\displaystyle\int \frac{\sqrt{9-4x}}{x^2}\,dx$

d) $\displaystyle\int \frac{\cos^{-1}\sqrt{x}}{\sqrt{x}}\,dx$

e) $\displaystyle\int \frac{\sqrt{x-x^2}}{x}\,dx$

f) $\displaystyle\int \frac{1}{t\sqrt{4+(\ln t)^2}}\,dt$

g) $\displaystyle\int \frac{x^2}{\sqrt{x^2-4x+5}}\,dx$

h) (†) $\displaystyle\int \frac{3x^2+7x-2}{x^3-x^2-2x}\,dx$

2. The work w done by a gas as its volume V changes is given by $w = \displaystyle\int P\,dV$ where P is the pressure. Calculate the work done in increasing the volume of a van der Waals gas where

$$\left(P - \frac{an^2}{V^2}\right)(V - nb) = nRT,$$

assuming all other variables are constant with respect to P and V. (Hint: solve for P in terms of V.)

3. The square-well potential for the interaction of two spherically symmetric molecules separated by a distance r is given by

$$u(r) = \begin{cases} \infty & 0 < r < \sigma \\ -\varepsilon & \sigma < r < \lambda\sigma \\ 0 & r > \lambda\sigma \end{cases},$$

where σ, λ, and ε are constants that are characteristic of the molecule. The second virial coefficient of imperfect gas theory is given by

$$B(T) = -2\pi \int_0^\infty \left(e^{-u(r)/(k_B T)} - 1\right) r^2\, dr,$$

where k_B is the Boltzmann constant and T is the kelvin temperature. Derive an expression for $B(T)$ (i.e., evaluate the integral).

3.2 Answers to Practice Problems

1. a) $\int e^{4x}\sqrt{1+e^{2x}}\, dx$. Let $u = e^{2x}$, then $du = 2e^{2x}\, dx$, which implies $\frac{1}{2} u\, du = e^{4x}\, dx$. So we have

$$\int e^{4x}\sqrt{1+e^{2x}}\, dx = \frac{1}{2}\int u\sqrt{1+u}\, du.$$

This looks like number 24 in the table with $u = x$, $a = b = 1$. Then we have

$$\begin{aligned} \int e^{4x}\sqrt{1+e^{2x}}\, dx = \frac{1}{2}\int u\sqrt{1+u}\, du = \frac{1}{2}\left(\frac{2}{15}(-2+u+3u^2)\sqrt{1+u}\right) + C \\ = \frac{2}{30}(-2+e^{2x}+3e^{4x})\sqrt{1+e^{2x}}) + C. \end{aligned}$$

b) $\int \frac{1}{\sqrt{2x^3+3x^2}}\, dx$. We have

$$\int \frac{1}{\sqrt{2x^3+3x^2}}\, dx = \int \frac{1}{x\sqrt{2x+3}}\, dx,$$

which appears as number 27 in the table with $a = 3$ and $b = 2$. Then

$$\int \frac{1}{\sqrt{2x^3+3x^2}}\, dx = \int \frac{1}{x\sqrt{2x+3}}\, dx = \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{2x+3}-\sqrt{3}}{\sqrt{2x+3}+\sqrt{3}}\right| + C.$$

c) $\int \frac{2x^2-x+4}{x^3+4x}\, dx$. Note that the denominator factors as $x^3+4x = x(x^2+4)$. So the integrand has a partial fraction decomposition of

$$\frac{2x^2-x+4}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}.$$

This means that

$$2x^2 - x + 4 = Ax^2 + 4A + Bx^2 + Cx.$$

Thus $A = 1$, $B = 1$, and $C = -1$. Then

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx = \int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right) dx = \int \left(\frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}\right) dx$$
$$= \ln|x| + \frac{1}{2}\ln(x^2+4) - \arctan\left(\frac{x}{2}\right) + C.$$

d) $\displaystyle\int \frac{x^3}{x^2 - x - 2}\,dx$. First, perform long division of polynomials to get

$$\frac{x^3}{x^2 - x - 2} = x + 1 + \frac{3x+2}{x^2 - x - 2}.$$

Also, $x^2 - x - 2 = (x+1)(x-2)$. Then we need to perform a partial fraction decomposition of the last term above. We have

$$\frac{3x+2}{x^2-x-2} = \frac{A}{x+1} + \frac{B}{x-2} = \frac{Ax - 2A + Bx + B}{x^2 - x - 2}.$$

This means that $A + B = 3$ and $B - 2A = 2$. Thus $A = 1/3$ and $B = 8/3$. Thus we have

$$\int \frac{x^3}{x^2-x-2}\,dx = \int \left(x + 1 + \frac{(1/3)}{x+1} + \frac{(8/3)}{x-2}\right) dx \frac{x^2}{2} + x + \frac{1}{3}\ln|x+1| + \frac{8}{3}\ln|x+2| + C.$$

Tables of Integrals

Elementary Integrals

1. $\int k\,dx = kx + C$
2. $\int (ax+b)^n\,dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C \quad (n \neq -1)$
3. $\int \dfrac{1}{ax+b}\,dx = \dfrac{1}{a}\ln|ax+b| + C$
4. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C$
5. $\int \sin x\,dx = -\cos x + C$
6. $\int \cos x\,dx = \sin x + C$
7. $\int \tan x\,dx = -\ln|\cos x| + C$
8. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
9. $\int \sec x \tan x\,dx = \sec x + C$
10. $\int \sec^2 x\,dx = \tan x + C$
11. $\int \cot x\,dx = \ln|\sin x| + C$
12. $\int \csc x\,dx = -\ln|\csc x - \cot x| + C$
13. $\int \csc x \cot x\,dx = -\csc x + C$
14. $\int \csc^2 x\,dx = -\cot x + C$
15. $\int u\,dv = uv - \int v\,du$

Forms Involving $a+bx$

16. $\int \dfrac{dx}{a+bx} = \dfrac{1}{b}\ln|a+bx| + C$
17. $\int \dfrac{x}{a+bx}\,dx = \dfrac{1}{b^2}(bx - a\ln|a+bx|) + C$
18. $\int \dfrac{x^2}{a+bx}\,dx = \dfrac{1}{2b^3}\left(b^2x^2 - 2abx + 2a^2\ln|a+bx|\right) + C$
19. $\int \dfrac{dx}{x(a+bx)} = \dfrac{1}{a}\ln\left|\dfrac{x}{a+bx}\right| + C$
20. $\int \dfrac{dx}{x^2(a+bx)} = -\dfrac{1}{ax} - \dfrac{b}{a^2}\ln\left|\dfrac{x}{a+bx}\right| + C$
21. $\int \dfrac{x}{(a+bx)^2}\,dx = \dfrac{1}{b^2}\left(\dfrac{a}{a+bx} + \ln|a+bx|\right) + C$
22. $\int \dfrac{dx}{x(a+bx)^2} = \dfrac{1}{a(a+bx)} + \dfrac{1}{a^2}\ln\left|\dfrac{x}{a+bx}\right| + C$
23. $\int \dfrac{x^2}{(a+bx)^2}\,dx = \dfrac{1}{b^3}\left(bx - \dfrac{a^2}{a+bx} - 2a\ln|a+bx|\right) + C$
24. $\int x\sqrt{a+bx}\,dx = \dfrac{2}{15b^2}(3bx-2a)(a+bx)^{3/2} + C$
25. $\int \dfrac{x}{\sqrt{a+bx}}\,dx = \dfrac{2}{3b^2}(bx-2a)\sqrt{a+bx} + C$
26. $\int \dfrac{x^2}{\sqrt{a+bx}}\,dx = \dfrac{2}{15b^2}\left(8a^2 + 3b^2x^2 - 4abx\right)\sqrt{a+bx} + C$
27. $\int \dfrac{dx}{x\sqrt{a+bx}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|\dfrac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right| + C & (a>0) \\ \dfrac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\dfrac{a+bx}{-a}} + C & (a<0) \end{cases}$
28. $\int \dfrac{\sqrt{a+bx}}{x}\,dx = 2\sqrt{a+bx} + a\int \dfrac{dx}{x\sqrt{a+bx}}$
29. $\int \dfrac{\sqrt{a+bx}}{x^2}\,dx = -\dfrac{\sqrt{a+bx}}{x} + \dfrac{b}{2}\int \dfrac{dx}{x\sqrt{a+bx}}$
30. $\int x^n\sqrt{a+bx}\,dx = \dfrac{2}{b(2n+3)}\left[x^n(a+bx)^{3/2} - na\int x^{n-1}\sqrt{a+bx}\,dx\right]$

31. $\displaystyle\int \frac{x^n}{\sqrt{a+bx}}\,dx = \frac{2x^n\sqrt{a+bx}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{x^{n-1}}{\sqrt{a+bx}}\,dx$

32. $\displaystyle\int \frac{dx}{x^n\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{a(n-1)x^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{dx}{x^{n-1}\sqrt{a+bx}}$

Forms Involving $a^2 + x^2$

33. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

34. $\displaystyle\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right) + C$

35. $\displaystyle\int x^2\sqrt{a^2+x^2}\,dx = \frac{x}{8}(a^2+2x^2)\sqrt{a^2+x^2} - \frac{a^4}{8}\ln\left(x+\sqrt{a^2+x^2}\right) + C$

36. $\displaystyle\int \frac{\sqrt{a^2+x^2}}{x}\,dx = \sqrt{a^2+x^2} - a\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right| + C$

37. $\displaystyle\int \frac{\sqrt{a^2+x^2}}{x^2}\,dx = -\frac{\sqrt{a^2+x^2}}{x} + \ln\left(x+\sqrt{a^2+x^2}\right) + C$

38. $\displaystyle\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left(x+\sqrt{a^2+x^2}\right) + C$

39. $\displaystyle\int \frac{x^2}{\sqrt{a^2+x^2}}\,dx = \frac{x}{2}\sqrt{a^2+x^2} - \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right) + C$

40. $\displaystyle\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right| + C$

41. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2+x^2}} = -\frac{\sqrt{a^2+x^2}}{a^2x} + C$

42. $\displaystyle\int (a^2+x^2)^{3/2}\,dx = \frac{x}{8}(5a^2+2x^2)\sqrt{a^2+x^2} + \frac{3a^4}{8}\ln\left(x+\sqrt{a^2+x^2}\right) + C$

43. $\displaystyle\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}} + C$

Forms Involving $a^2 - x^2$

44. $\displaystyle\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C$

45. $\displaystyle\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

46. $\displaystyle\int x^2\sqrt{a^2-x^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2} + \frac{a^4}{8}\sin^{-1}\left(\frac{x}{a}\right) + C$

47. $\displaystyle\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$

48. $\displaystyle\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\frac{\sqrt{a^2-x^2}}{x} - \sin^{-1}\left(\frac{x}{a}\right) + C$

49. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

50. $\displaystyle\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

51. $\displaystyle\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$

52. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C$

53. $\displaystyle\int (a^2-x^2)^{3/2}\,dx = -\frac{x}{8}(2x^2-5a^2)\sqrt{a^2-x^2} + \frac{3a^4}{8}\sin^{-1}\left(\frac{x}{a}\right) + C$

54. $\displaystyle\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}} + C$

Forms Involving $x^2 - a^2$

55. $\displaystyle\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

56. $\displaystyle\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$

57. $\displaystyle\int x^2\sqrt{x^2-a^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{x^2-a^2} - \frac{a^4}{8}\ln\left|x+\sqrt{x^2-a^2}\right| + C$

58. $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\sec^{-1}\left|\frac{x}{a}\right| + C$

59. $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x^2}\,dx = -\frac{\sqrt{x^2-a^2}}{x} + \ln\left|x+\sqrt{x^2-a^2}\right| + C$

60. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + C$

61. $\displaystyle\int \frac{x^2}{\sqrt{x^2-a^2}}\,dx = \frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$

62. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$

63. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x} + C$

64. $\displaystyle\int (x^2-a^2)^{3/2}\,dx = \frac{x}{8}(2x^2-5a^2)\sqrt{x^2-a^2} + \frac{3a^4}{8}\ln\left|x+\sqrt{x^2-a^2}\right| + C$

65. $\displaystyle\int \frac{dx}{(x^2-a^2)^{3/2}} = -\frac{x}{a^2\sqrt{x^2-a^2}} + C$

Forms Involving $\sqrt{2ax-x^2}$

66. $\displaystyle\int \sqrt{2ax-x^2}\,dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$

67. $\displaystyle\int x\sqrt{2ax-x^2}\,dx = \frac{2x^2-ax-3a^2}{6}\sqrt{2ax-x^2} + \frac{a^3}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$

68. $\displaystyle\int \frac{\sqrt{2ax-x^2}}{x}\,dx = \sqrt{2ax-x^2} + a\sin^{-1}\left(\frac{x-a}{a}\right) + C$

69. $\displaystyle\int \frac{\sqrt{2ax-x^2}}{x^2}\,dx = -\frac{2\sqrt{2ax-x^2}}{x} - \sin^{-1}\left(\frac{x-a}{a}\right) + C$

70. $\displaystyle\int \frac{dx}{\sqrt{2ax-x^2}} = \sin^{-1}\left(\frac{x-a}{a}\right) + C$

71. $\displaystyle\int \frac{x}{\sqrt{2ax-x^2}}\,dx = -\sqrt{2ax-x^2} + a\sin^{-1}\left(\frac{x-a}{a}\right) + C$

72. $\displaystyle\int \frac{x^2}{\sqrt{2ax-x^2}}\,dx = -\frac{x+3a}{2}\sqrt{2ax-x^2} + \frac{3a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$

73. $\displaystyle\int \frac{dx}{x\sqrt{2ax-x^2}} = -\frac{1}{ax}\sqrt{2ax-x^2} + C$

Forms Involving Trigonometric Functions

74. $\displaystyle\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$

75. $\displaystyle\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a} + C$

76. $\displaystyle\int \sin^3 ax\,dx = -\frac{3\cos ax}{4a} + \frac{\cos 3ax}{12a} + C$

77. $\displaystyle\int \cos ax\,dx = \frac{1}{a}\sin ax + C$

78. $\displaystyle\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a} + C$

79. $\displaystyle\int \cos^3 ax dx = \frac{3\sin ax}{4a} + \frac{\sin 3ax}{12a} + C$

80. $\displaystyle\int \sin^n x = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx \quad (n \text{ positive})$

81. $\displaystyle\int \cos^n x = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx \quad (n \text{ positive})$

82. $\displaystyle\int \tan^n x = \frac{1}{n-1}\tan^{n-1}x - \int \tan^{n-2}x\,dx \quad (n \geq 2)$

83. $\displaystyle\int \frac{1}{\sin^n x}\,dx \left(= \int \operatorname{cosec}^n x\,dx\right) = \frac{-\cos x}{(n-1)\sin^{n-1}x} + \frac{n-2}{n-1}\int \frac{1}{\sin^{n-2}x}\,dx \quad (n \neq 1, n \geq 2)$

84. $\int \frac{1}{\cos^n x}\, dx \left(=\int \sec^n x\, dx\right) = \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{1}{\cos^{n-2} x}\, dx \quad (n \ge 2)$

85. $\int \cot^n x \left(=\int \frac{1}{\tan^n x}\, dx\right) = \frac{-1}{n-1}\cot^{n-1} x - \int \cot^{n-2} x\, dx \quad (n \ge 2)$

86. $\int \cos x \sin x\, dx = \frac{1}{2}\sin^2 x + C = -\frac{1}{2}\cos^2 x + C = -\frac{1}{4}\cos 2x + C$

87. $\int \cos ax \sin bx\, dx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)} + C, \quad (a \ne b)$

88. $\int \sin ax \sin bx\, dx = \frac{\sin[(a-b)x]}{2(a-b)} - \frac{\sin[(a+b)x]}{2(a+b)} + C, \quad (a \ne b)$

89. $\int \cos ax \cos bx\, dx = \frac{\sin[(a-b)x]}{2(a-b)} + \frac{\sin[(a+b)x]}{2(a+b)} + C, \quad (a \ne b)$

90. $\int \sin^2 ax \cos bx\, dx = -\frac{\sin[(2a-b)x]}{4(2a-b)} + \frac{\sin bx}{2b} - \frac{\sin[(2a+b)x]}{4(2a+b)} + C \quad (2a \ne b)$

91. $\int \cos^2 ax \sin bx\, dx = \frac{\cos[(2a-b)x]}{4(2a-b)} - \frac{\cos bx}{2b} - \frac{\cos[(2a+b)x]}{4(2a+b)} + C \quad (2a \ne b)$

92. $\int \sin^2 ax \cos^2 bx dx = \frac{x}{4} - \frac{\sin 2ax}{8a} - \frac{\sin[2(a-b)x]}{16(a-b)} + \frac{\sin 2bx}{8b} - \frac{\sin[2(a+b)x]}{16(a+b)} + C \quad (a \ne b)$

93. $\int x \sin ax\, dx = -\frac{x\cos ax}{a} + \frac{n}{a^2}\sin ax\, dx$

94. $\int x^n \sin ax\, dx = -\frac{x^n \cos ax}{a} + \frac{n}{a}\int x^{n-1}\cos ax\, dx$

95. $\int x \cos ax\, dx = \frac{x\sin ax}{a} + \frac{n}{a^2}\cos ax\, dx.$

96. $\int x^n \cos ax\, dx = \frac{x^n \sin ax}{a} - \frac{n}{a}\int x^{n-1}\sin ax\, dx$

97. $\int e^{bx}\sin ax\, dx = \frac{1}{a^2+b^2}e^{bx}(b\sin ax - a\cos ax) + C$

98. $\int e^{bx}\cos ax\, dx = \frac{1}{a^2+b^2}e^{bx}(a\sin ax + b\cos ax) + C$

99. $\int xe^x \sin x\, dx = \frac{1}{2}e^x(\cos x - x\cos x + x\sin x) + C$

100. $\int xe^x \cos x\, dx = \frac{1}{2}e^x(x\cos x - \sin x + x\sin x) + C$

101. $$\int \sin^m x \cos^n x\, dx = \begin{cases} \int (1-\cos^2 x)^k \sin x \cos^n x\, dx = -\int (1-u^2)^k u^n\, du, & m = 2k+1, u = \cos x, du = -\sin x dx \\ \int \sin^m x(1-\sin^2 x)^k \cos x\, dx = \int (1-u^2)^k u^m\, du, & n = 2k+1, u = \sin x, du = \cos x dx \\ \int \left(\frac{1-\cos 2x}{2}\right)^{k_1}\left(\frac{1+\cos 2x}{2}\right)^{k_2} dx & m = 2k_1, n = 2k_2 \\ -\frac{\sin^{n-1} x \cos^{m+1} x}{n+m} - \frac{n-1}{n+m}\int \sin^{n-2} x \cos^m\, dxx & n \ge 2 \\ \frac{\sin^{n+1} x \cos^{m-1} x}{n+m} + \frac{m-1}{n+m}\int \sin^n x \cos^{m-2} x\, dx & m \ge 2 \end{cases}$$

Forms Involving Exponential and Logarithmic Functions

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102. $\int xe^x\, dx = (x-1)e^x + C$

103. $\int x^n e^x\, dx = x^n e^x - \int x^{n-1}e^x\, dx$

104. $\int xe^{-ax^2}\, dx = -\frac{1}{2a}e^{-ax^2} + C$

105. $\int \ln x\, dx = x\ln x - x + C$

106. $\int \frac{\ln x}{x}\, dx = \frac{1}{2}(\ln x)^2 + C$

107. $\int x^n \ln x\, dx = x^{n+1}\left(\frac{\ln x}{n+1} - \frac{1}{(n+1)^2}\right) + C, \quad (n \ne -1)$

108. $\int \frac{1}{x\ln x}\, dx = \ln|\ln x| + C$

109. $\int \ln(x^2+a^2)\, dx = x\ln(x^2+a^2) + 2a\tan^{-1}\left(\frac{x}{a}\right) - 2x + C$

110. $\int \ln(x^2-a^2)\, dx = x\ln(x^2-a^2) + a\ln\left|\frac{x+a}{x-a}\right| - 2x + C$

Forms Involving Inverse Trigonometric Functions

111. $\displaystyle\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1-x^2} + C$

112. $\displaystyle\int \cos^{-1} x\,dx = x\cos^{-1} x - \sqrt{1-x^2} + C$

113. $\displaystyle\int \tan^{-1} x\,dx = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$

114. $\displaystyle\int \cot^{-1} x\,dx = x\cot^{-1} x + \frac{1}{2}\ln(1+x^2) + C$

115. $\displaystyle\int \sec^{-1} x\,dx = x\sec^{-1} x - \ln\left|x+\sqrt{x^2-1}\right| + C$

116. $\displaystyle\int \csc^{-1} x\,dx = x\csc^{-1} x + \ln\left|x+\sqrt{x^2-1}\right| + C$

117. $\displaystyle\int x\sin^{-1} dx = \frac{2x^2-1}{4}\sin^{-1} x + \frac{x\sqrt{1-x^2}}{4} + C$

118. $\displaystyle\int x\cos^{-1} dx = \frac{2x^2-1}{4}\cos^{-1} x - \frac{x\sqrt{1-x^2}}{4} + C$

119. $\displaystyle\int x\tan^{-1} dx = \frac{x^2+1}{2}\tan^{-1} x - \frac{x}{2} + C$

120. $\displaystyle\int x^n \sin^{-1} x\,dx = \frac{1}{n+1}\left(x^{n+1}\sin^{-1} x - \int \frac{x^{n+1}}{\sqrt{1-x^2}}\,dx\right), \quad (n \neq -1)$

121. $\displaystyle\int x^n \cos^{-1} x\,dx = \frac{1}{n+1}\left(x^{n+1}\cos^{-1} x + \int \frac{x^{n+1}}{\sqrt{1-x^2}}\,dx\right), \quad (n \neq -1)$

122. $\displaystyle\int x^n \tan^{-1} x\,dx = \frac{1}{n+1}\left(x^{n+1}\tan^{-1} x - \int \frac{x^{n+1}}{1+x^2}\,dx\right), \quad (n \neq -1)$

3.3 Improper Integrals and Indeterminate Forms

Objectives and Concepts:

- To evaluate an improper integral, replace the corresponding integration bound with a limit.
- A limit may result in an indeterminate form, which may require l'Hôpital's Rule to evaluate.

References: OSC-2 §3.7 and CET §7.8 for Improper Integrals; OSC-1 §4.8 and CET §4.4 for Indeterminate Forms and l'Hôpital's Rule.

3.3.1 Improper Integrals

The definite integral $\int_a^b f(x)\,dx$ is always defined when the integrand f has only finitely many jump discontinuities in $[a,b]$. However, if we wish to compute the integral of a function f over the interval from a to b when one or both of a and b are infinite, or if the function contains an infinite discontinuity between a and b, then we have to take a special approach.

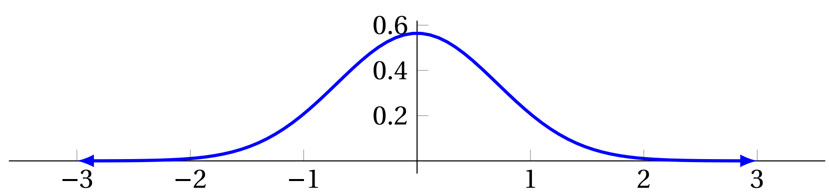
> **Definition:** An **improper integral** is an integral with an infinite limit of integration or an integral where the integrand has an infinite discontinuity.

While it might seem that integrals of this type might represent an infinite amount of area, there are many examples which are meaningful and actually quite important. One example where this arises is in applications involving the **Gauss function** (or **Gaussian**) f defined by

$$f(x) = ae^{-(x-b)^2/(2c^2)} + d.$$

For example, the classic "bell curve" is the Gaussian with $a = 1/\sqrt{\pi}$, $b = 0$, $c = 1/2$, and $d = 0$. In this case, it is actually true that

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-x^2}\,dx = 1.$$



3.3.2 Improper Integrals with Infinite Limits of Integration

1. Let f be a continuous function on $[a,\infty)$. Define

$$\int_a^\infty f(x)\,dx \quad \text{to be} \quad \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. Let f be a continuous function on $(-\infty,b]$. Define

$$\int_{-\infty}^b f(x)\,dx \quad \text{to be} \quad \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

3. Let f be a continuous function on $(-\infty,\infty)$. Let c be any real number; define

$$\int_{-\infty}^\infty f(x)\,dx \quad \text{to be} \quad \lim_{a\to-\infty}\int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx.$$

Definition: An improper integral is said to **converge** (or is **convergent**) if its corresponding limit exists; otherwise, it **diverges** (or is **divergent**).

The improper integral in part 3 above converges if and only if both of its limits exist.

Example 1: Evaluate $\displaystyle\int_1^\infty e^{-x}\,dx$.

Example 2: Evaluate $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$.

Example 3: For what values of p does $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converge?

If our goal is simply to determine if a given integral converges or diverges, we can often compare the integral to another one whose convergence behavior is known. This is useful when the integrands are related via an inequality. If we know the integral of a smaller function diverges, then the integral of the larger function must also diverge. If we know the integral of a larger function converges, then the integral of a smaller function must also converge.

The Direct Comparison Test for Improper Integrals:
Let f and g be continuous on $[a,\infty)$ where $0 \le f(x) \le g(x)$ for all x in $[a,\infty)$.

1. If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ converges.
2. If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then $\displaystyle\int_a^\infty g(x)\,dx$ diverges.

Example 4: Show that $\displaystyle\int_1^\infty e^{-x^2}\,dx$ converges.

3.3.3 Indeterminate Forms and l'Hôpital's Rule.

Definition: An **indeterminate form** is an expression that cannot be evaluated, usually obtained in the context of limits. Indeterminate forms could represent any numerical value, as well as $\pm\infty$.

The following are all examples of indeterminate forms:

$$0/0, \quad \pm\infty/\pm\infty, \quad \infty\cdot 0, \quad \infty-\infty, \quad 0^0, \quad \infty^0, \quad 1^\infty$$

The following are all **not** examples of indeterminate forms (some of these are *undefined* forms):

$$1/0, \quad 0/\pm\infty, \quad 1/\pm\infty, \quad \pm\infty/0, \quad \pm\infty/1, \quad 0-0, \quad \infty+\infty, \quad \infty\cdot\infty, \quad 0^1, \quad 1^0, \quad 0^\infty, \quad 0^{-\infty}$$

When dealing with a quotient indeterminate form, we can use l'Hôpital's Rule to gain more information:

l'Hôpital's Rule: Let a represent any real number or $\pm\infty$. If $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ produces an indeterminate form of type $0/0$ or $\pm\infty/\pm\infty$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Note that you should always check to see if you actually have one of the valid indeterminate forms, as applying l'Hôpital's Rule incorrectly usually produces wrong answers.

Example 5: Evaluate $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$.

3.3.4 (†) Improper Integration with Infinite Discontinuities

Let $f(x)$ be a continuous function on $[a,b]$ except at c, $a\le c\le b$, where $x=c$ is a vertical asymptote of f. Define

$$\int_a^b f(x)\,dx = \lim_{t\to c^-}\int_a^t f(x)\,dx + \lim_{t\to c^+}\int_t^b f(x)\,dx.$$

Example 6: Evaluate $\displaystyle\int_0^3 \frac{1}{x-1}\,dx$

Example 7: For what values of p does the integral $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converge?

Theorem:

1. The improper integral $\int_1^{\infty} \frac{1}{x^p}\, dx$ converges when $p > 1$ and diverges when $p \le 1$.
2. The improper integral $\int_0^{1} \frac{1}{x^p}\, dx$ converges when $p < 1$ and diverges when $p \ge 1$.

The Direct Comparison Test for Improper Integrals, Part II:
Let f and g be continuous functions on $[a, b]$ except at $x = c$, where each has a vertical asymptote, and $0 \le f(x) \le g(x)$ for all x in $[a, b]$, $x \ne c$.

- If $\int_a^b g(x)\, dx$ converges, then $\int_a^b f(x)\, dx$ converges.
- If $\int_a^b f(x)\, dx$ diverges, then $\int_a^b g(x)\, dx$ diverges.

3.3 Review of Concepts

- Know how to evaluate improper integrals with infinite limits of integration and infinite discontinuities in the integrand.
- Know how to use the Direct Comparison Test for Improper Integrals.

3.3 Practice Problems

1. Evaluate the following integrals or explain why they are divergent:

a) $\int_0^{\infty} xe^{-x^2}\, dx$

b) $\int_1^{2} \ln(x-1)\, dx$

c) $\int_0^{\infty} \frac{\arctan x}{1+x^2}\, dx$

d) $\int_2^{3} \frac{1}{\sqrt{3-x}}\, dx$

3.3 Exercises

1. Evaluate the following limits:

a) $\lim_{x\to 0^+} x\ln(x)$

b) $\lim_{x\to (\pi/2)^-} \left(x - \frac{\pi}{2}\right)\tan(x)$

c) $\lim_{x\to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

d) $\lim_{x\to\infty} \left(xe^{1/x} - x\right)$

2. Evaluate the following integrals or explain why they are divergent:

a) $\displaystyle\int_2^{\infty} \frac{1}{x \ln x}\, dx$

b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+5}$

c) $\displaystyle\int_0^{\ln 3} \frac{e^y}{(e^y-1)^{2/3}}\, dy$

d) $\displaystyle\int_0^{3} \frac{1}{\sqrt{9-x^2}}\, dx$

3. Use the Direct Comparison Test to determine if the integral $\displaystyle\int_0^{\infty} \frac{e^{-x}}{1+x^2}\, dx$ converges or diverges.

4. In gas kinetic theory, the probability that a molecule of mass m in a gas at temperature T has speed v is given by

$$f(v) = 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2k_B T}\right),$$

where k_B is the Boltzmann constant. The mean speed is given by

$$\overline{v} = \int_0^{\infty} v f(v)\, dv = 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_0^{\infty} v^3 e^{-mv^2/(2k_B T)}\, dv.$$

Calculate $\overline{v}$. (**Hint:** let $u = -mv^2/(2k_B T)$, pull as many constants out of the integral as possible, and you should be left with something like $\int ue^u\, du$.)

3.3 Answers to Practice Problems

1. a) $\displaystyle\int_0^{\infty} xe^{-x^2}\, dx$

Let $u = -x^2$, then $-du/2 = x\, dx$. Then we have

$$\int_0^{\infty} xe^{-x^2}\, dx = -\frac{1}{2}\int_0^{-\infty} e^u\, du = \frac{1}{2}\int_{-\infty}^{0} e^u\, du.$$

We must use a limit for the lower integration bound:

$$\frac{1}{2}\int_{-\infty}^{0} e^u\, du = \frac{1}{2}\lim_{a\to-\infty}\int_a^0 e^u\, du = \frac{1}{2}\lim_{a\to-\infty} e^u\Big|_a^0 = \frac{1}{2}\left(e^0 - \lim_{a\to-\infty} e^a\right) = \frac{1}{2}(1-0) = \frac{1}{2}.$$

b) $\displaystyle\int_1^{2} \ln(x-1)\, dx$

Note that the integrand has a vertical asymptote at $x = 1$. Thus we have

$$\begin{aligned}\int_1^2 \ln(x-1)\, dx &= \lim_{a\to1^+}\int_a^2 \ln(x-1)\, dx = \lim_{a\to1^+}\left[(x-1)\ln(x-1)-(x-1)\right]\Big|_a^2 \\ &= [(2-1)\ln(2-1)-(2-1)] - \lim_{a\to1^+}[(a-1)\ln(a-1)-(a-1)] = -1 - \lim_{a\to1^+}(a-1)\ln(a-1).\end{aligned}$$

The limit on the right is an indeterminate of the form $0\cdot\pm\infty$. To remedy this, write the $(a-1)$ as $1/(a-1)$ in the denominator and then use l'Hôpital's Rule:

$$\lim_{a\to1^+}(a-1)\ln(a-1) = \lim_{a\to1^+}\frac{\ln(a-1)}{\frac{1}{(a-1)}} \overset{\infty/\infty}{=} \lim_{a\to1^+}\frac{\frac{1}{(a-1)}}{\frac{-1}{(a-1)^2}} = \lim_{a\to1^+} -(a-1) = 0.$$

Thus we have

$$\int_1^2 \ln(x-1)\,dx = -1.$$

c) $\displaystyle\int_0^\infty \frac{\arctan x}{1+x^2}\,dx$

Let $u = \arctan x$, then $du = dx/(1+x^2)$. The antiderivative of the integrand is then $\frac{1}{2}(\arctan x)^2$ and we have

$$\begin{aligned}\int_0^\infty \frac{\arctan x}{1+x^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{\arctan x}{1+x^2}\,dx = \frac{1}{2}\lim_{b\to\infty}(\arctan x)^2\Big|_0^b \\ = \frac{1}{2}\left(\lim_{b\to\infty}(\arctan b)^2 - (\arctan 0)^2\right) = \frac{1}{2}\left(\left(\frac{\pi}{2}\right)^2 - 0\right) = \frac{\pi^2}{8}.\end{aligned}$$

d) $\displaystyle\int_2^3 \frac{1}{\sqrt{3-x}}\,dx$

Note that the integrand is not defined at $x = 3$. So we have

$$\int_2^3 \frac{1}{\sqrt{3-x}}\,dx \;=\; \lim_{b\to 3^-}\int_2^b \frac{1}{\sqrt{3-x}}\,dx \;=\; \lim_{b\to 3^-}(-2\sqrt{3-x})\Big|_2^b \;=\; -2\lim_{b\to 3^-}\sqrt{3-b} + 2\sqrt{1} \;=\; -2\cdot 0 + 2 \;=\; 2.$$

3.4 (Infinite) Sequences and Series

Objectives and Concepts:

- A sequence is an infinite list of numbers and can be viewed as a function whose domain is the set of natural numbers.
- A sequence that has a finite limit is convergent, otherwise it is divergent.
- An infinite series is the sum of the terms in a sequence, and a partial sum of a series is the sum of only finitely many terms in the sequence.
- An infinite series converges when its sum is finite, otherwise it diverges.

References: OSC-2 Section 5.1 and CET Section 11.1 for Sequences, OSC-2 Section 5.2 and CET Section 11.2 for Series; also see TCMB Chapter 7.

3.4.1 Sequences

Reference: OSC-2 Section 5.1, CET §11.1; TCMB §7.2.

Definition: A **sequence** is a list of numbers, each called a **term**, written in a definite order:

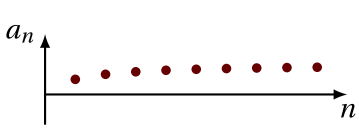
$$a_1, a_2, a_3, a_4, \ldots, a_{n-1}, a_n, a_{n+1}, \ldots$$

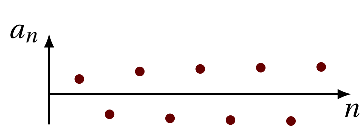
Notations used for this sequence are $\{a_1, a_2, a_3, \ldots\}$ or $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

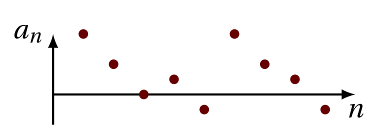
Since for every positive integer n there is a corresponding number a_n, a sequence can be viewed as a function $f(n)$ whose domain is $\mathbb{N}$, the set of natural numbers ($\mathbb{N} = \{1, 2, 3, \ldots\}$).

Example 1: Write the first four terms of the sequence $\{a_n\}$ where $a_n = \dfrac{n}{n+1}$.

The **graph of a sequence** is the graph of the function whose ordered pairs consist of individual points (n, a_n) for some integers n. Below are graphs of some different sequences.







Definition: A sequence a_n has the **limit** L if we can make terms a_n as close to L as we like by taking n sufficiently large. In this case we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty.$$

If the limit $\lim\limits_{n\to\infty} a_n$ exists and is finite, we say the sequence **converges**. Otherwise, we say the sequence **diverges**.

Example 2: Determine the limit of the sequence $\left\{\dfrac{n}{n+1}\right\}$. Is it convergent or divergent?

If we think of a sequence a_n as a function $f(n)$ with a domain of the positive integers, then we can see the connection between our definition of the limit of a sequence and our definition of the limit of a function.

Theorem: Suppose f is a function such that $f(n) = a_n$ for all positive integers n. If $\lim\limits_{x\to\infty} f(x) = L$, then the $\lim\limits_{n\to\infty} a_n = L$.

Concept Check: The sequences $a_n = 2^n$ and $b_n = \cos(n\pi)$ both diverge, but for different reasons. How are they different?

Example 3: Does $a_n = \dfrac{\ln n}{n}$ converge?

Example 4: Find $\lim\limits_{n\to\infty} \cos(1/n)$.

All of the properties of limits of continuous functions also apply to the limits of sequences.

Theorem: Suppose that a_n and b_n are convergent sequences and c is a constant. Then:

1. $\lim\limits_{n\to\infty} [a_n \pm b_n] = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$
2. $\lim\limits_{n\to\infty} [ca_n] = c \lim\limits_{n\to\infty} a_n$
3. $\lim\limits_{n\to\infty} c = c$
4. $\lim\limits_{n\to\infty} [a_n b_n] = \left(\lim\limits_{n\to\infty} a_n\right)\cdot\left(\lim\limits_{n\to\infty} b_n\right)$
5. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, provided $\lim\limits_{n\to\infty} b_n \neq 0$
6. $\lim\limits_{n\to\infty} [a_n]^p = \left[\lim\limits_{n\to\infty} a_n\right]^p$ if $p > 0$ and $a_n > 0$

Theorem: If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

Example 5: Does the sequence defined by $a_n = \dfrac{(-1)^n}{2n+1}$ converge or diverge?

Theorem: If $\lim\limits_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim\limits_{n\to\infty} f(a_n) = f(L)$.

The Squeeze Theorem: If $a_n \le b_n \le c_n$ for all $n \ge n_0$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.

Example 6: Does the sequence defined by $a_n = \dfrac{\sin(2n)}{1+\sqrt{n}}$ converge or diverge?

Definition:

- A sequence a_n is called **increasing** if $a_n < a_{n+1}$ for all $n \ge 1$.
- A sequence a_n is called **nondecreasing** if $a_n \le a_{n+1}$ for all $n \ge 1$.
- A sequence a_n is called **decreasing** if $a_n > a_{n+1}$ for all $n \ge 1$.
- A sequence a_n is called **nonincreasing** if $a_n \ge a_{n+1}$ for all $n \ge 1$.

A sequence a_n is called **monotonic** if it is either nondecreasing or nonincreasing.

Example 7: Label each sequence below as increasing, decreasing, nondecreasing, nonincreasing, or none of these.

- $0, -1, -2, -3, \ldots$
- $1, -2, 3, -4, 5, -6, \ldots$
- $0, 1, 2, 3, \ldots$
- $0, 1, 1, 2, 3, 4, \ldots$
- $3, 2, 1, 0, 0, 0, 0, \ldots$
- $0, 1, 2, 1, 2, 1, 2, \ldots$

Example 8: Show the sequence defined by $a_n = \dfrac{3n}{n+5}$ is monotonic.

Definition: A sequence a_n is called **bounded** if there are numbers m and M such that $m \le a_n \le M$ for all terms. It is called **bounded above** if there is a number M such that $a_n \le M$ for all n and it is called **bounded below** if there is a number m such that $m \le a_n$ for all n.

Monotone Convergence Theorem: If a sequence a_n is bounded and monotonic, then it converges.

Example 9: Find a formula for the general term a_n for each sequence (assuming that the patterns continue indefinitely).

a) $\left\{-\dfrac{1}{4}, \dfrac{2}{9}, -\dfrac{3}{16}, \dfrac{4}{25}, \ldots\right\}$

b) $\{-2, 5, 12, 19, \ldots\}$

c) $\{3, 6, 12, 24, 48, \ldots\}$

Note that some sequences don't have a simple defining equation. If we let a_n be the digit in the nth decimal place of π, then $\{a_n\}$ is the sequence whose first few terms are $\{1,4,1,5,9,2,\ldots\}$.

Definition: A **recursive sequence** is a sequence where the a_n term is defined in terms of $a_{n-1}, a_{n-2}, \ldots, a_2, a_1$.

Example 10: Let $a_0 = 1$, $a_1 = 1$, and define $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. Write out the first 8 terms of this sequence.

The above sequence is known as the **Fibonacci sequence** and is prevalent in many areas of mathematics and nature.

Concept Check: It is clear that the Fibonacci sequence itself diverges. Define a new sequence $\{b_n\}$ where $b_n = \dfrac{a_{n+1}}{a_n}$ for $n \geq 1$. Does $\{b_n\}$ converge?

3.4.2 Infinite Series (a.k.a. Infinite Sums)

OSC-2 Section 5.2, CET §11.2, TCMB §7.4.

If we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an **infinite series** (or just a **series**). For notation, we use

$$\sum_{n=1}^{\infty} a_n \qquad \text{or} \qquad \sum a_n.$$

The **partial sums** of the series is the sequence $\{s_n\}$ given by

$$s_1 = a_1, \qquad s_2 = a_1 + a_2, \qquad s_3 = a_1 + a_2 + a_3, \quad \ldots, \quad s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

Example 11: Find the first three partial sums of the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2-1}{n^2+1}$. Do you think the sequence of partial sums is convergent?

Definition: Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a series with partial sums $s_n = \displaystyle\sum_{i=1}^{n} a_i$. If the sequence $\{s_n\}$ converges with limit s (where s is a real number), then the series $\sum a_n$ is called **convergent** and we write

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s.$$

The number s is called the **sum** of the series. Otherwise, the series is called **divergent.**

A series is convergent if and only if the sequence of partial sums of the series converges to a finite value.

Example 12: Find the first three partial sums of the series

$$\sum_{n=1}^{\infty} 9\left(\frac{1}{10^n}\right) = 0.9 + 0.09 + 0.009 + 0.0009 + \cdots.$$

Definition: A **geometric series** is a series in which each term is a constant multiple of the one before. A **finite geometric series** has the form

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

and an **infinite geometric series** has the form

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n + \cdots.$$

The number r is called the **common ratio** of successive terms.

Theorem: The geometric series $\sum_{k=0}^{\infty} ar^k$ diverges if $|r| \geq 1$. If $|r| < 1$, the geometric series converges. Its sum is

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \qquad \text{provided } |r| < 1.$$

Concept Check: Rewrite the infinite series given in the previous example so that it is a geometric series (starting index of 0, terms of the form ar^n). Does the series converge? If so, what is its sum?

Example 13: Describe the infinite repeating decimal

$$0.\overline{41} = 0.414141414141\ldots$$

as a geometric series using sigma notation. Use the geometric series sum formula to find the sum of the series you described.

Example 14: Determine if the series $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$ converges or diverges, and if it converges, find its sum.

Example 15: The series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ is a geometric series. Write the first few terms of the series, write it in the general form of an infinite series, and determine whether or not the series converges.

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Warning: The converse is NOT true. Just because $a_n \to 0$ does not necessarily imply that $\sum a_n$ converges. If you know $\lim_{n\to\infty} a_n = 0$, you cannot conclude anything about the series $\sum a_n$.

Concept Check: The **harmonic series** defined by

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

diverges. Why?

The Test for Divergence:

- If $\lim_{n\to\infty} a_n$ does not exist, then the series $\sum a_n$ is divergent.
- If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ is divergent.

Example 16: Determine whether each of the following series converges or diverges. If it converges, find its sum.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1+3^n}{2^n}$

b) $\displaystyle\sum_{k=1}^{\infty} \frac{k(k+2)}{(k+3)^2}$

c) $\displaystyle\sum_{\ell=1}^{\infty} \frac{e^\ell}{\ell^2}$

Properties of Series: Suppose $\sum a_n$ and $\sum b_n$ are convergent series and c is a constant.

i) The series $\sum ca_n$ converges and its sum is $c\sum a_n$.

ii) The series $\sum (a_n \pm b_n)$ converges and its sum is $(\sum a_n) \pm (\sum b_n)$.

iii) Changing a finite number of terms in a series does not change whether or not it converges (but it might change its sum, if it does converge).

(†) **Telescoping Sums.** The convergence behavior of some series can be determined by examining their partial sums. One particular kind of series that can be handled this way is a **telescoping series**.

Example 17: (†) Determine whether the series $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$ converges or diverges by analyzing its partial sums.
Hint: Note that $\frac{2}{n^2+4n+3} = \frac{1}{n+1} - \frac{1}{n+3}$.

3.4 Review of Concepts

- Terms to know: sequence, convergent, divergent, monotonic, increasing, nonincreasing, decreasing, nondecreasing, series, partial sum, geometric series, harmonic series, telescoping series.
- Know how to find a general formula for a sequence with a pattern.
- Know how to determine if a sequence converges or diverges.
- Know how to analyze the partial sums of a series.
- Know how to determine if a geometric series converges or diverges.

3.4 Practice Problems

1. Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

a) $\{5, 8, 11, 14, 17, \dots\}$

b) $\{-3, 2, -4/3, 8/9, -16/27, \dots\}$

c) $\left\{\frac{1}{1}, \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{6}, \dots\right\}$

2. Is the sequence defined by $a_n = n + (-1)^n$ monotonic?

3. Determine whether each sequence converges or diverges. If it converges, find the limit.

a) $a_n = \frac{n^3}{n^3+1}$

b) $a_n = \frac{3^{n+2}}{5^n}$

c) $a_n = \frac{(-1)^n n^3}{n^3+2n^2+1}$

d) $a_n = \tan\left(\frac{2n\pi}{1+8n}\right)$

e) $a_n = \frac{(-3)^n}{n!}$

4. Express the number $1.5\overline{342}$ as a rational number.

5. Determine whether each series converges or diverges. If it converges and it is possible, find its sum.

a) (†) $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$

b) $\sum_{n=1}^{\infty} \sqrt[n]{2}$

c) $2 + \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots$

3.4 Exercises

1. Determine whether each sequence converges or diverges. If it converges, find the limit.

a) (†) $a_n = n - \sqrt{n^2-1}$ b) $a_n = \dfrac{(-1)^n}{0.9^n}$ c) $a_n = \dfrac{n^2+4n+1}{\sqrt{4n^4+1}}$

2. Determine whether each series converges or diverges. If it converges and it is possible, find its sum.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{3^{n+1}}$

b) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n 5^{3-n}$

c) (†) $\displaystyle\sum_{n=1}^{\infty} \left(\arcsin\left(\frac{1}{n}\right) - \arcsin\left(\frac{1}{n+1}\right)\right)$

d) (†) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+4)}$

e) (†) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{16n^2+8n-3}$

f) $1 + \dfrac{1}{\pi} + \dfrac{1}{\pi^2} + \dfrac{1}{\pi^3} + \cdots$

3. Give an example of a sequence that is bounded but that does not converge. Give an example of a sequence that is monotonic but that does not converge. Can you give an example of a sequence that converges but is not monotonic or bounded?

4. (†) Let the Fibonacci sequence be denoted by $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. Show that

$$\frac{1}{f_{n-1}f_{n+1}} = \frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}}.$$

Then use this fact to prove that

$$\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1}f_{n+1}} = 2.$$

5. The energy of a quantum-mechanical harmonic oscillator is given by $\varepsilon_n = \left(n + \dfrac{1}{2}\right) h\nu$, $n = 0, 1, 2, \ldots$, where h is the Planck constant and ν is the fundamental frequency of the oscillator. The average vibrational energy of a harmonic oscillator is given by

$$\varepsilon_{\text{vib}} = \left(1 - e^{-h\nu/(k_B T)}\right) \sum_{n=0}^{\infty} \varepsilon_n e^{-nh\nu/(k_B T)},$$

where k_B is the Boltzmann constant and T is the kelvin temperature. Show that

$$\varepsilon_{\text{vib}} = \frac{h\nu}{2} + \frac{h\nu e^{-h\nu/(k_B T)}}{1 - e^{-h\nu/(k_B T)}}.$$

3.4 Answers to Practice Problems

1. Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

a) $\{5, 8, 11, 14, 17, \ldots\}$

Note that all of the terms are increasing by 3. Thus we have $a_n = 2 + 3n$, $n \geq 1$.

b) $\{-3, 2, -4/3, 8/9, -16/27, \ldots\}$

We see that the terms are alternating in sign, so there is a factor of $(-1)^n$ in the formula. Also, notice that the denominators seem to be powers of 3. If you look at the first term, note it is $-1 \cdot 3^{-1}$ for $n = 1$ and after that, the exponent is $n - 2$. The second term is $2 \cdot 3^0$. But the $4, 8, 16$ in subsequent terms means there's a power of two with exponent $n - 1$. The general formula is $a_n = (-1)^n \dfrac{2^{n-1}}{3^{n-2}}$, $n \geq 1$.

c) $\left\{\frac{1}{1}, \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{6}, \ldots\right\}$

It almost looks like we have two separate sequences. Assuming our starting index is 1,if you look at the odd-indexed terms, we simply have $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ and the even-indexed terms are $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$. For the odd ones, since n is odd, it can be written as $n = 2k + 1$. For the even ones, note that when $n = 2$ we need the denominator to be 3, and when $n = 4$ we need it to be 4, and so on. Then we have

$$a_n = \begin{cases} \frac{1}{k} & \text{if } n = 2k - 1, \\ \frac{1}{k+2} & \text{if } n = 2k. \end{cases}$$

2. The sequence defined by $a_n = n + (-1)^n$ is not monotonic as

$$\{a_n\} = 0, 3, 2, 5, 4, 7, 6, 9, \ldots.$$

3. Determine whether each sequence converges or diverges. If it converges, find the limit.

a) $a_n = \dfrac{n^3}{n^3 + 1}$

We have

$$\lim_{n\to\infty} \frac{n^3}{n^3 + 1} = \lim_{n\to\infty} \frac{1}{1 + \frac{1}{n^3}} = \frac{1}{1 + 0} = 1.$$

b) $a_n = \dfrac{3^{n+2}}{5^n}$

Note that $\dfrac{3^{n+2}}{5^n} = 3 \cdot \dfrac{3^n}{5^n} = 3\left(\dfrac{3}{5}\right)^n$. As $n \to \infty$, $\left(\dfrac{3}{5}\right)^n \to 0$, so the sequence converges and its limit is 0.

c) $a_n = \dfrac{(-1)^n n^3}{n^3 + 2n^2 + 1}$

We can use the absolute value theorem and divide everything by n^3:

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^3}{n^3 + 2n^2 + 1} = \lim_{n\to\infty} \frac{1}{1 + \frac{2}{n} + \frac{1}{n^3}} = 1.$$

However, this doesn't tell us anything as the limit of $|a_n|$ did not turn out to be 0. In fact, if you look at only the odd-indexed terms as a sequence, those terms converge to -1, while the even-indexed terms converge to 1. Thus the sequence must diverge.

d) $a_n = \tan\left(\dfrac{2n\pi}{1+8n}\right)$

Now as $n \to \infty$, we have $\dfrac{2n\pi}{1+8n} = \dfrac{2\pi}{\frac{1}{n}+8} \to \dfrac{\pi}{4}$. Thus $a_n \to \tan(\pi/4) = 1$.

e) $a_n = \dfrac{(-3)^n}{n!}$

First we examine $|a_n|$. Note that, if $n \geq 2$,

$$\frac{3^n}{n!} = \frac{3}{1}\cdot\frac{3}{2}\cdot\frac{3}{3}\cdot\frac{3}{4}\cdots\frac{3}{n-1}\cdot\frac{3}{n} \leq \frac{3}{1}\cdot\frac{3}{2}\cdot\frac{3}{n} = \frac{27}{2n},$$

as the product of the terms

$$\frac{3}{3}\cdot\frac{3}{4}\cdots\frac{3}{n-1} \leq 1.$$

Now we know that $|a_n| > 0$, and we also know that $\dfrac{27}{2n} \to 0$ as $n \to \infty$. Thus the Squeeze Theorem shows that $|a_n| \to 0$, which implies $a_n \to 0$.

4. Express the number $1.5\overline{342}$ as a rational number.

Now

$$\begin{aligned}
1.5\overline{342} &= 1.5342342342342\ldots \\
&= 1.5 + .0342 + .0000342 + .0000000342 + \cdots \\
&= \frac{3}{2} + \frac{342}{10^4} + \frac{342}{10^7} + \frac{342}{10^{10}} + \cdots \\
&= \frac{3}{2} + \frac{342}{10}\left(\frac{1}{10^3} + \frac{1}{10^6} + \frac{1}{10^9} + \cdots\right) \\
&= \frac{3}{2} + \frac{342}{10}\sum_{k=1}^{\infty}\frac{1}{1000^k} \\
&= \frac{3}{2} + \frac{342}{10}\left(\frac{-1}{1} + \sum_{k=0}^{\infty}\frac{1}{1000^k}\right) \\
&= \frac{3}{2} + \frac{342}{10}\left(-1 + \frac{1}{1-\frac{1}{1000}}\right) \\
&= \frac{3}{2} + \frac{342}{10}\left(-1 + \frac{1000}{999}\right) \\
&= \frac{3}{2} + \frac{342}{10}\cdot\frac{1}{999} = \frac{14985}{9990} + \frac{342}{9990} = \frac{15327}{9990}.
\end{aligned}$$

5. Determine whether each series converges or diverges. If it converges and it is possible, find its sum.

a) $\displaystyle\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$

Note that

$$\sum_{n=1}^{\infty} \ln \frac{n}{n+1} = \sum_{n=1}^{\infty} (\ln n - \ln(n+1)).$$

Examining the partial sums, we see that

$$s_1 = \ln 1 - \ln 2, \qquad s_2 = \ln 1 - \ln 2 + \ln 2 - \ln 3, \qquad s_3 = \ln 1 - \ln 2 + \ln 2 - \ln 3 + \ln 3 - \ln 4,$$

$$s_n = \ln 1 - \ln(n+1) = -\ln(n+1).$$

Since

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} -\ln(n+1) = -\infty,$$

we have that the series diverges.

b) $\displaystyle\sum_{n=1}^{\infty} \sqrt[n]{2}.$

Note that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^{1/n} = 2^0 = 1.$$

Since the limit of the terms in the series is not 0, the series diverges by the Test for Divergence.

c) $2 + \dfrac{1}{2} + \dfrac{1}{8} + \dfrac{1}{32} + \cdots$

This is a geometric series. Note that

$$\begin{aligned}
2 + \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots &= 2 + \frac{1}{2^1} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots \\
&= 2 + 2\frac{1}{4^1} + 2\frac{1}{4^2} + 2\frac{1}{4^3} + \cdots \\
&= 2\left(\frac{1}{4^0} + \frac{1}{4^1} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right) \\
&= 2\sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \\
&= 2 \cdot \frac{1}{1-\frac{1}{4}} = 2 \cdot \frac{4}{3} = \frac{8}{3}.
\end{aligned}$$

3.5 Tests for Convergence or Divergence of Series (Infinite Sums)

Objectives and Concepts:

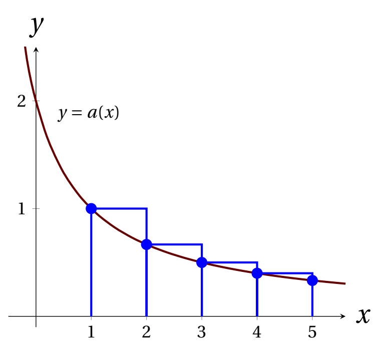
- Given an infinite series, several techniques can be applied to determine whether or not the series converges.
- A series with both positive and negative terms may be absolutely convergent, conditionally convergent, or divergent.

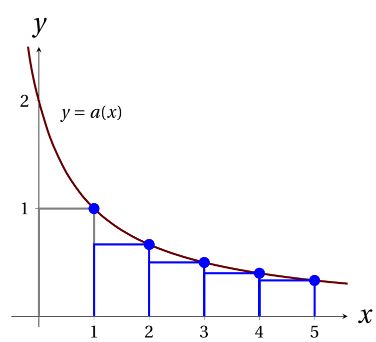
References: OSC-2 Chapter 5, §§3 to 6; CET Chapter 11 §§3 to 7; TCMB §7.5.

3.5.1 (†) The Integral Test

References: OSC-2 §5.3, CET §11.3; TCMB §7.5.

Recall that integrals themselves are actually the limit of a sum. If we construct an approximation to the integral $\int_1^\infty f(x)\,dx$ using rectangles of width 1 and a height given by $a_n = f(n)$, looking at the figure on the left, we see that the area of the rectangles is larger than the integral. On the other hand, if we use the right endpoint of each interval as the height of the rectangles, we see that the integral is greater than the area of the rectangles.





Thus we have

$$\int_1^\infty f(x)\,dx \le \sum_{n=1}^\infty a_n \qquad \text{and} \qquad \int_1^\infty f(x)\,dx + a_1 \ge \sum_{n=1}^\infty a_n.$$

(The single term a_1 cannot affect the convergence behavior of the series. Thus, we see an equivalence

$$\sum_{n=1}^\infty a_n \le a_1 + \int_1^\infty f(x)\,dx \le a_1 + \sum_{n=1}^\infty a_n,$$

in the area under the curve $y = f(x)$ and the sum of the terms of the series, which means the area under the curve and the series either both converge or both diverge.

(†) The Integral Test: Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words,

i) If $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

ii) If $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Warning: The sum of the series is not necessarily the value of the integral! This test will only give you the convergence behavior of the series.

Example 1: Use the Integral Test to show that the harmonic series diverges.

Convergence of p-Series: The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges when $p \leq 1$.

Important Note: The Integral Test requires four things: $f(x) > 0$ for $x \geq 1$, $f(x)$ continuous for $x \geq 1$, $f(n) = a_n$ for $n \geq 1$, and $f(x)$ decreasing for $x \geq 1$.

It may be easy to check the first three, but checking the last condition can be tricky. One method is to find $f'(x)$ and demonstrate $f'(x) < 0$ for $x \geq 1$.

Example 2: Determine if the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

3.5.2 (†) Comparison Tests

References: OSC-2 §5.4, CET §11.4, TCMB §7.5.

Recall that we have a Direct Comparison Test for Improper Integrals - an analogous test exists for series.

(†) Direct Comparison Test: Let $\{a_n\}$ and $\{b_n\}$ be positive sequences where $a_n \leq b_n$ for all $n \geq N$, for some $N \geq 1$.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

The Direct Comparison Test is most useful when there is a straightforward comparison.

Example 3: Does the series $\sum_{n=1}^{\infty} \frac{n^3}{2n^4-1}$ converge or diverge? Note that as $n \to \infty$, $\frac{n^3}{2n^4-1}$ behaves like $\frac{1}{2n}$. Now we know $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges (why?), thus, since

$$\frac{1}{2n} = \frac{n^3}{2n^4} \leq \frac{n^3}{2n^4-1},$$

(because when the denominator becomes smaller the whole fraction becomes larger) we can conclude that the series $\sum_{n=1}^{\infty} \frac{n^3}{2n^4-1}$ also diverges.

The above example demonstrates that *some* series are easy to compare to a series whose convergence behavior is known. However, this is not always so easy. For example, what if the denominator was $2n^4+1$ instead of $2n^4-1$? Then the above procedure would not work. Another related test can be used when it is difficult to bound one sequence above or below by one whose convergence behavior is known.

(†) Limit Comparison Test: Let $\{a_n\}$ and $\{b_n\}$ be positive sequences.

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where L is a positive real number, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then if $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then if $\sum_{n=1}^{\infty} b_n$ diverges, then so does $\sum_{n=1}^{\infty} a_n$.

Example 4: Does the series $\sum_{n=1}^{\infty} \frac{n^3}{2n^4+3n}$ converge or diverge? Notice that the inequality we used in the prior example does not necessarily hold, as our denominator becomes larger when you add the $3n$ term (as opposed to becoming smaller when subtracting 1). However, we have that

$$\lim_{n\to\infty} \frac{\frac{n^3}{2n^4+3n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^4}{2n^4+3n} = \lim_{n\to\infty} \frac{1}{2+\frac{3}{n^3}} = \frac{1}{2},$$

so the series in question behaves the same way that the harmonic series does. Thus $\sum_{n=1}^{\infty} \frac{n^3}{2n^4+3n}$ diverges.

3.5.3 The Ratio [and (†)Root] Tests

References: OSC-2 §5.6, CET §11.6; TCMB §7.5

Recall that the harmonic series diverges even though the terms of the series go to 0 because the rate of decrease of the terms is just slightly too slow. The Ratio and Root Tests are tests designed to determine if the terms in a positive series go to 0 "fast enough."

The Ratio Test: Let $\{a_n\}$ be a positive sequence where $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \rho$.

1. If $\rho < 1$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.
2. If $\rho > 1$ or $\rho = \infty$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.
3. If $\rho = 1$, the Ratio Test is inconclusive.

(†) The Root Test: Let $\{a_n\}$ be a positive sequence, and let $\lim\limits_{n\to\infty} (a_n)^{1/n} = \rho$.

1. If $\rho < 1$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.
2. If $\rho > 1$ or $\rho = \infty$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.
3. If $\rho = 1$, the Root Test is inconclusive.

We will see many applications of the Ratio Test when we study Power Series in the next section, and it is particularly useful when dealing with the factorial function $n! = 1\cdot 2\cdot 3\cdots(n-1)\cdot n$.

3.5.4 Absolute Convergence and Alternating Series

References: OSC-2 §5.5; CET §11.5 for Alternating Series; CET §11.6 for Absolute Convergence.

So far all of our tests are strictly for series with positive terms. What if some of the terms are negative? We need a new definition.

Definition:

- A series $\sum a_n$ is called **absolutely convergent** if the series $\sum |a_n|$ converges.
- A series $\sum a_n$ is called **conditionally convergent** if $\sum a_n$ converges and $\sum |a_n|$ diverges.
- A series $\sum a_n$ is called **divergent** if both $\sum a_n$ and $\sum |a_n|$ diverge.

Note that if a series is absolutely convergent, then it is also convergent (but the converse is not true). One common type of series with some negative terms is an alternating series.

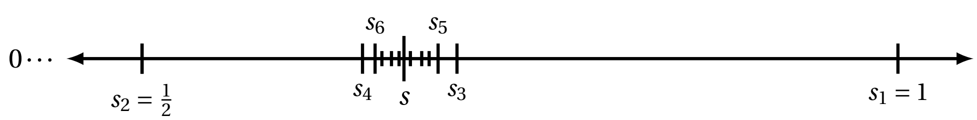
Definition: Let $\{a_n\}$ be a positive sequence. An **alternating series** is a series of either the form

$$\sum_{n=1}^{\infty}(-1)^n a_n \qquad \text{or} \qquad \sum_{n=1}^{\infty}(-1)^{n+1} a_n.$$

We have seen before that the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. Does the **alternating harmonic series**

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

converge or diverge? It's not immediately clear - let's look at the partial sums on a number line:



If you examine the odd-indexed sequence of partial sums $\{s_1, s_3, s_5, \ldots\}$ of the alternating harmonic series you will notice that it is a decreasing sequence that is bounded below by s. Similarly, the even-indexed sequence of partial sums is increasing and bounded above by s. This leads to an idea for a test for alternating series:

Alternating Series Test: Let $\{a_n\}$ be a positive, decreasing sequence where $\lim_{n\to\infty} a_n = 0$. Then

$$\sum_{n=1}^{\infty}(-1)^n a_n \qquad \text{and} \qquad \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$

are convergent.

Example 5: Does the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$ converge absolutely, converge conditionally, or diverge?

Note that $a_n = \frac{1}{\sqrt{n}}$ is a positive, decreasing sequence that satisfies $\lim_{n\to\infty} a_n = 0$. Thus the alternating series itself is convergent. However, we have that

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

is a p-series with $p = 1/2 \le 1$, which diverges. Thus the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$ is conditionally convergent.

3.5.5 Procedure for Testing Series

We have seen several tests to help us determine if a series converges or diverges. The tricky part is knowing which test to use on which series. In this respect, testing series is similar to integrating functions. It is not a good idea to make a list of the tests and apply them in a specific order. Instead, the main strategy is to classify the series according to its form and to know which test to try with which form.

The following suggestions may be helpful:

1. If the series has form $\sum ar^{n-1}$ or $\sum ar^n$, it is a Geometric Series:
   It converges if $|r| < 1$ and diverges if $|r| \geq 1$.
2. If the series has form $\sum \dfrac{1}{n^p}$, it is a p-series: It converges if $p > 1$ and diverges if $p \leq 1$.
3. If you can see that $\lim\limits_{n\to\infty} a_n \neq 0$, then the Test for Divergence can be used.
4. If the series has form $\sum(-1)^{n-1}b_n$ or $\sum(-1)^n b_n$, then the Alternating Series Test should be used.
   As a special case, the Alternating p-series $\sum \dfrac{(-1)^{n-1}}{n^p}$ converges if $p > 0$.
5. If a_n contains factors like r^n and/or factorials — along with possibly powers of n or other polynomial factors — then the Ratio Test may be useful.
   Note: This is the end of the most commonly useful series convergence tests; the ones covered most fully in this course.
6. (†) If a_n is a rational or algebraic function, then try a Comparison Test (Direct or Limit), usually with a p-series. Remember that the comparison tests require all a_n to be positive. If $\sum a_n$ has some negative terms, then we can apply the comparison tests to $\sum |a_n|$ and test for absolute convergence instead.
7. (†) If $a_n = f(n) > 0$, where $\int_1^\infty f(x)\,dx$ is easily evaluated and $f(x)$ is also decreasing, then the Integral Test is effective.
8. (†) If a_n is of the form $(b_n)^n$ (and the Ratio Test does not help), then the Root Test may be useful.

3.5 Review of Concepts

- Terms to know: (†) integral test, p-series, (†) limit comparison test, (†) direct comparison test, ratio test, (†) root test, absolute convergence, conditional convergence, alternating series, alternating series test.
- Know how to apply the tests for series to determine the convergence behavior of a particular series.

3.5 Practice Problems

1. Determine if the following series are absolutely convergent, conditionally convergent, or divergent:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ (†)

b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$

d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{1+2\sqrt{n}}$

f) $\displaystyle\sum_{n=0}^{\infty} n\left(\frac{2}{3}\right)^n$

3.5 Exercises

1. The vibrational partition function of a diatomic molecule is given by the series

$$q(T) = \sum_{n=0}^{\infty} e^{-\left(n+\frac{1}{2}\right)h\nu/(k_B T)},$$

where h is the Planck constant, ν is the frequency of the oscillator, k_B is the Boltzmann constant, and T is the kelvin temperature. Does this geometric series converge, and if it converges, what is its sum?

2. Determine if the following series are absolutely convergent, conditionally convergent, or divergent:

a) $\displaystyle\sum_{n=1}^{\infty} n e^{-n^2}$ (†)

b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(4n)}{n^{3/2}}$ (†)

d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3}$

e) $\displaystyle\sum_{n=4}^{\infty} \frac{2}{n^2-10}$ (†)

f) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^3}$

g) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$ (†)

h) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n^2)}{n^2}$ (†)

3.5 Answers to Practice Problems

1. Determine if the following series are absolutely convergent, conditionally convergent, or divergent:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$

This looks like a good candidate for a Limit Comparison Test. Note that the dominant power of n in the denominator is $n^{5/2}$, so it might be good to choose the sequence $b_n = n^2/n^{5/2} = 1/n^{1/2}$ for comparison. With $a_n = \dfrac{2n^2+3n}{\sqrt{5+n^5}}$, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} \cdot \frac{n^{1/2}}{1} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{\sqrt{5+n^5}}$$
$$= \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{\sqrt{5+n^5}} \cdot \frac{\frac{1}{n^{5/2}}}{\frac{1}{n^{5/2}}} = \lim_{n\to\infty} \frac{2+\frac{3}{n}}{\sqrt{\frac{5}{n^5}+1}} = \frac{2+0}{\sqrt{0+1}} = 2.$$

Since $\sum b_n$ is divergent and $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a positive real number, we have that $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ is also divergent.

b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

We will analyze $\displaystyle\int_2^{\infty} \frac{1}{x\ln x}\,dx$. Let $u = \ln x$, then $du = dx/x$. Note that if $x = 2$, $u = \ln 2$, and if $x \to \infty$, $u \to \infty$ as well. The integral is then

$$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \int_{\ln 2}^{\infty} \frac{1}{u}\,du = \lim_{b\to\infty} \ln u\Big|_{\ln 2}^{b} = \lim_{n\to\infty} \ln b - \ln(\ln 2) = \infty.$$

Since the integral diverges, the series also diverges.

c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$

Note that for $n \geq 1$, $0 < \pi/n \leq \pi$, and sin is always nonnegative in the upper half of the unit circle. In fact, for $n \geq 2$, $\sin(\pi/n) > 0$. To use the Alternating Series Test, we must show that $\lim_{n\to\infty} \sin(\pi/n) = 0$ and that $\sin(\pi/n)$ is a decreasing sequence. We have

$$\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \lim_{n\to\infty} \sin(0) = 0.$$

To show $\sin(\pi/n)$ is decreasing, note that

$$\frac{d}{dx}\left(\sin\left(\frac{\pi}{x}\right)\right) = \cos\left(\frac{\pi}{x}\right) \cdot \left(\frac{-\pi}{x^2}\right),$$

which is always negative if $x > 2$. Thus $\sin(\pi/n)$ is a decreasing sequence for $n > 2$, and therefore the series converges by the Alternating Series Test.

Does the series converge absolutely, i.e., is $\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{n}\right)$ convergent? One important concept is that, for values of x near 0, $\sin x \approx x$ (why?). So it seems that $\sin(\pi/n) \approx \pi/n$, which would be a constant multiple of the sequence $1/n$. So let's use the limit comparison test with $b_n = 1/n$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{1}{n}} \stackrel{0/0}{=} \lim_{n\to\infty} \frac{\frac{-\pi}{n^2}\cos\left(\frac{\pi}{n}\right)}{\frac{-1}{n^2}} \lim_{n\to\infty} \pi\cos\left(\frac{\pi}{n}\right) = \pi\cdot 1 = \pi.$$

Since $\sum 1/n$ diverges and the limit is a positive constant, the series $\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{n}\right)$ diverges. Therefore the original series $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$ is conditionally convergent.

d) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

As there is a factorial involved, let's try the Ratio Test. We have

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)n!}\cdot\frac{n!}{n^n} &= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} \\ &= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e.\end{aligned}$$

Since $e > 1$, the series diverges by the Ratio Test.

e) $\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{1+2\sqrt{n}}$

This is an alternating series, so we check the limit of the sequence $\frac{\sqrt{n}}{1+2\sqrt{n}}$. Now

$$\lim_{n\to\infty} \frac{\sqrt{n}}{1+2\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt{n}}+2} = \frac{1}{2},$$

which implies that the limit

$$\lim_{n\to\infty} \frac{(-1)^n\sqrt{n}}{1+2\sqrt{n}}$$

does not exist. Thus the series diverges by the Test for Divergence.

f) $\sum_{n=0}^{\infty} n\left(\frac{2}{3}\right)^n$

Using the Ratio Test, we have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)\left(\frac{2}{3}\right)^{n+1}}{n\left(\frac{2}{3}\right)^n} = \lim_{n\to\infty} \frac{n+1}{n}\left(\frac{2}{3}\right) = 1\cdot\left(\frac{2}{3}\right) = \frac{2}{3}.$$

Since the limit is less than 1, the series converges by the Ratio Test.

Summary of Tests for Convergence or Divergence of Series

These are in the order that I suggest trying them in, starting with the ones that are easiest to use when they work.

Geometric Series: The series $\sum_{n=0}^{\infty} ar^n$ diverges if $|r| \geq 1$ and converges if $|r| < 1$ with sum $\dfrac{a}{1-r}$.

p**-Series:** The series $\sum_{n=0}^{\infty} \dfrac{1}{n^p}$ diverges if $p \leq 1$ and converges if $p > 1$.

Test for Divergence: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. If $\lim_{n\to\infty} a_n = 0$, there is no conclusion.

Alternating Series Test: If $\sum(-1)^n a_n$ is an alternating series with a_n positive, decreasing, and $\lim_{n\to\infty} a_n = 0$, then the alternating series converges.

Ratio Test: Let $\sum a_n$ be a series and let $\rho = \lim_{n\to\infty} |a_{n+1}/a_n|$.
If $\rho < 1$, the series converges absolutely; if $\rho > 1$ the series diverges; if $\rho = 1$ this test gives no conclusion.

(†) **Limit Comparison Test:** Let $\sum a_n$ and $\sum b_n$ be two series with positive terms. Let $\rho = \lim_{n\to\infty} \dfrac{a_n}{b_n}$. If $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge. If $L = 0$ and $\sum b_n$ converges, $\sum a_n$ converges too. If $L = \infty$ and $\sum b_n$ diverges, $\sum a_n$ diverges too.

(†) **Direct Comparison Test:** Let $\sum a_n$ and $\sum b_n$ be series with non-negative terms such that $a_n \leq b_n$. If $\sum b_n$ converges, then $\sum a_n$ converges. If $\sum a_n$ diverges, then $\sum b_n$ diverges.

(†) **Integral Test:** Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms such that $a_n = f(n)$ for a continuous function $f(x)$ which is decreasing for x sufficiently large. Then $\sum a_n$ and $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge.

(†) **Root Test:** [*Note: This mostly applies in the same cases as the Ratio Test above.*]
Let $\sum a_n$ be a series and let $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.
If $\rho < 1$, the series converges absolutely; if $\rho > 1$ the series diverges; if $\rho = 1$ this test gives no conclusion.

3.6 Power Series and Taylor Series

Objectives and Concepts:

- A power series is a function defined in terms of an infinite series. A power series has an interval upon which it converges.
- If f is a continuous function with derivatives of all orders, then it has a Taylor Series representation.

References: OSC-2 Chapter 6, CET §§11.8–10, TCMB §§7.6–8.

3.6.1 Power Series

References: OSC-2 §6.1, CET §§11.8,9; TCMB §§7.6,8.

We've seen that an infinite geometric series is a series of the form $\sum_{n=0}^{\infty} ar^n$ and that this series converges to the sum $\frac{a}{1-r}$ provided $|r| < 1$. When $a = 1$ we have $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots + r^n + \cdots = \frac{1}{1-r}$.

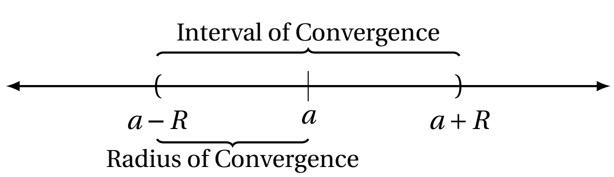
Replacing r with the variable x, this series becomes a new representation of a common function:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots = \frac{1}{1-x}, \qquad \text{provided } |x| < 1.$$

Definition: A **power series** in $(x-a)$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where x is a variable and a and c_k are real numbers. The c_k's are the **coefficients** of the power series and a is the **center** of the power series. The set of values of x for which the series converges is called the **interval of convergence**. The **radius of convergence** of the power series, denoted R, is the distance from the center of the series to the boundary of the interval of convergence.



The power series is also called a **power series centered** at a or a **power series about** a.

Theorem: For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

a) The series converges only when $x = a$.

b) The series converges for all x.

c) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

One of the most common tests used in determining if and where a power series is convergent is the Ratio Test, as the limit it utilizes often has many factors that can cancel. To find the radius and interval of convergence, we must use the Ratio Test within the context of absolute convergence, as some of the terms will be negative when $x < a$:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

One important note about convergence of power series - often the behavior of the series at the endpoints of the interval of convergence must be checked individually.

Example 1: Determine the radius of convergence and the interval of convergence of the power series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}$.

We can use a power series to define a function:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

where the domain of f is a subset of the interval of convergence of the power series. One can apply calculus techniques to such functions; in particular, we can find derivatives and antiderivatives.

Theorem: Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ be a function defined by a power series, with radius of convergence R.

1. $f(x)$ is continuous and differentiable on $(a-R, a+R)$.
2. $f'(x) = \sum_{n=1}^{\infty} c_n \cdot n \cdot (x-a)^{n-1}$, with radius of convergence R.
3. $\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$, with radius of convergence R.

A few notes:

1. The theorem states that differentiation and integration do not change the radius of convergence. It does not state anything about the *interval* of convergence. They are not always the same.
2. Notice how the summation for $f'(x)$ starts with $n = 1$. This is because the constant term c_0 of $f(x)$ goes to 0.
3. Differentiation and integration are simply calculated term–by–term using the Power Rules.

Example 2: Find the interval of convergence of $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, then find $f'(x)$ and $\int f(x)\,dx$.

Theorem: Suppose the power series $\sum c_n x^n$ and $\sum d_n x^n$ converge to $f(x)$ and $g(x)$, respectively, on an interval I. Then:

1. **Sum and Difference:** The power series $\sum (c_n \pm d_n)\, x^n$ converges to $f(x) \pm g(x)$ on I.
2. **Multiplication by a Power:** Suppose m is an integer so that $n + m \geq 0$ for all terms of the power series $x^m \sum c_n x^n = \sum c_n x^{n+m}$. This series converges to $x^m f(x)$ for all $x \neq 0$ in I. When $x = 0$, the series converges to $\lim_{x \to 0} x^m f(x)$.
3. **Composition:** If $h(x) = bx^m$, where m is a positive integer and b is a nonzero real number, the power series $\sum c_n (h(x))^n$ converges to the composite function $f(h(x))$, for all x such that $h(x)$ is in I.

Example 3: Find power series representations for $f(x) = \frac{3x^2}{1-4x}$ and $g(x) = \frac{1}{1+x^2}$.

Example 4: Find a power series representation for $f(x) = \tan^{-1} x$.

Concept Check: Given your knowledge of the function $f(x) = \tan^{-1} x$, find a series representation for the number π.

3.6.2 Taylor and Maclaurin Series

References: OSC-2 §6.3; CET §§11.10,11; TCMB §§7.6,7.

The two series we found for e^x and $\tan^{-1} x$ above show that there are power series representations for many complicated functions. When a function has infinitely many continuous derivatives, we can find its **Taylor Series** representation.

Definition: Let $f(x)$ have derivatives of all orders at $x = c$.

1. The **Taylor Series** of $f(x)$, centered at c is
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$
2. Setting $c = 0$ gives the **Maclaurin Series** of $f(x)$:
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Example 5: To see why the Maclaurin Series of e^x was what we found above, note that every derivative of

$f(x) = e^x$ is $f^{(n)}(x) = e^x$, and we have $f^{(n)}(0) = e^0 = 1$. Thus the Maclaurin Series for e^x is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Example 6: Find the Maclaurin Series of $\cos x$.

Example 7: Verify that the nth derivative of $\ln x$ is $(-1)^{n+1}\dfrac{(n-1)!}{x^n}$. Use this to find the Taylor Series for $\ln x$ centered at $c = 1$.

So when is a function actually equal to its power/Taylor/Maclaurin series representation?

Taylor's Theorem:

1. Let f be a function whose $n+1^{\text{st}}$ derivative exists on an interval I and let c be in I. Then, for each x in I, there exists z_x between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x),$$

where $R_n(x) = \dfrac{f^{(n+1)}(z_x)}{(n+1)!}(x-c)^{(n+1)}$.

2. $|R_n(x)| \leq \dfrac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|(x-c)^{(n+1)}\right|$

Taylor's Theorem establishes bounds for the error in using the nth degree **Taylor polynomial** as an approximation to the function $f(x)$. Taylor polynomials are incredibly useful in a wide variety of applications. If this error vanishes as the number of terms in the polynomial goes to infinity (i.e., if we use the entirety of the Taylor series), then we can conclude that the function is actually equal to its Taylor series on its interval of convergence.

Theorem: Let $f(x)$ have derivatives of all orders at $x = c$, let $R_n(x)$ be as stated in Taylor's Theorem, and let I be an interval on which the Taylor series of $f(x)$ converges. If $\lim\limits_{n\to\infty} R_n(x) = 0$ for all x in I containing c, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \quad \text{on } I.$$

Important Taylor and Maclaurin Series:

Function and Series	First Few Terms	Interval of Convergence
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$(-\infty, \infty)$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$(-\infty, \infty)$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$(-\infty, \infty)$
$\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$	$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$	$(0, 2]$
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + x^3 + \cdots$	$(-1, 1)$
$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!} x^n$	$1 + kx + \frac{k(k-1)}{2!} x^2 + \cdots$	$(-1, 1)$[a]
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$[-1, 1]$

[a]Convergence at $x = \pm 1$ depends on the value of k.

Example 8: What is the Maclaurin Series for e^{ix} where $i = \sqrt{-1}$?

Although it looks intimidating at first, the product of two power series can be computed as well. If power series $\sum a_n x^n$ and $\sum b_n x^n$ converge to $f(x)$ and $g(x)$, respectively, on an interval I, then

$$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} \big(a_0 b_n + a_1 b_{n-1} + \ldots a_n b_0\big) x^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} x^n.$$

3.6 Review of Concepts

- Terms to know: power series, interval of convergence, radius of convergence, Taylor series, Maclaurin series, Taylor polynomial
- Know how to find the radius and interval of convergence of a given power series.
- Know how to find new power series based on manipulation, differentiation, or integration of a given power series.
- Know how to find the Taylor or Maclaurin series of a given function.

3.6 Practice Problems

1. Determine the radius of convergence and the interval of convergence for the following power series:

a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(x-2)^n}{4^n}$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$ c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n(n+1)}$

2. Using the known Taylor and Maclaurin series, find series representations of the following functions:

a) $\dfrac{x^4}{2-x}$ b) $\dfrac{x}{(1-x)^2}$ c) $\ln\left(\dfrac{1+x}{1-x}\right)$

3. Evaluate $\displaystyle\int \arctan(x^2)\,dx$ as a power series.

3.6 Exercises

1. Find the radius and interval of convergence for each of the following power series.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-2x)^n}{\sqrt{n+3}}$

2. Find power series representations for the following functions and determine the radius and interval of convergence for each.

a) $\dfrac{1}{2+x}$ b) $\dfrac{1}{1+5x^2}$ c) $\dfrac{1}{x^2+2x}$

3. Find the first four terms of the Maclaurin series for $f(x) = \sqrt{1+x}$. Can you come up with a closed-form (i.e. summation) representation of the series?

4. Describe (in words) two ways (other than using the definition) to obtain the Maclaurin series for $f(x) = \cos^2 x$, and compute the first four terms of the series.

5. The indefinite integral $\displaystyle\int e^{-x^2}\,dx$ cannot be evaluated using traditional means. Find the Maclaurin series for the function $f(x) = e^{-x^2}$ and use it to derive a series representation of $\displaystyle\int f(x)\,dx$.

6. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\pi^{2n+1}}{4^{2n+1}(2n+1)!}$. (Hint: this is a known power series evaluated at a particular value of x.)

3.6 Answers to Practice Problems

1. Determine the radius of convergence and the interval of convergence for the following power series:

a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{4^n}$

Using the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{x-2}{4}\right| = \frac{|x-2|}{4}.$$

Now this limit is less than 1 when $|x-2| < 4$, so the radius of convergence is 4. The interval of convergence contains $-2 < x < 6$, but we need to check the endpoints. When $x = -2$, we have that

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n (-2-2)^n}{4^n} = \sum_{n=0}^{\infty} \frac{(4)^n}{4^n} = \sum_{n=0}^{\infty} 1,$$

which diverges by the Test for Divergence, so the interval does not contain $x = -2$. At $x = 6$, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n (6-2)^n}{4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{4^n} = \sum_{n=0}^{\infty} (-1)^n,$$

which also diverges by the Test for Divergence, so the interval does not contain $x = 6$. Thus the interval of convergence is $(-2, 6)$.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$

Using the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(x-1)^n}\right| = \lim_{n\to\infty} |x-1|\sqrt{\frac{n+1}{n}} = |x-1|.$$

So the power series converges when $|x-1| < 1$, which implies that the radius of convergence is 1. We need to check the endpoints $x = 0$ and $x = 2$. If $x = 0$, we have

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}},$$

which converges by the Alternating Series Test. If $x = 2$, then

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(2-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},$$

which diverges (p-series). Thus the interval of convergence is $[0, 2)$.

c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n (n+1)}$

Using the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{3^{n+1}(n+2)}\cdot\frac{3^n(n+1)}{(-1)^n x^n}\right| = \lim_{n\to\infty}\left|\frac{x(n+1)}{3(n+2)}\right| = \frac{|x|}{3}.$$

Thus the power series converges when $|x| < 3$, so the radius is 3. We must check $x = 3$ and $x = -3$. If $x = 3$ we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{3^n(n+1)} = \sum_{n=0}^{\infty}\frac{(-1)^n 3^n}{3^n(n+1)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1},$$

which converges by the Alternating Series Test. If $x = -3$ we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{3^n(n+1)} = \sum_{n=0}^{\infty}\frac{(-1)^n(-3)^n}{3^n(n+1)} = \sum_{n=0}^{\infty}\frac{1}{n+1},$$

which diverges (it is actually the harmonic series, re-indexed to start at 0). Thus the interval of convergence is $(-3,3]$.

2. Using the known Taylor and Maclaurin series, find series representations of the following functions:

a) $\dfrac{x^4}{2-x}$

To get this in the form of $\dfrac{1}{1-x}$ we need to factor a 2 out of the denominator:

$$\frac{x^4}{2-x} = \frac{1}{2}\cdot\frac{x^4}{1-\frac{x}{2}}.$$

The power series for $\dfrac{x^4}{1-\frac{x}{2}}$ can then be found by multiplying the power series for $\dfrac{1}{1-\frac{x}{2}}$ by x^4:

$$\frac{x^4}{1-\frac{x}{2}} = x^4\left(\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n\right) = x^4\left(\sum_{n=0}^{\infty}\frac{x^n}{2^n}\right) = \sum_{n=0}^{\infty}\frac{x^{n+4}}{2^n}.$$

Thus

$$\frac{x^4}{2-x} = \sum_{n=0}^{\infty}\frac{x^{n+4}}{2^{n+1}}.$$

b) $\dfrac{x}{(1-x)^2}$

Notice that if we differentiate $\dfrac{1}{1-x}$ we get $\dfrac{1}{(1-x)^2}$. So we can find the desired power series by differentiating the power series for $\dfrac{1}{1-x}$ and multiplying by x:

$$\frac{x}{(1-x)^2} = x\cdot\frac{d}{dx}\left(\frac{1}{1-x}\right) = x\cdot\frac{d}{dx}\left(\sum_{n=0}^{\infty}x^n\right) = x\sum_{n=1}^{\infty}nx^{n-1} = \sum_{n=1}^{\infty}nx^n.$$

c) $\ln\left(\dfrac{1+x}{1-x}\right)$

Using the laws of logs, we have $\ln\left(\dfrac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$. The power series for $\ln(1-x)$ can be obtained by integrating the power series for $\dfrac{1}{1-x}$, which gives

$$\int \frac{1}{1-x}\,dx = \int \sum_{n=0}^{\infty} x^n\,dx = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1},$$

which implies that

$$\int \frac{1}{1-x}\,dx = -\ln(1-x) = C + \sum_{n=1}^{\infty} \frac{x^n}{n},$$

and $\ln 1 = 0$ which means $C = 0$, so

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots.$$

Then as $\ln(1+x) = \ln(1-(-x))$, we have

$$\ln(1+x) = -\sum_{n=1}^{\infty} \frac{(-x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots.$$

Then we have

$$\begin{aligned}
\ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} - \left(-\sum_{n=1}^{\infty} \frac{x^n}{n}\right) \\
&= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\right) \\
&= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) \\
&= 2\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}.
\end{aligned}$$

3. Using the representation $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$, we have that

$$\begin{aligned}
\int \arctan(x^2)\,dx &= \int \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1}\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}\,dx \\
&= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)(4n+3)} = C + \frac{x^3}{1\cdot 3} - \frac{x^7}{3\cdot 7} + \frac{x^{11}}{5\cdot 11} - \frac{x^{15}}{7\cdot 15} + \cdots.
\end{aligned}$$

3.7 Periodic Functions and Fourier Series

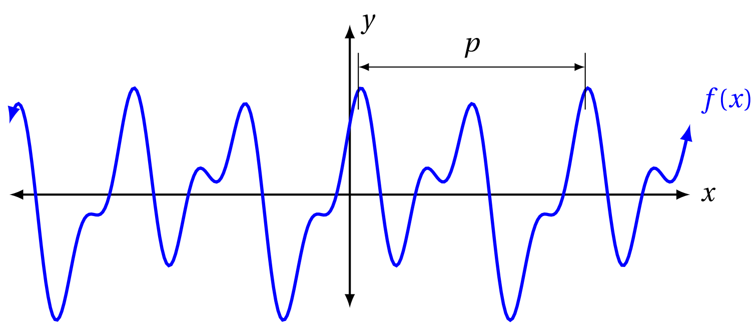
Objectives and Concepts:

- A periodic function is a function that repeats its values in regular intervals.
- A Fourier series is an infinite series of trigonometric functions that represents a periodic function.

References: TCMB §15.4.

3.7.1 Periodic Functions

Several interesting functions display periodic behavior, which means that a specific pattern over a small interval is repeated over the whole real line:



Definition: A function $f(x)$ is called a **periodic function** if $f(x)$ is defined for all real x (except possibly at some points), and if there is some positive constant p, called a **period** of f, such that

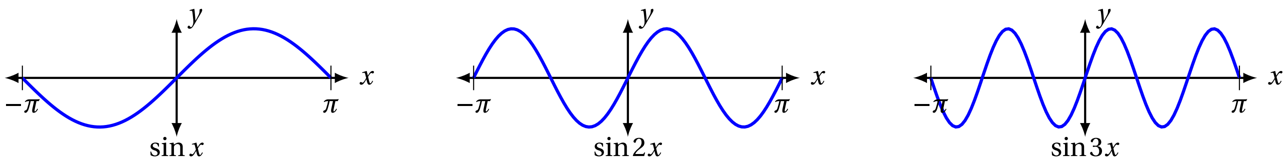
$$f(x+p) = f(x)$$

for all x.

There are several classical examples of periodic functions that we are familiar with, including all trigonometric functions. Any linear combination of periodic functions f and g, both with period p, also has period p. It is also true that any function that has period p also has period np for any positive integer n:

$$f(x+p) = f(x) \quad \Longrightarrow \quad f(x+np) = f(x).$$

Recall that $\sin x$, $\cos x$, $\tan x$, etc., have period 2π. While the functions $\sin nx, \cos nx$ (where n is any integer) have period $2\pi/n$, they also have period 2π as well.



For the remainder of this section, we only consider functions with period 2π, however the results here apply to **any** periodic function with period $p = 2L$. We can translate from a period of length 2π to a period of length $p = 2L$ by the mapping

$$v = \frac{p}{2\pi}x \quad \Longrightarrow \quad x = \frac{\pi}{L}v.$$

Thus $x = \pm\pi$ corresponds to $v = \pm L$.

3.7.2 Fourier Series

Around the turn of the 19th century while studying the distribution of heat in solid bodies, the French mathematician and physicist Jean-Baptiste Joseph Fourier described temperature distributions as finite or infinite sums of wave-like functions (sines and cosines). Fourier described solutions to the *heat equation* (a partial differential equation) as (possibly infinite) linear combinations of *eigensolutions*, which were simple sine and cosine waves. Essentially, the simple sine and cosine functions serve as a *basis* for the space of piecewise continuous periodic functions.

Theorem: The trigonometric system

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$$

is *orthogonal* on the interval $-\pi \le x \le \pi$ (hence also on $0 \le x \le 2\pi$ or any other interval of length 2π); that is, the integral of the product of any two functions in the above system over that interval is 0, so that for any integers n and m,

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0, \qquad \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \qquad (n \ne m),$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \qquad (n \ne m \text{ or } n = m).$$

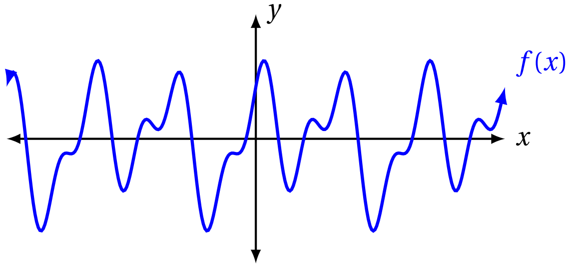
Whereas the Taylor series of a continuous and infinitely differentiable function $f(x)$ aims to represent the function in terms of a *basis* consisting of the polynomial functions $1, x, x^2, x^3, \ldots$, a **Fourier series** strives to represent a periodic function in terms of the *trigonometric basis functions*

$$1, \sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \cos 3x, \ldots.$$

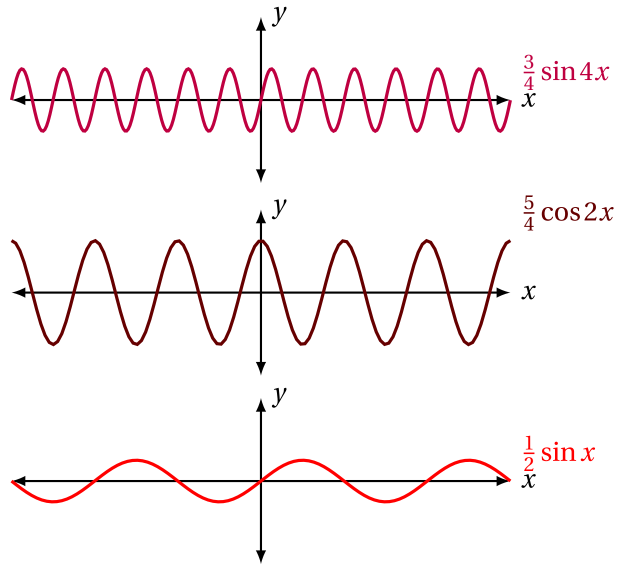
Example 1: The wave given at the beginning of the section is the simple combination

$$f(x) = \frac{3}{4}\sin 4x + \frac{5}{4}\cos 2x + \frac{1}{2}\sin x.$$

The graphs below represent the combination of these individual wave functions.



$$f(x) = \frac{3}{4}\sin 4x + \frac{5}{4}\cos 2x + \frac{1}{2}\sin x.$$



Definition: If $f(x)$ has period 2π and the series below converges, then the **Fourier series** of $f(x)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

The $a_0, a_1, b_1, a_2, b_2, \ldots$ are the **Fourier coefficients** of the Fourier series, and are found by the *Euler formulas*

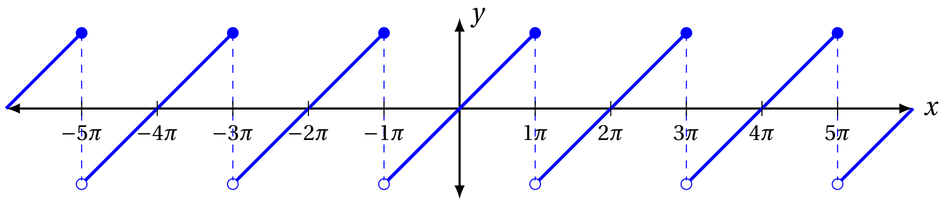
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

For the function $f(x) = \frac{3}{4}\sin 4x + \frac{5}{4}\cos 2x + \frac{1}{2}\sin x$, we have that $b_1 = 1/2$, $a_2 = 5/4$, $b_4 = 3/4$, and all other a_n, b_n are zero.

Theorem: Let $f(x)$ be periodic with period 2π and piecewise continuous in the interval $-\pi \le x \le \pi$. Furthermore, let $f(x)$ have a left-hand derivative and a right-hand derivative at each point of that interval. Then the Fourier series of $f(x)$ converges. Its sum is $f(x)$, except at points x_0 where $f(x)$ is discontinuous. There the sum of the series is the average of the left- and right-hand limits of $f(x)$ at x_0.

Example 2: Find the Fourier coefficients and Fourier series of the *sawtooth function* defined by

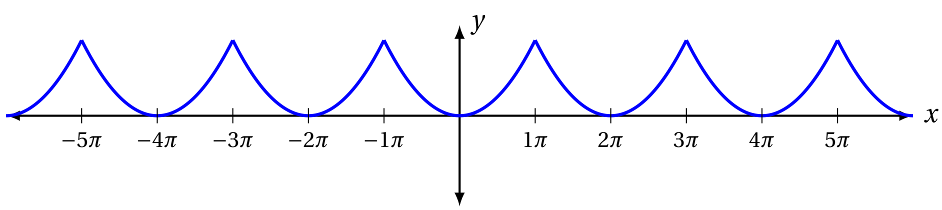
$$f(x) = x \quad \text{for } -\pi < x \le \pi, \qquad f(x+2\pi) = f(x).$$



> **Theorem:** If $f(-x) = f(x)$ (i.e., f is even) and f has period 2π, then the Fourier series of f is the **Fourier cosine series**
>
> $$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx,$$
>
> i.e., $b_n = 0$ for all n.
>
> If $f(-x) = -f(x)$ (i.e., f is odd) and f has period 2π, then the Fourier series of f is the **Fourier sine series**
>
> $$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$
>
> i.e., $a_n = 0$ for all n.

Example 3: Find the Fourier coefficients and Fourier series of the function defined by

$$f(x) = x^2 \quad \text{for } -\pi \le x \le \pi, \qquad f(x+2\pi) = f(x).$$



> **Theorem:** The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 and f_2.
>
> The Fourier coefficients of cf are c times the corresponding Fourier coefficients of f.

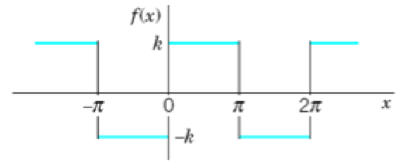
3.7 Review of Concepts

- Terms to know: periodic function, Fourier series, Fourier coefficients
- Know how to find the Fourier series of a periodic function.

3.7 Practice Problems

1. Find the Fourier coefficients of the periodic function $f(x)$ given by

$$f(x) = \begin{cases} -k & \text{if } -\pi < x < 0, \\ k & \text{if } \quad 0 < x < \pi, \end{cases} \quad \text{and} \quad f(x+2\pi) = f(x).$$



This function represents a “square wave” in many applications.

3.7 Exercises

1. Find the Fourier coefficients of the “triangular wave” function $f(x)$ given by

$$f(x) = \begin{cases} -x & \text{if } -\pi < x < 0, \\ x & \text{if } \quad 0 < x < \pi, \end{cases} \quad \text{and} \quad f(x+2\pi) = f(x).$$

3.7 Answers to Practice Problems

1. Note that this function is actually an odd function, as

$$f(-x) = \begin{cases} -k & \text{if } -\pi < -x < 0, \\ k & \text{if } \quad 0 < -x < \pi, \end{cases} = \begin{cases} -k & \text{if } \quad \pi > x > 0, \\ k & \text{if } \quad 0 > x > -\pi, \end{cases} = -f(x).$$

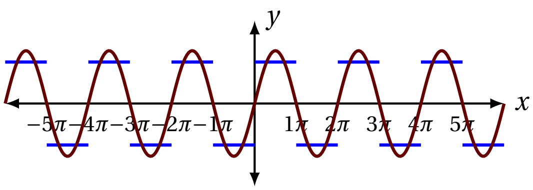
Thus, we know $a_n = 0$ for $n = 0, 1, 2, 3, \ldots$. To find the b_n, we have

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\ &= \frac{1}{\pi}\left(\int_{-\pi}^{0} (-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right) \\ &= \frac{1}{\pi}\left((-k)\frac{-\cos nx}{n}\Big|_{-\pi}^{0} + k\frac{-\cos nx}{n}\Big|_{0}^{\pi}\right) \\ &= \frac{1}{\pi}\left(\frac{k\cos 0 - k\cos(-n\pi)}{n} + \frac{-k\cos(n\pi) + k\cos 0}{n}\right) \\ &= \frac{1}{n\pi}\left(2k\cos 0 - 2k\cos(n\pi)\right) \\ &= \begin{cases} \frac{4k}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

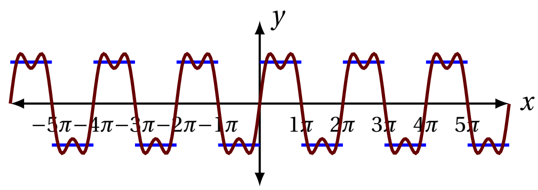
Thus, using the fact that $2n-1$ is odd for all n, we have

$$f(x) = \sum_{n=1}^{\infty} \frac{4k}{(2n-1)\pi}\sin((2n-1)x).$$

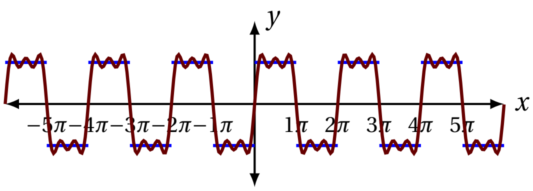
Below we have a plot of $f(x)$ (blue) with $S_m(x) = \sum_{n=1}^{m} \frac{4k}{(2n-1)\pi} \sin((2n-1)x)$ (dark red) for the first few terms.



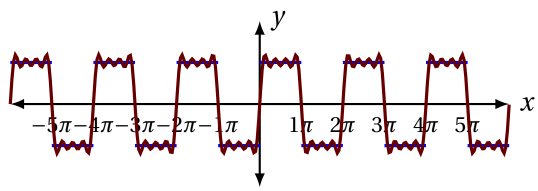
$S_1(x)$



$S_2(x)$



$S_3(x)$



$S_4(x)$

Chapter 4

Multivariable Calculus

4.1 Multivariate Functions and Partial Derivatives

Objectives and Concepts:

- Derivatives of functions of n independent variables are defined with respect to an individual variable and represent the rate of change of the function in the direction of that variable's axis.
- The partial derivative of a function with respect to a variable is found by using normal differentiation rules while holding all other variables constant.

References: OSC-3 Chapter 4, §§1–5; CET §§14.1–5; TCMB §§9.1–3,5–7, 10.3.

4.1.1 Multivariate Functions

References: OSC-3 §4.1; CET §§14.1,2; TCMB §§9.1,2, 10.3.

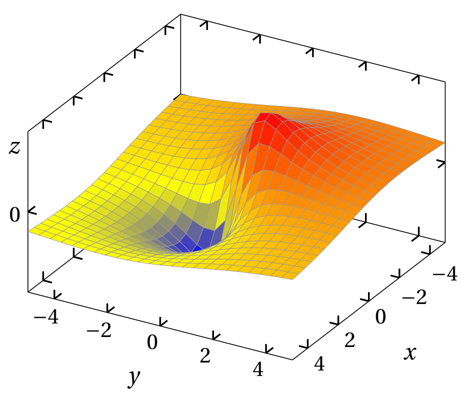
> **Definition:** Let D be a subset of $\mathbb{R}^n$ for some positive integer n. A **function** f of the n variables $x_1, x_2, \ldots, x_n$ is a rule that assigns to each n-tuple $(x_1, x_2, \ldots, x_n)$ in D a value $f(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}$. The set D is the **domain** of f and the set of all outputs is the **range** of f.

While functions can be described for inputs of size any positive integer n, the most commonly discussed multivariate functions are functions of two variables $z = f(x, y)$, three variables $w = f(x, y, z)$, and four variables $w = f(t, x, y, z)$ (the latter being the case where the output value w depends on the position (x, y, z) of a point in 3-D space and time t).

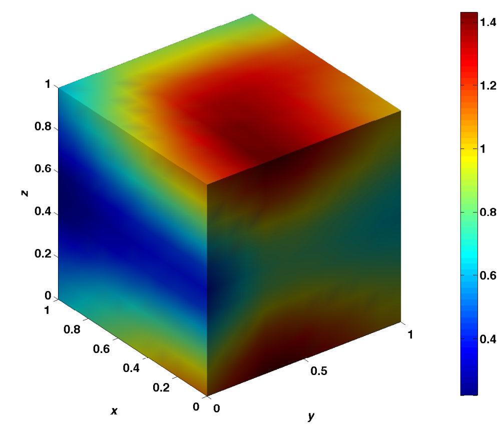
Example 1: Let $f(x,y,z) = \dfrac{x^2+z+3\sin y}{x+2y-z}$. Find the domain of f and evaluate $f(4,0,2)$. Can you describe the range of f?

> **Definition:** The **graph** of a multivariate function $f : \mathbb{R}^n \to \mathbb{R}$ is the set of all points $(x_1, x_2, \ldots, x_n, f(x_1, x_2, \ldots, x_n))$ in $\mathbb{R}^{n+1}$.

Graphs of functions of two variables can be visualized as surfaces in $\mathbb{R}^3$. However, we have some trouble visualizing functions of three or more variables. One approach for plotting a function $f(x,y,z)$ is to fill the domain with colors at every point, each point representing a different color that represents the output of the function.



$$z = \frac{y-x}{x^2+y^2+1}$$



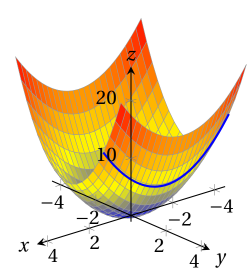
$$w = \frac{y\sin(\pi x^2) + \cos(\pi z)^2 + y^{3/10}}{x^2+y^2+1}$$

4.1.2 Partial Derivatives

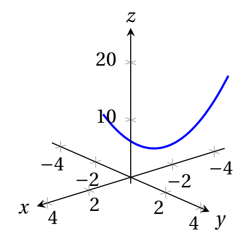
References: OSC-3 §4.3; CET §14.3; TCMB §9.3.

Let y be a function of x. We have studied in great detail the derivative of y with respect to x, that is, $\frac{dy}{dx}$, which measures the rate at which y changes with respect to x. For a multivariate function $z = f(x,y)$, it makes sense to want to know how z changes with respect to x and/or y.

Consider the function $z = f(x,y) = x^2 + 2y^2$. By fixing $y = 2$, we focus our attention to all points on the surface where the y-value is 2, shown in both figures. These points form a curve in space: $z = f(x,2) = x^2 + 8$ which is a function of just one variable. We can take the derivative of z with respect to x along this curve and find equations of tangent lines, etc.



The key notion to extract from this example is: by treating y as constant (it does not vary) we can consider how z changes with respect to x. In a similar fashion, we can hold x constant and consider how z changes with respect to y. This is the underlying principle of **partial derivatives**. We state the formal, limit–based definition first, then show how to compute these partial derivatives without directly taking limits.



Definition: Let $z = f(x, y)$ be a continuous function on an open set D in $\mathbb{R}^2$.

1. The **partial derivative of f with respect to** x is $f_x(x, y) = \lim\limits_{h\to 0} \dfrac{f(x+h, y) - f(x, y)}{h}$.
2. The **partial derivative of f with respect to** y is $f_y(x, y) = \lim\limits_{h\to 0} \dfrac{f(x, y+h) - f(x, y)}{h}$.

Alternate notations for $f_x(x, y)$ include:

$$\frac{\partial}{\partial x} f(x, y), \quad \frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x}, \quad \text{and } z_x,$$

with similar notations for $f_y(x, y)$. For ease of notation, $f_x(x, y)$ is often abbreviated f_x. In many chemistry textbooks, it is often the case that the variables that are to be held constant are denoted as subscripts outside parentheses of the fractional partial notation. For example, the derivatives

$$\left(\frac{\partial f}{\partial x}\right)_{y,z} \qquad \text{and} \qquad \left(\frac{\partial V}{\partial n}\right)_{P,T}$$

represent the derivatives of f with respect to x while holding y and z constant, and the derivative of V with respect to n while holding P and T constant.

Since partial derivatives are defined using the limit definition, all of the standard rules techniques for computing derivatives of single-variable functions apply. Thus all one has to do when finding a partial derivative is treat all other variables as constants.

Example 2: If $f(x, y) = \sin\left(\dfrac{x}{1+y}\right)$, find f_x and f_y.

Example 3: If $g(x, y, z) = e^{xy} \ln z$, find g_x, g_y, and g_z.

Just as higher-order derivatives are found for single-variable functions, they can be found for multivariable functions. However, we now have the idea of **mixed partials** - partial derivatives of on

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \qquad f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \qquad f_{xxy} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right).$$

Example 4: Compute the second partials of the function $f(x, y) = \cos\left(2x^2 + 3y\right)$.

Clairaut's Theorem: Suppose f is defined on a set D that contains the point (a,b). If the function f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)$. (In some contexts this is known as the **Euler reciprocity relation**.)

Example 5: If $f(\rho,\theta,\phi) = 4\rho^2 \sin\phi \sin\theta + \cos(\theta\phi^2)$, find $f_{\theta\phi\rho}$.

Concept Check: How many different third-order partial derivatives does a function of two variables have? What about a function of three variables?

4.1.3 The Chain Rule

References: OSC-3 §4.5; CET §14.5; TCMB §9.6.

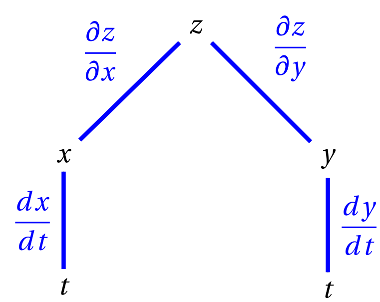
Recall that when $y = f(x)$ and $x = g(t)$ where f and g are differentiable functions, that y is also a differentiable function of t as $y = f(g(t))$ and thus the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

The same is true for multivariate functions, however there may be more than one "path" to the independent variable(s).

The Chain Rule (Case I): Suppose that $z = f(x,y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

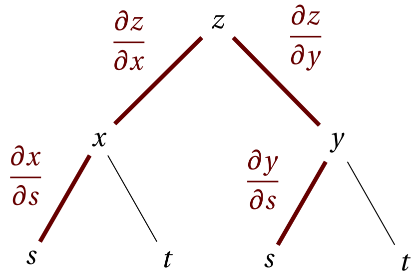
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$



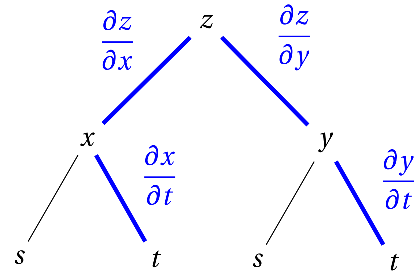
Example 6: Find $\dfrac{dz}{dt}$ if $z = \arctan(y/x)$, $x = e^t$, and $y = 1 - e^{-t}$.

The Chain Rule (Case II): Suppose that $z = f(x,y)$ is a differentiable function of x and y, where $x = g(s,t)$ and $y = h(s,t)$ are both differentiable functions of s and t. Then z is a differentiable function of s and t, and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$



$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example 7: Let $z = x^2 + xy^3$ where $x = wv^2 + w$ and $y = u + ve^w$.
Find $\dfrac{\partial z}{\partial u}$, $\dfrac{\partial z}{\partial v}$, and $\dfrac{\partial z}{\partial w}$ when $u = 2, v = 1, w = 0$.

4.1.4 Implicit Differentiation

References: OSC-3 §4.5; CET §14.5.

If $z = f(x, y)$ is an implicitly-defined function by an equation of the form $F(x, y, z) = 0$, then we can use the chain rule to compute the derivative with respect to x or y implicitly:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0.$$

Since $\partial x/\partial x = 1$ and $\partial y/\partial x = 0$, we have

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0,$$

which can be solved for $\dfrac{\partial z}{\partial x}$.

> **Theorem:** If $F(x, y, z) = 0$ defines z implicitly as a function of x and y, and the below partial derivatives all exist, then
> $$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

This can also by used to find $\dfrac{dy}{dx}$ when $F(x, y) = 0$: $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$.

Example 8: Find $\dfrac{\partial z}{\partial x}$ if $xyz = \cos(x + y + z)$.

Recalling from single-variable implicit differentiation the **reciprocal identity** $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$, we can arrive at the following result.

Euler's Cyclic Relation: If $f(x, y, z) = 0$ defines each variable as a function of the other variables, we have

$$\left(\frac{\partial x}{\partial y}\right)\left(\frac{\partial y}{\partial z}\right)\left(\frac{\partial z}{\partial x}\right) = -1.$$

This formula is also known as the **cycle rule** or **triple product rule**.

4.1.5 The Total Differential

References: OSC-3 §4.4; CET §14.4; TCMB §9.5.

Recall that if $y = f(x)$ and f is differentiable, then, $dy = f'(x)dx$. One important use of this differential is in Integration by Substitution. Another important application is approximation. Let $\Delta x = dx$ represent a change in x. When dx is small, $dy \approx \Delta y$, the change in y resulting from the change in x. Fundamental in this understanding is this: as dx gets small, the difference between Δy and dy goes to 0. Another way of stating this: as dx goes to 0, the *error* in approximating Δy (the actual change in y) with dy (the differential) goes to 0.

We extend this idea to functions of two variables. Let $z = f(x, y)$, and let $\Delta x = dx$ and $\Delta y = dy$ represent changes in x and y, respectively. Let $\Delta z = f(x+dx, y+dy) - f(x, y)$ be the change in z over the change in x and y. Recalling that f_x and f_y give the instantaneous rates of z-change in the x- and y-directions, respectively, we can approximate Δz with $dz = f_x dx + f_y dy$; in words, the total change in z is approximately the change caused by changing x plus the change caused by changing y. In a moment we give an indication of whether or not this approximation is any good. First we give a name to dz.

Definition: Let $z = f(x, y)$ be continuous on an open set D. Let dx and dy represent changes in x and y, respectively. Where the partial derivatives f_x and f_y exist, the **total differential of** z is

$$dz = f_x(x, y)dx + f_y(x, y)dy.$$

Example 9: Let $z = x^4 e^{3y}$. Find dz.

Differentials can be used to determine the sensitivity of a particular quantity to changes in measurement of the input variables.

Example 10: A cylindrical steel storage tank is to be built that is 10ft tall and 4ft across in diameter. It is known that the steel will expand/contract with temperature changes; is the overall volume of the tank more sensitive to changes in the diameter or in the height of the tank?

A cylindrical solid with height h and radius r has volume $V = \pi r^2 h$. We can view V as a function of two variables, r and h. We can compute partial derivatives of V:

$$\frac{\partial V}{\partial r} = V_r(r, h) = 2\pi r h \qquad \text{and} \qquad \frac{\partial V}{\partial h} = V_h(r, h) = \pi r^2.$$

The total differential is $dV = (2\pi rh)dr + (\pi r^2)dh$. When $h = 10$ and $r = 2$, we have $dV = 40\pi dr + 4\pi dh$. Note that the coefficient of dr is $40\pi \approx 125.7$; the coefficient of dh is a tenth of that, approximately 12.57. A small change in radius will be multiplied by 125.7, whereas a small change in height will be multiplied by 12.57. Thus the volume of the tank is more sensitive to changes in radius than in height. Note that this analysis only applies to a tank of those dimensions. A tank with a height of 1ft and radius of 5ft would be more sensitive to changes in height than in radius.

4.1.6 (‡) Exact Differentials

Reference: TCMB §9.7.

Differentials play an important role in physical chemistry, as they can indicate whether certain processes are *independent of path*. Relevant to this is the concept of an *exact* differential.

Definition: A differential of the form $du = M(x, y)\,dx + N(x, y)\,dy$ is an **exact differential** if it is the differential of some function $f(x, y)$. Otherwise du is an **inexact differential**.

The differential df of a function $f(x, y)$ satisfies an important property: if $df = M(x, y)\,dx + N(x, y)\,dy$, then $M_y(x, y) = N_x(x, y)$ (why?). This gives us a way to determine if any given differential $M(x, y)\,dx + N(x, y)\,dy$ is exact.

Example 11: Determine if the differential $du = \left(2xy + \dfrac{9x^2}{y}\right) dx + \left(x^2 - \dfrac{3x^2}{y^2}\right) dy$ is exact or inexact.

For a differential in three independent variables, a test for exactness can be derived from Clairaut's Theorem. If

$$du = M(x, y, z)\,dx + N(x, y, z)\,dy + P(x, y, z)\,dz$$

then du is an exact differential provided

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \qquad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}.$$

4.1 Review of Concepts

- Terms to know: partial derivative, mixed partial, Clairaut's Theorem, reciprocal rule, Euler's cylcic relation, total differential, exact differential
- Know how to find the partial derivative of a multivariable function with respect to a single variable.
- Know how to apply the Chain Rule to find derivatives of composite multivariable functions.
- Know how to compute the total differential of a given quantity.
- Know how to determine if a given differential is exact.

4.1 Practice Problems

1. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $z = \sin(x+y)e^{x-y}$.

2. Find all nonzero partials of $f(x, y) = 3x^2 + y^2 + 2xy^3$.

3. If $xyz + x^2 + y^2 + z^2 = 0$, find $\dfrac{\partial x}{\partial y}$.

4. If $x - z = \arctan(yz)$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial x}$.

5. Find $\dfrac{\partial u}{\partial s}$ and $\dfrac{\partial u}{\partial t}$ if $u = e^{x+y}$, $x = te^s$, and $y = \sin s$.

6. If $w = e^x \ln y \sin z$, find the differential dw.

7. Determine if $du = 2xe^{axy}\,dx + 2ye^{axy}\,dy$ is an exact differential.

4.1 Exercises

1. Find the indicated partial derivatives:

a) $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ where $f(x, y) = ae^{-b(x^2+y^2)} + c\sin(x^2 y)$

b) $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y^2}$ where $f(x, y) = \cos(x/y)$

c) $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ where $yz = \ln(x+z)$

d) $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ where $z = e^{x+2y}$, $x = s/t$, and $y = t/s$

2. Test the following differentials for exactness:

a) $(4x+3y)\,dx + (3x+8y)\,dy$ b) $y\cos x\,dx + \sin x\,dy$ c) $y\ln x\,dx + x\ln y\,dy$

3. For a gas obeying the van der Waals equation of state

$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT,$$

Find the partials $\left(\dfrac{\partial V}{\partial P}\right)_{T,n}$, $\left(\dfrac{\partial V}{\partial T}\right)_{P,n}$, $\left(\dfrac{\partial T}{\partial P}\right)_{V,n}$, and $\left(\dfrac{\partial V}{\partial n}\right)_{P,T}$ (here R is always a constant).

4. For a certain system, the thermodynamic energy U is given as a function of S, V, and n by

$$U(S,V,n) = Kn^{5/3}V^{-2/3}e^{2s/(3nR)},$$

where S is the entropy, V is the volume, n is the number of moles, K is a constant, and R is the gas constant. Find the differential dU in terms of dS, dV, and dn.

5. If $z = f(x,y)$, $x = r\cos\theta$, and $y = r\sin\theta$, find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$ and show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

6. Using the $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ you found in the previous exercise, find (in terms of r and θ and partials of z with respect to those variables) the partials $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$. Functions that satisfy the Laplace's equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

are known as **harmonic functions** and they arise in many applications in the physical sciences, including fluid flow, heat conduction, and gravitational and electrostatic potential theory.

4.1 Answers to Practice Problems

1. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $z = \sin(x+y)e^{x-y}$.

We have

$$\frac{\partial z}{\partial x} = \sin(x+y)e^{x-y} + \cos(x+y)e^{x-y}, \qquad \frac{\partial z}{\partial y} = -\sin(x+y)e^{x-y} + \cos(x+y)e^{x-y}.$$

2. Find all nonzero partials of $f(x,y) = 3x^2 + y^2 + 2xy^3$.

$$f_x = 6x + 2y^3, \qquad f_y = 2y + 6xy^2,$$
$$f_{xx} = 6, \qquad f_{xy} = 6y^2, \qquad f_{yy} = 2 + 12xy,$$
$$f_{yyx} = 12y, \qquad f_{yyy} = 12x, \qquad f_{xyy} = 12y, \qquad f_{yyxy} = 12.$$

3. If $xyz + x^2 + y^2 + z^2 = 0$, find $\dfrac{\partial x}{\partial y}$.

Let $F(x,y,z) = xyz + x^2 + y^2 + z^2$. Then

$$\frac{\partial x}{\partial y} = -\frac{F_y}{F_x} = -\frac{xy + 2y}{yz + 2x}.$$

4. If $x - z = \arctan(yz)$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial x}$.

Let $F(x, y, z) = x - z - \arctan(yz)$. Then $F_x = 1$, $F_y = -\dfrac{z}{1 + y^2 z^2}$, and $F_z = -1 - \dfrac{y}{1 + y^2 z^2} = \dfrac{-1 - y^2 z^2 - y}{1 + y^2 z^2}$. Thus

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{1 + y^2 z^2}{1 + y^2 z^2 + y}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{-z}{1 + y^2 z^2 + y}.$$

5. Find $\dfrac{\partial u}{\partial s}$ and $\dfrac{\partial u}{\partial t}$ if $u = e^{x+y}$, $x = te^s$, and $y = \sin s$.

We have

$$\frac{\partial u}{\partial x} = e^{x+y} = \frac{\partial u}{\partial y}, \qquad \frac{\partial x}{\partial s} = te^s, \qquad \frac{\partial x}{\partial t} = e^s, \qquad \frac{\partial y}{\partial s} = \cos s, \qquad \frac{\partial y}{\partial t} = 0.$$

Thus

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = \left(e^{x+y}\right)\left(te^s\right) + \left(e^{x+y}\right)(\cos s) = (te^s + \cos s)e^{x+y},$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = \left(e^{x+y}\right)\left(e^s\right) + \left(e^{x+y}\right)(0) = e^s e^{x+y}.$$

6. If $w = e^x \ln y \sin z$, find the differential dw.

$$dw = w_x\,dx + w_y\,dy + w_z\,dz = e^x \ln y \sin z\,dx + \frac{e^x \sin z}{y}\,dy + e^x \ln y \cos z\,dz.$$

7. Determine if $du = 2xe^{axy}\,dx + 2ye^{axy}\,dy$ is an exact differential.

To do this, we find

$$\frac{\partial}{\partial y}\left(2xe^{axy}\right) = 2ax^2 e^{axy},$$

and

$$\frac{\partial}{\partial x}\left(2ye^{axy}\right) = 2ay^2 e^{axy},$$

which are not the same, so du is not an exact differential.

4.2 Multiple Integrals

Objectives and Concepts:

- Integrals over rectangular regions in $\mathbb{R}^2$ or $\mathbb{R}^3$ can be computed in any order.
- When multiple integrals over general regions have variable limits of integration, those limits must be in terms of subsequent variables of integration.
- Changing variables can reduce the complexity of a multiple integral.

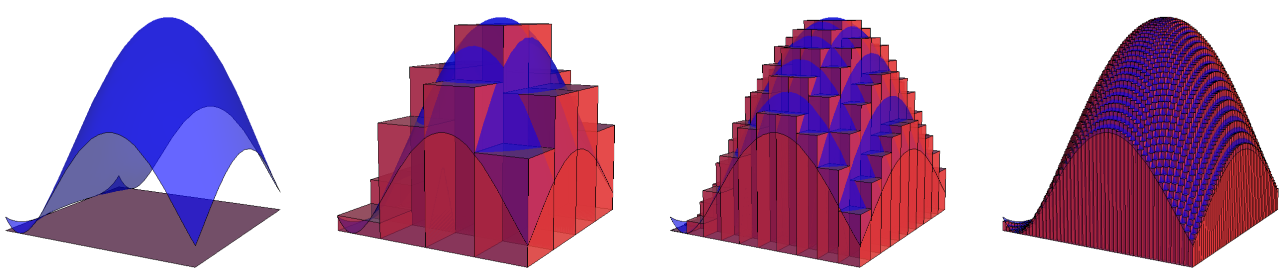
References: OSC-3 Chapter 5, §§1–5 & 7; CET §§15.1–3,6–8; TCMB §§9.9–11

4.2.1 Multiple Integrals

Recall that the definite integral $\int_a^b f(x)\,dx$ represented the net area between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$. The **double integral** $\iint_R f(x, y\,dA$ represents the **net volume** between the surface $z = f(x, y)$ and the xy-plane that lies above or below the the region R. Just as the single integral was really a limit of a sum of areas of rectangles, the double integral can be thought of as a limit of a sum of volumes of rectangular columns:

$$\iint_R f(x, y)\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$

where ΔA_i represents the area of the base of the ith column.



It is important to note that in this situation, the **region of integration** is a region in the xy-plane. We can extend this concept to even more dimensions, however the visualization becomes a little more difficult. A **triple integral** $\iiint_V f(x, y, z)\,dV$ of $f(x, y, z)$ over a volume V actually represents the net *hypervolume* between the surface $w = f(x, y, z)$ and the xyz-space where all of the points (x, y, z) belong to the 3-D region V.

As we will see in subsequent examples, evaluation of multiple integrals is executed much in the same way that it is for single integrals. However, the process of setting up the integral may be a challenge. We begin with the simplest context, in which the region of integration is a **rectangle** in $\mathbb{R}^n$,

Now to differentiate functions of several variables with respect to one variable, we simply treat all the other variables as constants or coefficients. The same is true for **partial integration** - we simply treat another variable as a constant. To illustrate, if $f(x,y) = x^2 y$, then

$$\int f(x,y)\,dx = \int x^2 y\,dx = \frac{1}{3}x^3 y + C_x, \quad \text{and} \quad \int f(x,y)\,dy = \int x^2 y\,dy = \frac{1}{2}x^2 y^2 + C_y.$$

But note that the "constants" C_x and C_y above are not necessarily constants, there are merely constant *with respect to a variable*, i.e., they may contain expressions of other variables. For example, note that both $F(x,y) = \frac{1}{3}x^3 y + 2$ and $G(x,y) = \frac{1}{3}x^3 y + 4\sin(y^2) - y$ are both partial antiderivatives of $f(x,y)$, as $F_x(x,y) = G_x(x,y) = f(x,y)$. So in reality, C_x is some function of y, and C_y is some function of x and we have

$$\int f(x,y)\,dx = \int x^2 y\,dx = \frac{1}{3}x^3 y + C_x(y), \quad \text{and} \quad \int f(x,y)\,dy = \int x^2 y\,dy = \frac{1}{2}x^2 y^2 + C_y(x).$$

4.2.2 Multiple Integrals Over Rectangles

When the region of integration is bounded by constant values in all directions, we can evaluate a multiple integral using **iterated** (or **repeated**) **integration**. For example, if $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$, then we have

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy, \quad \text{or} \quad \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left(\int_c^d f(x,y) dy \right) dx.$$

Example 1: Evaluate $\displaystyle\int_{-1}^{1} \int_0^2 \left(4 + 9x^2 y^2\right)\,dy\,dx$.

Fubini's Theorem: Let f be continuous on the rectangle $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$. Then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

Fubini's Theorem says that the order of integration in an iterated integral over a rectangle does not matter. This way, if it is more convenient to perform integration with respect to one variable as opposed to another, then the order of integration can be switched (as long as the corresponding limits of integration are switched as well).

Concept Check: How many different ways are there to evaluate $\displaystyle\iiint_V f(x,y,z)\,dV$ if V is a rectangular box in $\mathbb{R}^3$?

Example 2: Evaluate $\displaystyle\int_0^{\sqrt{\ln 2}}\int_0^{\sqrt{\ln 4}}\int_0^1 xyze^{-x^2-y^2}\,dz\,dy\,dx.$

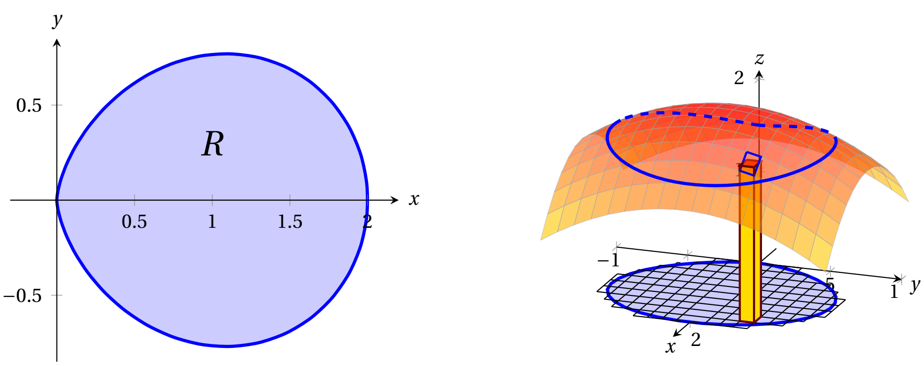
It is possible to greatly simplify multiple integrals over rectangles when the integrand can be factored into products of single-variable functions.

$$\int_a^b\int_c^d f(x)g(y)\,dy\,dx=\left(\int_a^b f(x)\,dx\right)\left(\int_c^d g(y)\,dy\right)$$

Example 3: Evaluate $\displaystyle\int_0^1\int_1^2 \frac{xe^x}{y}\,dy\,dx.$

4.2.3 Multiple Integrals Over General Regions

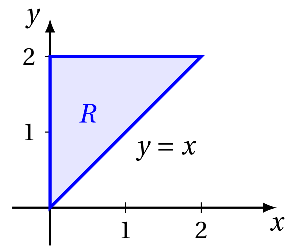
While multiple integration over rectangles is fairly straightforward, many applications of multiple integrals will require us to integrate over a more general region.

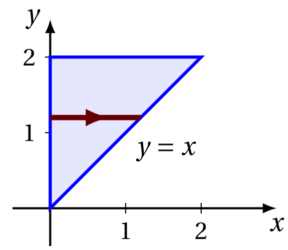


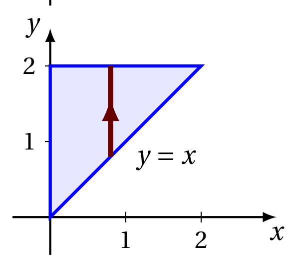
In these cases, we must take care to accurately describe the limits of integration for each variable, as these limits may depend on other variable(s). One important rule to follow is that the outermost integral **must** have constant limits of integration. Often there is more than one way to set up an integral over a general region.

Example 4: Find $\displaystyle\iint_R (x^2y-x)\,dA$, where R is the region in the xy-plane bounded by the y-axis, the line $y=x$, and the line $y=2$.

First, we must draw the region R so we can get a good understanding of what it looks like. This region is actually a triangle with one side on the y-axis.







If we decide to integrate with respect to x first (if $dA = dx\,dy$), then we draw a line in region R in the direction of the x-axis. Note that the smallest value of x is 0 but the largest will be y. Since the y-values then range from 0 to 2, the integral is written as

$$\int_0^2 \int_0^y (x^2y - x)\,dx\,dy.$$

If we integrate with respect to y first, note that the smallest value of y will always be x and the largest will be 2. Then x will range from 0 to 2:

$$\int_0^2 \int_x^2 (x^2y - x)\,dy\,dx.$$

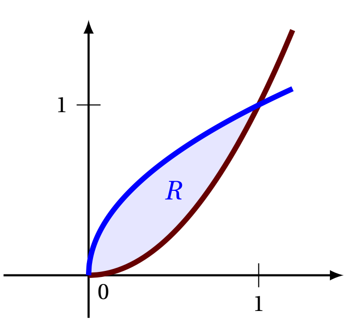
Using the first integral, we have

$$\int_0^2 \int_0^y (x^2y - x)\,dx\,dy = \int_0^2 \left(\frac{1}{3}x^3y - \frac{1}{2}x^2\right)\Big|_0^y\,dy = \int_0^2 \left(\frac{1}{3}y^4 - \frac{1}{2}y^2\right)dy$$
$$= \left(\frac{1}{15}y^5 - \frac{1}{6}y^3\right)\Big|_0^2 = \frac{32}{15} - \frac{8}{6} = \frac{64-40}{30} = \frac{4}{5}.$$

Using the second integral, we have

$$\int_0^2 \int_x^2 (x^2y - x)\,dy\,dx = \int_0^2 \left(\frac{1}{2}x^2y^2 - xy\right)\Big|_x^2\,dx = \int_0^2 \left(2x^2 - 2x - \frac{1}{2}x^4 + x^2\right)dx$$
$$= \int_0^2 \left(3x^2 - 2x - \frac{1}{2}x^4\right)dx = \left(x^3 - x^2 - \frac{1}{10}x^5\right)\Big|_0^2 = 8 - 4 - \frac{32}{10} = \frac{4}{5}.$$

Example 5: Set up the double integral of $f(x, y)$ over R two different ways, where R is the region between the curves $y = \sqrt{x}$ and $y = x^2$ in the first quadrant.



Concept Check: What is the area of the region R in the above example?

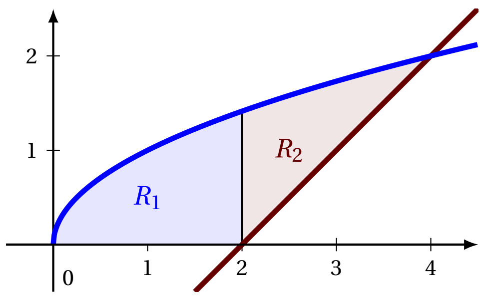
Example 6: Evaluate $\iint_R x^3\,dA$ where R is the region in the first quadrant bounded by the curve $y = \ln x$ and the line $x = e$.

Example 7: Evaluate $\iint_R \sin x \cos y\,dA$ where R is the triangle with vertices $(-1,0)$, $(1,0)$, and $(0,1)$.

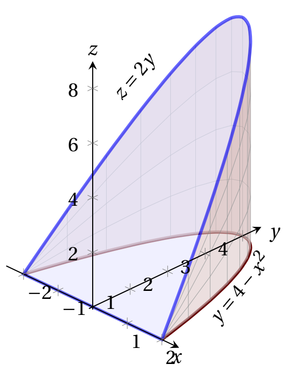
Sometimes it is necessary to break R into multiple regions, for example $R = R_1 \cup R_2$, and use the fact that

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA.$$

Example 8: Set up the double integral of $f(x,y)$ over $R = R_1 \cup R_2$ which is the area bounded by the x-axis, the curve $y = \sqrt{x}$, and the line $y = x - 2$.



Example 9: Set up the triple integral of $f(x,y,z)$ over R is the volume bounded by the xy-plane, the surface $y = 4 - x^2$, and the plane $z = 2y$.



Example 10: Write five other integrals that are equal to the integral $\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\,dz\,dy\,dx$.

4.2.4 Change of Variables in Multiple Integrals

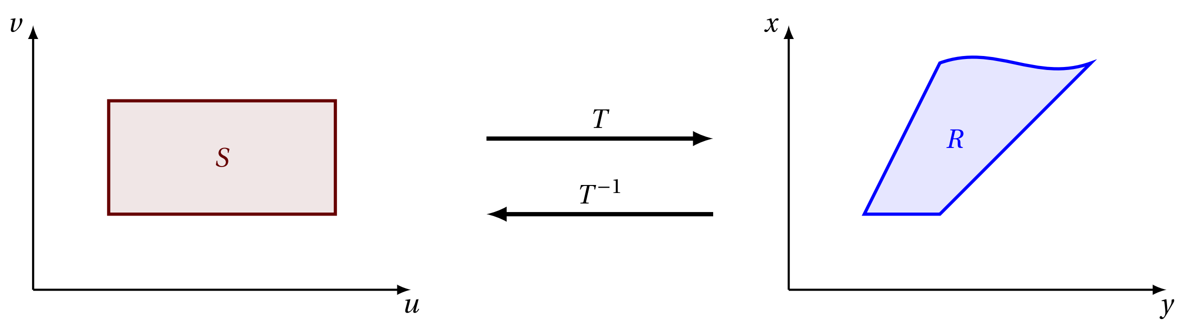
References: OSC-3 §5.7; CET §§15.3,6–8; TCMB §9.11.

Often it is useful to make a change of variables (substitutions) to make either the integral easier to set up or evaluate, or both. When we make a substitution, we must replace everything in the integral with the new variables, including the area differential dA or volume differential dV. Let's examine a simple u-substitution with the roles of x and u somewhat reversed. For example, if $x = g(u)$, then $dx = g'(u)\,du$, so we can write

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du = \int_c^d f(g(u))\frac{dx}{du}\,du,$$

where we transform the interval $[c,d]$ on the u-axis to the interval $[a,b]$ on the x-axis using $a = g(c)$ and $b = g(d)$. In reality, we are applying a **transformation** to the variable u to get the variable x - specifically, the transformation defined by $x = g(u)$. The factor of $\dfrac{dx}{du}$ represents the **scaling** of the x-interval to the u-interval.

How does this work for multiple substitutions? Let's suppose that we want to transform a region S in the uv-plane to a region R in the xy-plane. Let T represent this transformation, so that $T(u,v) = (x,y)$ is defined by using the equations $x = g(u,v)$, $y = h(u,v)$. Provided T is a one-to-one transformation, T^{-1} exists.



Example 11: Let the transformation T be defined by $T(u,v) = (x,y)$ where $x = u^2 - v^2$ and $y = 2uv$. Find the image of the square $S = \{(u,v) \mid 0 \le u \le 1, 0 \le v \le 1\}$ under T.

Now to change variables in an integral, we must also make the appropriate change of the area differential dA - we must include a factor that gives the relative scaling of different coordinate systems. Recall that in the single-variable case, this factor was $\dfrac{dx}{du}$. For transformations involving multiple variables, we need a matrix.

Definition: Let T be a transformation from the uv-plane to the xy-plane defined by $T(u,v) = (g(u,v), h(u,v)) = (x,y)$ so $x = g(u,v)$ and $y = h(u,v)$. The **Jacobian** of T is the 2×2 matrix $\boldsymbol{J}(u,v)$ defined by

$$\boldsymbol{J}(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix}.$$

The **Jacobian determinant** of T is the absolute value of the determinant of $\boldsymbol{J}(u,v)$:

$$|\det(\boldsymbol{J}(u,v))| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right|.$$

The notation $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$ is most commonly used to denote the absolute value of the determinant of the Jacobian, and it is exactly the scaling factor that gives the relationship between the area differential in the xy-plane and the area differential in the uv-plane:

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

Thus to make the change of variables from x and y to u and v defined by $x = g(u,v)$, $y = h(u,v)$, we have

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(g(u,v),h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

For a change of three variables, the same relationship holds:

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_W f(g(u,v,w),h(u,v,w),k(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2ex] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

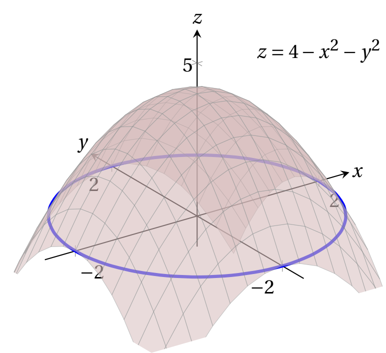
For certain functions and geometries, the polar, cylindrical, and spherical coordinate systems are very convenient in describing the regions of integration.

Example 12: Compute the Jacobian and its determinant for the change of variables $x = r\cos\theta,\ y = r\sin\theta$.

Example 13: Find the volume under the surface $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$ and above the circle $x^2 + y^2 = 1$.

Example 14: Compute the Jacobian and its determinant for the change of variables from Cartesian to Cylindrical coordinates.

Example 15: Set up integrals that give the volume of the solid bounded by the surface $z = 4 - x^2 - y^2$ and the xy-plane in both Cartesian coordinates and Cylindrical coordinates. Evaluate one of the integrals.



Example 16: Compute the Jacobian and its determinant for the change of variables from Cartesian to Spherical Coordinates.

Example 17: Evaluate $\iiint_V e^{(x^2+y^2+z^2)^{3/2}}\,dV$ where V is the sphere located at the origin with radius 1.

Change of Coordinates to Polar, Cylindrical, and Spherical Form:

Polar: $\iint_R f(x, y)\,dx\,dy = \iint_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

Cylindrical: $\iiint_V f(x, y, z)\,dx\,dy\,dz = \iiint_W f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$

Spherical:

$$\iiint_V f(x, y, z)\,dx\,dy\,dz = \iiint_W f(\rho\sin\phi\cos\theta, r\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

4.2 Review of Concepts

- Terms to know: multiple integral, Jacobian
- Know how to evaluate multiple integrals.
- Know how to set up multiple integrals over general regions.
- Know how to make a change of variable and represent an integral in a different coordinate system.

4.2 Practice Problems

1. Evaluate the following multiple integrals:

a) $\displaystyle\int_1^2\int_0^\pi y\sin(xy)\,dy\,dx$

b) $\displaystyle\int_0^1\int_0^1 \frac{1+x^2}{1+y^2}\,dy\,dx$

c) $\displaystyle\int_0^1\int_0^1 xy\sqrt{x^2+y^2}\,dy\,dx$

d) $\displaystyle\int_0^1\int_0^z\int_0^y ze^{-y^2}\,dx\,dy\,dz$

e) $\displaystyle\int_0^{\pi/2}\int_0^{\cos\theta} e^{\sin\theta}\,dr\,d\theta$

2. Evaluate $\iint_R xy^2\,dA$ where R is the region enclosed by $x=0$ and $x=\sqrt{1-y^2}$.

3. Set up $\iint_R f(x,y)\,dA$ where R is the region enclosed by the lines $x=0$, $x+y=2$, and $x=y$ in two different ways.

4. Change the order of integration of the integral $\displaystyle\int_1^2\int_0^{\ln x} f(x,y)\,dy\,dx$.

5. Set up double and triple integrals that would each give the volume enclosed by the three coordinate planes and the plane $2x+y+3z=6$ in the first quadrant.

6. Set up the triple integrals that find the volume of the region above the xy-plane bounded by the surfaces $x^2+y^2=1$, $z=0$ and $z=-y$.

7. Evaluate $\iiint_V e^{\sqrt{x^2+y^2+z^2}}\,dV$ where V is the region in the first octant enclosed by the sphere $x^2+y^2+z^2=9$.

4.2 Exercises

1. Evaluate the following multiple integrals:

a) $\displaystyle\int_0^2\int_y^{2y} xy\,dx\,dy$

b) $\displaystyle\int_0^4\int_{\sqrt{x}}^2 \frac{1}{1+y^3}\,dy\,dx$

c) $\displaystyle\int_0^1\int_x^1 e^{x/y}\,dy\,dx$

d) $\displaystyle\int_0^3\int_0^1\int_0^{\sqrt{1-z^2}} ze^y\,dx\,dz\,dy$

e) $\displaystyle\int_0^1\int_x^{2x}\int_0^y 2xyz\,dz\,dy\,dx$

f) $\displaystyle\int_0^1\int_0^z\int_0^{x+z} 6xz\,dy\,dx\,dz$

g) $\displaystyle\int_0^{\sqrt{\pi}}\int_0^x\int_0^{xz} x^2\sin y\,dy\,dz\,dx$

h) $\displaystyle\int_1^e\int_1^x\int_0^{x+y} \frac{1}{x}\,dz\,dy\,dx$

2. Let R be the region in the xy-plane bounded by the line $x=2$, the line $y=2$, and the line $y=2-x$. Set up the double integral $\displaystyle\iint_R f(x,y)\,dA$ two different ways.

3. Change the order of integration of the integral $\displaystyle\int_0^4\int_{y/2}^2 e^{x^2}\,dx\,dy$ and evaluate the integral.

4. Evaluate $\displaystyle\iint_R x\cos y\,dA$ where R is the region bounded by $y=0$, $y=x^2$, and $x=1$.

5. Evaluate $\displaystyle\iint_R e^{-x^2-y^2}\,dA$ where R is the region bounded by the semicircle $x=\sqrt{4-y^2}$ and the y-axis.

6. Evaluate $\iiint_V y\,dV$ where V is the region bounded by the planes $x=0$, $y=0$, $z=0$, and $2x+2y+z=4$.

7. Evaluate $\iiint_V z\,dV$ where V is the region bounded by the cylinder $y^2+z^2=9$ and the planes $x=0$, $y=3x$, and $z=0$ in the first octant.

8. Evaluate $\iiint_V xyz\,dV$ where V is the region between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=16$ and above the cone $z^2=x^2+y^2$.

9. Evaluate $\iint_R \cos\left(\frac{y-x}{y+x}\right)\,dA$ where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2)$, and $(0,1)$.

10. One way to view the triple integral over all of $\mathbb{R}^3$ is to view it as the triple integral over a sphere as the radius of the sphere goes to infinity. Using this idea, convert $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dx\,dy\,dz$ to an integral in spherical coordinates and use it to evaluate

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}\,e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz.$$

4.2 Answers to Practice Problems

1. Evaluate the following multiple integrals:

a) $\int_1^2\int_0^\pi y\sin(xy)\,dy\,dx$

Note that the integration is easier if we integrate with respect to x first, as if $u=xy$, then $du=y\,dx$ so $du/y=dx$.

$$\int_1^2\int_0^\pi y\sin(xy)\,dy\,dx = \int_0^\pi\left(\int_1^2 y\sin(xy)\,dx\right)dy = \int_0^\pi\left(-\cos(xy)\Big|_1^2\right)dy$$
$$= \int_0^\pi\left(-\cos(2y)+\cos(y)\right)dy = \left(-\frac{1}{2}\sin(2y)+\sin(y)\Big|_0^\pi\right) = 0.$$

b) $\int_0^1\int_0^1 \frac{1+x^2}{1+y^2}\,dy\,dx$

We can factor the integrand into $(1+x^2)\cdot\frac{1}{1+y^2}$. Thus we have

$$\int_0^1\int_0^1 \frac{1+x^2}{1+y^2}\,dy\,dx = \left(\int_0^1 (1+x^2)\,dx\right)\left(\int_0^1 \frac{1}{1+y^2}\,dy\right) = \left(\left(x+\frac{x^3}{3}\right)\Big|_0^1\right)\left(\arctan y\Big|_0^1\right)$$
$$= \left(1+\frac{1}{3}-0-0\right)\left(\frac{\pi}{4}-0\right) = \frac{\pi}{3}.$$

c) $\int_0^1\int_0^1 xy\sqrt{x^2+y^2}\,dy\,dx$

Note that if $u = x^2 + y^2$, then $du/2 = y\,dy$. Also we will use $u = 1 + x^2$ so $du/2 = x\,dx$. Then

$$\int_0^1 \int_0^1 xy\sqrt{x^2+y^2}\,dy\,dx = \int_0^1 \left(\frac{1}{2}\int_{x^2}^{x^2+1} x\sqrt{u}\,du\right) dx = \int_0^1 \left(\frac{1}{2}\cdot \frac{2xu^{3/2}}{3}\bigg|_{x^2}^{x^2+1}\right) dx$$
$$= \frac{1}{3}\int_0^1 \left(x(x^2+1)^{3/2} - x^4\right) dx = \frac{1}{3}\left(\frac{1}{2}\cdot\frac{2(x^2+1)^{5/2}}{2} - \frac{x^5}{5}\right)\bigg|_0^1 = \frac{2^{5/2}}{15} - \frac{2}{15}.$$

d) $\displaystyle\int_0^1 \int_0^z \int_0^y ze^{-y^2}\,dx\,dy\,dz$

Note that if $u = -y^2$, then $-du/2 = y$. We have

$$\int_0^1 \int_0^z \int_0^y ze^{-y^2}\,dx\,dy\,dz = \int_0^1 \int_0^z \left(xze^{-y^2}\Big|_0^y\right) dy\,dz = \int_0^1 \int_0^z yze^{-y^2}\,dy\,dz$$
$$= \int_0^1 \left(\int_0^z yze^{-y^2}\,dy\right) dz = \int_0^1 \left(-\frac{1}{2}ze^{-y^2}\Big|_0^z\right) dz$$
$$= -\frac{1}{2}\int_0^1 z\left(e^{-z^2} - 1\right) dz = \left(\frac{e^{-z^2}}{4} + \frac{z^2}{4}\right)\bigg|_0^1 = \frac{1}{4e} - \frac{1}{4} + \frac{1}{4} = \frac{1}{4e}.$$

e) $\displaystyle\int_0^{\pi/2} \int_0^{\cos\theta} e^{\sin\theta}\,dr\,d\theta$

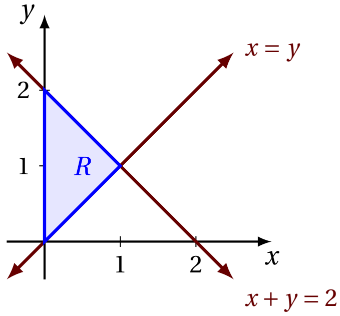
$$\int_0^{\pi/2} \int_0^{\cos\theta} e^{\sin\theta}\,dr\,d\theta = \int_0^{\pi/2} \left(re^{\sin\theta}\Big|_0^{\cos\theta}\right) d\theta = \int_0^{\pi/2} \cos\theta e^{\sin\theta}\,d\theta = e^{\sin\theta}\Big|_0^{\pi/2} = e - 1.$$

2. Evaluate $\iint_R xy^2\,dA$ where R is the region enclosed by $x = 0$ and $x = \sqrt{1-y^2}$.

Note that $x = \sqrt{1-y^2}$ is the part of the unit circle $x^2 + y^2 = 1$ to the right of the y-axis, so R is the right half of the unit circle. Then we can convert this integral to polar coordinates:

$$\iint_R xy^2\,dA = \int_{-\pi/2}^{\pi/2} \int_0^1 (r\cos\theta)(r^2\sin^2\theta) r\,dr\,d\theta = \left(\int_{-\pi/2}^{\pi/2} \cos\theta\sin^2\theta\,d\theta\right)\left(\int_0^1 r^4\,dr\right)$$
$$= \left(\frac{\sin^3\theta}{3}\bigg|_{-\pi/2}^{\pi/2}\right)\left(\frac{r^5}{5}\bigg|_0^1\right) = \frac{1}{3}\left(\sin^3\frac{\pi}{2} - \sin^3\left(-\frac{\pi}{2}\right)\right)\left(\frac{1}{5}\right) = \frac{2}{15}.$$

3. Set up $\iint_R f(x,y)\,dA$ where R is the region enclosed by the lines $x = 0$, $x + y = 2$, and $x = y$ in two different ways.



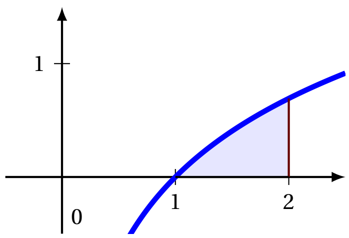
If we integrate with respect to y first, we can do this in a single integral:

$$\iint_R f(x,y)\,dA = \int_0^1 \int_x^{2-x} f(x,y)\,dy\,dx.$$

However, if we integrate with respect to x first, we need two integrals:

$$\iint_R f(x,y)\,dA = \int_0^1 \int_0^{y} f(x,y)\,dx\,dy + \int_1^2 \int_0^{2-y} f(x,y)\,dx\,dy.$$

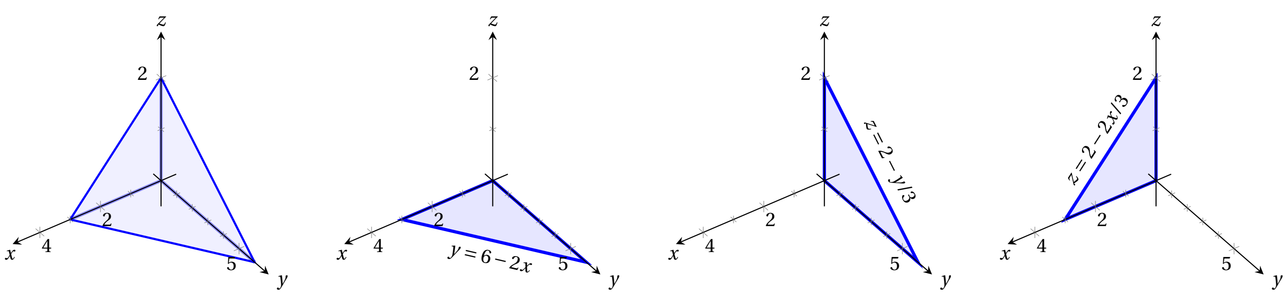
4. Change the order of integration of the integral $\displaystyle\int_1^2 \int_0^{\ln x} f(x,y)\,dy\,dx$.



The curve $y = \ln x$ is also the curve $x = e^y$. Thus, to change the order of integration we have

$$\int_1^2 \int_0^{\ln x} f(x,y)\,dy\,dx = \int_0^{\ln 2} \int_{e^y}^{2} f(x,y)\,dx\,dy.$$

5. Set up double and triple integrals that would each give the volume enclosed by the three coordinate planes and the plane $2x + y + 3z = 6$ in the first quadrant.



Looking a the figure on the left, we see that to find the volume we would just need to evaluate $\iint_R z\,dA$ where $z = 2 - x/3 - y/6$. This region is triangular in nature, and we would have

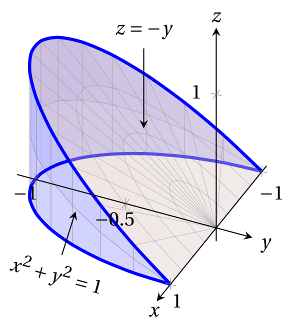
$$\iint_R z\,dA = \int_0^3 \int_0^{6-2x} \left(2 - \frac{x}{3} - \frac{y}{6}\right) dy\,dx = \int_0^6 \int_0^{3-y/2} \left(2 - \frac{x}{3} - \frac{y}{6}\right) dx\,dy.$$

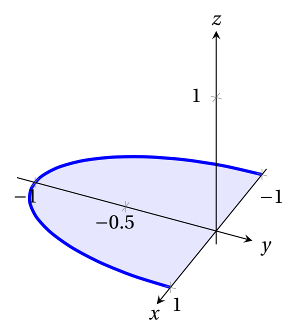
Otherwise, the volume can be computed by evaluating $\iiint_V 1\,dV$, which can be done in several ways, with

$$\iiint_V 1\,dV = \int_0^3 \int_0^{6-2x} \int_0^{2-x/3-y/6} 1\,dz\,dy\,dx,$$

being one way.

6. Set up the triple integrals in Cartesian and cylindrical coordinates that find the volume of the region above the xy-plane bounded by the surfaces $x^2 + y^2 = 1$, $z = 0$ and $z = -y$.





Note that the region of integration in the xy-plane is the bottom half of the unit circle. In Cartesian coordinates, we can find the volume using

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{0} \int_{0}^{-y} 1\, dz\, dy\, dx.$$

Converting to cylindrical coordinates, this becomes

$$\int_{\pi}^{2\pi} \int_{0}^{1} \int_{0}^{-r\sin\theta} r\, dz\, dr\, d\theta.$$

7. Evaluate $\iiint_V e^{\sqrt{x^2+y^2+z^2}}\, dV$ where V is the region in the first octant enclosed by the sphere $x^2 + y^2 + z^2 = 9$.

In the first octant, ϕ ranges from 0 to $\pi/2$, as does θ. Thus the integral is

$$\begin{aligned}
\iiint_V e^{\sqrt{x^2+y^2+z^2}}\, dV &= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 e^{\rho} \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi \\
&= \left(\int_0^{\pi/2} \sin\phi\, d\phi \right) \left(\int_0^{\pi/2} 1\, d\theta \right) \left(\int_0^3 e^{\rho} \rho^2\, d\rho \right) \\
&= \left(-\cos\phi \Big|_0^{\pi/2} \right) \left(\frac{\pi}{2} \right) \left((\rho^2 - 2\rho + 2) e^{\rho} \Big|_0^3 \right) \\
&= (-0+1) \left(\frac{\pi}{2} \right) \left((3^2 - 2\cdot 3 + 2) e^3 - 2e^0 \right) = \frac{\pi\left(5e^3 - 2\right)}{2}.
\end{aligned}$$

4.3 Parametric Curves and Vector-valued Functions

Objectives and Concepts:

- Parametric description of curves in the plane and in space
- Vector-valued functions: $\vec{F} : \mathbb{R} \to \mathbb{R}^n$ and $\vec{F} : [\alpha, \beta] \to \mathbb{R}^n$
- Calculus with curves, like definite integrals $\int_\alpha^\beta \vec{F}(t)\,dt$

References: OSC-3 §1.1, §1.2, §3.1, & §3.2; CET §§10.1,2, 13.1,2; TCMB §§9.8, 16.4.

4.3.1 Curves Described by Parametric Equations

References: OSC-3 §1.1

Many curves in the x-y plane cannot be described as the graph of a function, either $y = F(x)$ or $x = G(y)$: circles for example. A more flexible and general approach is to describe a curve C by specifying both the x and y coordinates as functions of another variable; a **parameter**, often called t and often with the meaning of time, or of distance traveled along the curve:

$$x = f(t),\ y = g(t).$$

The t values might be limited to an interval $\alpha \le t \le \beta$, in which case the curve has **initial point** $(a, b) = (f(\alpha), g(\alpha))$ and **terminal point** $(c, d) = (f(\beta), g(\beta))$. These are both **end points**, and can be the same, to describe a closed curve. Other common domains for the parameter are infinite intervals like $[\alpha, \infty)$ or $(\infty, \infty), = \mathbb{R}$.

Example 1: The equations

$$x = t\cos t,\ y = t\sin t, \quad t \ge 0$$

describe a spiral with each point at a distance t from the origin at time t, and rotating at a steady rate of one radian per unit time. In polar coordinates, it could also be described as

$$r = t, \theta = t, \quad t \ge 0$$

The same can be done in three dimensions of course, with

$$x = f(t),\ y = g(t),\ z = h(t).$$

Example 2: Describe in words the parametric space curve given by

$$x = t\cos t,\ y = t\sin t,\ z = t,\ t \geq 0$$

Hint: it looks nicer in cylindrical coordinates!

Example 3: A useful example for many topics is the **cycloid**; the trajectory of a point on a circle as it rolls along the x-axis in the x-y-plane; one revolution can be parametrized as

$$x = r(t - \sin t),\ y = r(1 - \cos t), 0 \leq t \leq 2\pi,$$

but note that it can continued for all t.

Sketch this, and then go a bit beyond both ends; say $-\pi/2 \leq t \leq 5\pi/2$.

4.3.2 Vector-valued Functions

Reference: OSC-3 §3.1

The above two and three-dimensional cases can be consolidated by using position vectors $\vec{r} = \langle x, y\rangle$ or $\vec{r} = \langle x, y, z\rangle$ that depend on a parameter t; for example, in 3D,

$$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle = \vec{\imath} f(t) + \vec{\jmath} g(t) + \vec{k} h(t).$$

Loosely, we can also think of these as "vectors", each of whose components is a function of t instead of a number.

Many familiar concepts of calculus like limits, continuity, derivatives and integrals extend simply to such vector functions, and they also have the nice geometrical significance of describing curves in the plane or in space.

Limits of Vector Functions. A limit of a vector function is built up from limits of its components, so that for example in 2D,

$$\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t)\right\rangle$$

Such a limit thus exists if and only if each of the component limits exist.

Continuity. With this notion of a limit, it makes sense to say that vector function $\vec{r}(t)$ is **continuous at *a*** if the limit exists there and equals the value there:

$$\lim_{t\to a} \vec{r}(t) = \vec{r}(a),$$

which is true if and only if all of the component functions are continuous at a.
Such a function is called simply **continuous** if it is continuous at each point a in its domain.

The Visual Meaning of Continuity. Continuity has a familiar geometrical meaning:

Intuitively, a vector function $\vec{r}(t)$ is continuous if the curve it describes has no "breaks". Thus, a continuous parametric plane curve is one that could be drawn without raising one's pen from the page.

Note that, when drawing parametric curves, arrowheads are typically used on the curve to indicate the direction of motion as the parameter value increases.

Example 4: Sketch the curve $x = \sin 2t,\ y = \cos 2t, \quad 0 \le t \le 2\pi$

Example 5: Find parametric equations for the circle with center (x_0, y_0) and radius R.

4.3.3 Derivatives and Integrals of Vector-valued Functions

Reference: OSC-3 §1.2 and §3.2

We can build the derivative of a vector function from the derivatives of its components, but the definition can also be done from first principles, with difference quotients:

Definition: The **derivative** $\vec{r}\,'$ of a vector function of variable t is given by

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

It can be checked that the derivative exists if and only if each of the component derivatives exist, and then it is the vector of derivatives of the components; for example, in 2D

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\big\langle f(t), g(t)\big\rangle = \big\langle f'(t), g'(t)\big\rangle.$$

Note: when $\vec{r}(t)$ gives the position vector of an object as a function of time t, $\vec{v}(t) = \vec{r}\,'(t)$ is its *velocity* and $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$ is its *acceleration*. *Speed* $s(t)$ is the magnitude of velocity, so $s(t) = \|\vec{v}(t)\| = \|\vec{r}\,'(t)\|$.

Differentiation Rules. The familiar differentiation rules for sums, products and compositions have natural counterparts for vector-valued functions:

Theorem: For $\vec{u}$ and $\vec{v}$ differentiable vector functions, f a differentiable scalar (real-valued) function, and c a scalar constant,

1. $\dfrac{d}{dt}[\vec{u}(t)+\vec{v}(t)] = \vec{u}'(t)+\vec{v}'(t)$
2. $\dfrac{d}{dt}[c\vec{u}(t)] = c\vec{u}'(t)$
3. $\dfrac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t)+f(t)\vec{u}'(t)$
4. $\dfrac{d}{dt}[\vec{u}(t)\cdot\vec{v}(t)] = \vec{u}'(t)\cdot\vec{v}(t)+\vec{u}(t)\cdot\vec{v}'(t)$
5. $\dfrac{d}{dt}[\vec{u}(t)\times\vec{v}(t)] = \vec{u}'(t)\times\vec{v}(t)+\vec{u}(t)\times\vec{v}'(t)$
6. $\dfrac{d}{dt}[\vec{u}(f(t))] = \vec{u}'(f(t))f'(t)$

As always with the cross product, the order matters in item 5.

Definite Integrals. Like derivatives, definite integrals of vector functions can be built from first principles (using Riemann sums and limits), and one gets the predictable result in terms of integrals of components, so for example:

$$\text{For } \vec{r}(t) = \big\langle f(t), g(t)\big\rangle,\ \int_{t=\alpha}^{\beta} \vec{r}(t)\,dt = \left\langle \int_{t=\alpha}^{\beta} f(t)\,dt, \int_{t=\alpha}^{\beta} g(t)\,dt \right\rangle$$

Note: here and in many places below, I will state the dummy variable of a integral explicitly in the limits of definite integrals and evaluations with notation like $\displaystyle\int_{t=\alpha}^{\beta} f(t)\,dt$ and $\Big[F(x)\Big]_{t=\alpha}^{\beta}$. This is because there will be many changes of variable (substitutions) and things can otherwise get confusing or ambiguous.
I actually like this in general, to avoid ambiguity; it also matches style with summation notation $\displaystyle\sum_{i=m}^{n} a_i$.

The Fundamental Theorem of Calculus for Vector Valued Functions. The following should be no surprise:

For $\vec{r}(t) = \langle r_1(t), r_2(t), \ldots\rangle$, if R_1, R_2, etc. are any anti-derivatives of the components $r_1(t)$, $r_2(t)$ etc., then $\vec{R}(t) = \langle R_1(t), R_2(t), \ldots\rangle$ is an antiderivative of $\vec{r}$; that is, $\vec{R}' = \vec{r}$. Hence,

$$\int_{t=\alpha}^{\beta} \vec{r}(t)\,dt = \vec{R}(\beta) - \vec{R}(\alpha) = \Big[\vec{R}(t)\Big]_{t=\alpha}^{\beta}$$

Other integral rules can be extended in equally intuitive ways, but it is best to deal with such integration problems as a collection of separate integrals, one for each component function.

Example 6: Evaluate

$$\bar{\vec{R}} = \frac{1}{2\pi}\int_{t=0}^{2\pi} \vec{r}(t)\,dt,$$

the "average position" along the spiral curve $\langle x, y\rangle = \langle t\cos t, t\sin t\rangle$ from Example 1. As a "sanity check", plot this point on the graph of that curve, and decide if it is plausible as an average.

Solution:

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \langle t\cos t, t\sin t\rangle\,dt &= \frac{1}{2\pi}\left([\langle t\sin t, -t\cos t\rangle]_0^{2\pi} - \int_0^{2\pi} \langle \sin t, -\cos t\rangle\,dt\right) \\
&= \frac{1}{2\pi}\left([\langle t\sin t, -t\cos t\rangle]_0^{2\pi} + [\langle \cos t, \sin t\rangle]_0^{2\pi}\right) \\
&= \frac{1}{2\pi}\left(\langle 0, -2\pi\rangle\right) = \langle 0, -1\rangle \\
&= \langle 0, -1\rangle
\end{aligned}$$

4.3 Review of Concepts

- Terms to know: parameter, parametric equations, initial and terminal points, end points
- Know how to evaluate derivatives and integrals of vector-valued functions

Examples and Exercises from OSC Volume 3

- Review the Examples in §1.1
- Study Examples 4 and 5 in§1.2, and the Checkpoints after each.
- Study all the Examples and Checkpoints in §3.1
- Study up to Example 6 and Checkpoint 6 in §3.2

4.3 Exercises

1. (CGS 12.1.1) Sketch the curve with parametric equations $x = t$, y=t^3. Find the velocity vector and the speed at $t = 1$.

2. (CGS 12.1.2) Sketch the path with parametric equations $x = 1 + t$, y=$1 - t$. Find the xy equation of the path and the speed along it.

3. (CGS 12.1.3) On the circle $x = \cos t$, $y = \sin t$, explain by the chain rule and then by geometry why $dy/dx = -\cot t$.

4. (CGS 12.1.5) Find the slope at any point (x, y) on the curve $x = e^t$, $y = e^{-t}$ without solving for the xy equation of this curve. Then find that xy equation, and verify your previous result.

5. (CGS 12.1.7) First find a parametric form for the line through points $P(1,2,4)$ and $Q(5,5,4)$. Probably your curve has speed 5: reparametrize to have speed 10. Probably your curve has initial point P; reparametrize to have initial point Q.

For further examples and exercises, look at “Calculus” by Gilbert Strang [CGS], Section 12.1 (in particular, Examples 5–8) and TCMB Sections 9.8 and 16.4.

4.4 Arc Length, Line Integrals, and Path Integrals

Objectives and Concepts:

- Compute the length of a curve.
- Integrate (sum) a quantity along of curve, like work done by a force.

References: OSC-3 §3.3 & §6.2; CET §§10.2, 13.3, 16.2,3; TCMB §9.8.

4.4.1 The Length of a Plane Curve; its Arc Length

Reference: OSC-3 §3.3

Just as calculus was used to compute the areas of regions with curved boundaries, starting with an approximation by the area of a collection of polygons, we will now make sense of the length of a curve through approximating a curve by a collection of short straight line segments, and so approximate its length by the total length of that collection, and then take a limit to get the exact result. The resulting length is called the *arc length* of the curve.

First, we do this for curves in parametric curves in the plane: a curve C that is the set of points (x, y) given by $x = f(t)$, $y = g(t)$ for $\alpha \le t \le \beta$. Below, we will then consider the length of curves in space, and indeed for curves in any n-dimensional space given by vector-valued functions.

Approximation by Polygonal Curves. The first step is familiar: divide the range of t values into n small intervals with values at spacing $\Delta t = (\beta - \alpha)/n$: $t_0, t_1, \ldots, t_n$ with $t_i = \alpha + i\Delta t$. (Note that $t_0 = \alpha$ and $t_n = \beta$.)

This then gives a sequence of points along the curve: $P_i(x_i, y_i)$ with $x_i = f(t_i)$, $y_i = g(t_i)$, $0 \le i \le n$.

Joining these points with line segments gives a "polygonal curve", of length

$$L_n = \sum_{i=1}^{n} |P_{i-1}P_i| = \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + \left(y_i - y_{i-1}\right)^2} = \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + \left(\Delta y_i\right)^2}$$

where $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$ are the horizontal and vertical steps respectively on the i-th edge.

Curve Length as a Limit. The exact length of the curve C is then defined as the limit of these approximations:

$$L = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \tag{4.1}$$

As with areas and volumes, we want to turn this into a definite integral in the parameter t by using the Fundamental Theorem of Calculus (Part 2). To get the needed factor of Δt, rearrange as

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t}\right)^2 + \left(\frac{\Delta y_i}{\Delta t}\right)^2}\,\Delta t$$

The difference quotients are approximately values of $f'(t_i)$ and $g'(t_i)$ respectively;
using the Mean Value Theorem, we can get

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\left(f'(t_i^*)\right)^2 + \left(g'(t_i^*)\right)^2}\,\Delta t, \quad \text{for suitable values } t_i^* \in (t_{i-1}, t_i).$$

Curve Length as an Integral. Finally, the Fundamental Theorem of Calculus gives:

> If $f'(t)$ and $g'(t)$ are continuous on $[\alpha, \beta]$ then the curve C of points $P(x, y)$ given by $x = f(t)$ and $y = g(t)$ for $\alpha \le t \le \beta$ has arc length
>
> $$L = \int_{t=\alpha}^{\beta} \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2}\,dt = \int_{t=\alpha}^{\beta} \sqrt{(x')^2 + (y')^2}\,dt = \int_{t=\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

The Arc Length Function. Sometimes we are interested not just in the total length, but the length of the part of a curve from the initial point up to a certain value of the parameter t. This will be useful for example when the curve is describing motion in time, and we want to compute distance traveled as a function of time.

Changing the dummy variable of integration from t to τ, the length of the curve from the initial point where $t = \alpha$ to the point where $t = x$ is a function of x, the **arc length function** is

$$s(t) = \int_{\tau=\alpha}^{t} \sqrt{\left(f'(\tau)\right)^2 + \left(g'(\tau)\right)^2}\,d\tau = \int_{\tau=\alpha}^{t} \sqrt{(x')^2 + (y')^2}\,d\tau$$

Arc Length Derivative and Differential. The rate of change of arc length as t increases is given by the Fundamental Theorem of Calculus (Part 1) as

$$\frac{ds}{dt} = \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

Note: if t is time, this is the speed of motion along the curve.

The differential of the arc length function can be useful; it is

$$ds = \frac{ds}{dt}dt = \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2}\,dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

which has a nicely mnemonic form if we square both sides:

$$(ds)^2 = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right](dt)^2 = \left(\frac{dx}{dt}dt\right)^2 + \left(\frac{dy}{dt}dt\right)^2 = (dx)^2 + (dy)^2.$$

So formally we have

$$ds = \sqrt{(dx)^2 + (dy)^2}, \tag{4.2}$$

which is intuitively what the Pythagorean Theorem gives for the length of a small piece of the curve of horizontal extent dx, vertical extent dy.

Arc Length Mnemonic: Integrate the Arc Length Differential. Perhaps the best way to remember the arc length formulas is to first remember this intuitive arc length differential formula (4.2) and then integrate:

$$s(t) = \int_{\tau=\alpha}^{t} ds = \int_{\tau=\alpha}^{t} \sqrt{\left(f'(\tau)\right)^2 + \left(g'(\tau)\right)^2}\, d\tau = \int_{\tau=\alpha}^{t} \sqrt{(x'(\tau))^2 + \left(y'(\tau)\right)^2}\, d\tau \tag{4.3}$$

$$L = s(\beta) = \int_{t=\alpha}^{\beta} ds = \int_{t=\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{4.4}$$

The Length of a Curve that is the Graph of a Function. Two common and convenient cases are when the curve is the graph of a function, like $y = f(x)$. Then the input argument x can be used at the parameter in place of t, which gives

$$ds = \sqrt{1 + \left(f'(x)\right)^2}\, dx = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{4.5}$$

$$s(x) = \int_{\tau=a}^{x} ds = \int_{\tau=a}^{x} \sqrt{1 + \left(f'(\tau)\right)^2}\, d\tau \tag{4.6}$$

$$L = \int_{t=a}^{b} ds = \int_{t=a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{4.7}$$

Example 1: Compute the length of the circumference of a circle of radius r, center (x_0, y_0), parametrized as

$$x = x_0 + r\cos t,\; y = y_0 + r\sin t,\; 0 \le t \le 2\pi.$$

Example 2: Compute the arc length of one arch of the cycloid $x = r(t - \sin t),\; y = r(1 - \cos t), 0 \le t \le 2\pi$.

4.4.2 The length of a Parametric Curve in Space, and in $\mathbb{R}^n$

Reference: OSC-3 §3.3

The procedure above can be applied to compute the length of a smooth parametrized curve in space, $x = f(t)$, $y = g(t)$, $z = h(t)$. The key formulas above become

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right](dt)^2 \tag{4.8}$$

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \qquad \text{the arc length differential} \tag{4.9}$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \qquad \text{the “speed”} \tag{4.10}$$

$$s(t) = \int_{\tau=\alpha}^{t} ds = \int_{\tau=\alpha}^{t} \sqrt{(x'(\tau))^2 + \left(y'(\tau)\right)^2 + (z'(\tau))^2}\, d\tau \qquad \text{the arc length function} \tag{4.11}$$

$$L = \int_{\tau=\alpha}^{\beta} ds = \int_{\tau=\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \qquad \text{the arc length} \tag{4.12}$$

The plane and space versions can both be put into a single compact form using the position vectors $\vec{r}(t) = \langle f(t), g(t)\rangle$ or $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$:

$$ds = \left\| \frac{d\vec{r}}{dt} \right\| dt \qquad \text{the arc length differential} \tag{4.13}$$

$$\frac{ds}{dt} = \left\| \frac{d\vec{r}}{dt} \right\| \qquad \text{the “speed”} \tag{4.14}$$

$$s(t) = \int_{\tau=\alpha}^{t} ds = \int_{t=\alpha}^{t} \|\vec{r}'(\tau)\|\, d\tau \qquad \text{the arc length function} \tag{4.15}$$

$$L = \int_{t=\alpha}^{\beta} ds = \int_{t=\alpha}^{\beta} \|\vec{r}'(t)\|\, dt \qquad \text{the arc length} \tag{4.16}$$

All if this requires the existence of the derivative; more precisely, if the curve can be broken into a finite number of differentiable pieces, their lengths can be added, so that is good enough. Thus the following definitions characterize curves that are “sufficiently nice”:

Definition: A space curve is **smooth** if it is given by $\vec{r}(t)$ on interval I with both $\vec{r}$ and $\vec{r}'$ continuous, and with $\vec{r}' \neq \vec{0}$ except possibly at the endpoints of I.
If the curve is continuous and its derivative is zero or non-existent at only a finite number of points, the curve is called **piecewise smooth**.

Concept Check: Why is the condition $\vec{r}' \neq \vec{0}$ is important for smoothness?

4.4.3 Line Integrals in the Plane

Reference: OSC-3 §6.2

It is often useful to "sum" (integrate) a quantity F along a curve, with the simplest example being the above computation of arc length. One important physical application is summation of force to get the work done as an object moves along a curve; another will be the change in entropy as a system like a gas or chemical solution changes state.

If a plane curve C is given parametrically as

$$x = x(t),\ y = y(t), \quad \alpha \le t \le \beta \tag{4.17}$$

or in vector form $\vec{r}(t) = \vec{\imath}x(t) + \vec{\jmath}y(t)$, then one can start by considering sums of the form

$$\sum_{i=1}^{n} F(x_i^*, y_i^*)\Delta s_i$$

where the interval $[\alpha, \beta]$ of t values is divided into n subintervals $[t_{i-1}, t_i]$, $1 \le i \le n$,
each point $\left(x_i^*, y_i^*\right) = \left(x(t_i^*), y\left(t_i^*\right)\right)$ is given by some t_i^* in $[t_{i-1}, t_i]$ and so lies on the i-th sub-arc,
and $\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$ is the approximate length of that sub-arc.

The now familiar limit process turns these approximations into an integral along the curve:

Definition: If F is defined at each point of a smooth curve C given by Eq. (4.17) then the **line integral of f along C** is

$$\int_C F(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*, y_i^*)\Delta s_i \tag{4.18}$$

This is also called the **line integral of F along C with respect to arc-length.**

Next, we can express this in terms of an integral in variable t over the interval $[\alpha, \beta]$.

Line Integral Formula. For $F(x, y) = 1$, the limit in Eq. (4.18) is the formula (4.1) for arc-length of a curve in the plane and it was seen there that in effect,

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \tag{4.19}$$

The same argument here gives

$$\int_C F(x,y)\,ds = \int_{t=\alpha}^{\beta} \left(F(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \right) dt \tag{4.20}$$

4.4.4 Integrals Along *Paths* in the Plane: Piecewise Smooth Curves

Reference: OSC-3 §6.2

A **path** is another name for a piecewise smooth curve C: a collection of smooth curves $C_1, C_2, \dots$ that join end to end. These are sometimes denoted $C = C_1 + C_2 + \cdots$

The **path integral** along a path C is simply the sum of the line integrals along each smooth piece.

Line and Path Integrals with Respect to the Coordinates, x and y. In some situations, the quantity to be summed is $F(x,y)\,\Delta x$ or $F(x,y)\,\Delta y$: for example if f is a force acting on an object in the x direction, $F(x,y)\,\Delta x$ is the work done by that force when the object moves distance Δx in that direction (no work is associated with movement perpendicular to the force).

Limits of sums give the work done by such a force in the x-direction as an object moves along curve C to be

$$\int_C F(x,y)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*, y_i^*)\Delta x_i \tag{4.21}$$

(or to be careful with piecewise smooth curves, the sum of such integrals for each smooth piece).
This is called the **integral of *F* along *C* with respect to *x***.

Much as with the line integral formula (4.20), these integrals can be computed with

$$\int_C F(x,y)\,dx = \int_{t=\alpha}^{\beta} \left(F(x(t), y(t)) \frac{dx}{dt} \right) dt \tag{4.22}$$

When the curve C is the graph of a function, $y = f(x)$, the parameter can just be x and the integral simplifies through the substitution rule:

$$\int_C F(x,y)\,dx = \int_{t=\alpha}^{\beta} F(x, f(x)) \frac{dx}{dt} dt = \int_{x=a}^{b} F(x, f(x))\,dx \tag{4.23}$$

Similarly the **integral of *F* along *C* with respect to *y*** is

$$\int_C F(x,y)\,dy = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*, y_i^*)\Delta y_i = \int_{t=\alpha}^{\beta} \left(F(x(t), y(t)) \frac{dy}{dt} \right) dt \tag{4.24}$$

and when the curve is a "sideways graph" of the form $x = g(y)$,

$$\int_C F(x,y)\,dy = \int_{t=\alpha}^{\beta} F(g(y), y) \frac{dy}{dt} dt = \int_{y=c}^{d} F(g(y), y)\,dy \tag{4.25}$$

Note: in practice, these are often easier to evaluate than line integrals like (4.20) due to the absence of the arc length square root term.

Paired Line Integrals, Vector Form, and Work Done by a Force. One often gets a sum of line integrals in both x and y, and these can be abbreviated

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy \tag{4.26}$$

This combination can be further abbreviated with a dot product notation:

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_C \langle P(x,y), Q(x,y)\rangle \cdot \langle dx, dy\rangle = \int_C \vec{F}\cdot d\vec{r}$$

where $\vec{F} = \langle P(x,y), Q(x,y)\rangle$, $\vec{r} = \langle x, y\rangle$, and we use the short-hand "$d\vec{r} = d\langle x,y\rangle = \langle dx, dy\rangle$".

Note: this function $\vec{F}(x,y)$ or $\vec{F}(\vec{r})$ is our first example of a *vector-valued function of several variables*, sometimes called a **vector field**; we will see more about these in Subsection 4.5.2.

Example 3: When P and Q are the components in the x and y directions of a force vector $\vec{F}$, this gives the total work done by the force as the object traverses curve C.

Reversing the Orientation of a Curve. A curve C described by a parameterization of a curve $x = f(t)$, $y = g(t)$ $\alpha \le t \le \beta$ has an *orientation*, meaning a direction of motion from initial point $(x(\alpha), y(\alpha))$ to final point $(x(\beta), y(\beta))$.

If we reverse the orientation, for example with new parameterization $u = -t$ so that $x = f(-u)$, $y = g(-u)$, $-\beta \le u \le -\alpha$, the new curve is denoted $-C$.

This reverses the order of limits on integration so that

$$\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx, \quad \int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy \tag{4.27}$$

However, arc length is not changed by this reversal, and

$$\int_{-C} f(x,y)\,ds = \int_C f(x,y)\,ds.$$

This is because Δs_i remains positive in Eq. (4.18) whereas Δx_i and Δy_i in Equations (4.21) and (4.24) change sign.

4.4.5 Line and Path Integrals in Space

Reference: OSC-3 §6.2

The above concepts extend easily to space curves C, given parametrically as

$$x = x(t),\ y = y(t),\ z = z(t), \quad \alpha \le t \le \beta \tag{4.28}$$

or in vector form, $\vec{r}(t) = \vec{\imath}x(t) + \vec{\jmath}y(t) + \vec{k}z(t)$.

We again define the **line integral of *F* along *C* (with respect to arc-length)** as

$$\int_C F(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*, y_i^*, z_i^*)\Delta s_i$$

where in the sum, the interval $[\alpha,\beta]$ is divided into n (equally long) subintervals with Δs_i the distance between the endpoints of the corresponding part of the curve, and (x_i^*, y_i^*, z_i^*) some point in that part of the curve.

Also as above, we use the Fundamental theorem of Calculus to turn this limit of a sum into a definite integral over interval $[\alpha,\beta]$:

$$\begin{aligned} \int_C F(x,y,z)ds &= \int_{t=\alpha}^{\beta} F(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt \\ &= \int_{t=\alpha}^{\beta} F(\vec{r}(t))\left|\frac{d\vec{r}}{dt}\right| dt. \end{aligned} \tag{4.29}$$

The last vector form is convenient as it covers both the 2D ad 3D versions.

The simple and important special case $F = 1$ gives a compact formula for the arc length of the curve,

$$\int_C ds = \int_{t=\alpha}^{\beta} \left|\frac{d\vec{r}}{dt}\right| dt.$$

Line Integrals With Respect to Space Coordinates. Line integrals with respect to each of the coordinate variables x, y and z can be defined, so that for example

$$\int_C F(x,y,z)\,dz = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*, y_i^*, z_i^*)\Delta z_i = \int_{\alpha}^{\beta} F(x(t),y(t),z(t))\frac{dz}{dt}\,dt \tag{4.30}$$

As with line integrals in the plane, these often occur as sums of one such integral in each variable. Using the shorthands $P(\vec{r})$ for $P(x,y,z)$ and so on, these can be abbreviated as

$$\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = \int_{t=\alpha}^{\beta} \left[P(\vec{r}(t))\frac{dx}{dt} + Q(\vec{r}(t))\frac{dy}{dt} + R(\vec{r}(t))\frac{dz}{dt}\right] dt \tag{4.31}$$

and in vector notation as

$$\int_C \langle P(\vec{r}), Q(\vec{r}), R(\vec{r})\rangle \cdot \langle dx, dy, dz\rangle = \int_{t=\alpha}^{\beta} \vec{F}\cdot d\vec{r} \tag{4.32}$$

with $\vec{r} = \langle x, y, z\rangle$ and $\vec{F} = \langle P, Q, R\rangle$

4.4 Review of Concepts

- Terms to know: arc length, smooth curve, piecewise smooth, line integral, path integral
- Know how to evaluate arc length and other line and path integrals

Examples and Exercises from OSC Volume 3

- In §3.3 study Examples 9 and 10 and (as always), then attempt the Checkpoint items that follow each; do a selection from Exercises 102–112.
- In §6.2 study up to Example 23; do a few Exercises from each of the following groups:
  - 39–43 (T/F),
  - 49 and 51–53 (work integrals, along smooth curves),
  - 55–58, 62, 85 (line integrals, along smooth curves),
  - 65 and 67–70, 72, 73 (more work integrals along smooth curves),
  - 76 and 77 (comparison of two line integrals between the same points),
  - 66, 71 (work "path" integrals, along piecewise smooth curves).

4.4 Exercises

1. (CGS 15.2.1) $\int_C ds$ for C the curve $x = t,\ y = 2t,\ 0 \le t \le 1$.

2. (CGS 15.2.2) $\int_C ds$, $\int_C x\,ds$ and $\int_C xy\,ds$ for C the curve $x = \cos t,\ y = \sin t,\ 0 \le t \le \pi/2$.

3. (CGS 15.2.3) $\int_C xy\,ds$ for C the piecewise smooth curve from $(0,0)$ to $(1,1)$ and then to $(1,0)$; straight between those points.

4. (CGS 15.2.4) $\int_C y\,dx - x\,dy$ for C any square path with sides of length 3.

5. (CGS 15.2.5) $\int_C dx$ and $\int_C y\,dx$ for C any closed circle of radius 3.

6. (CGS 15.2.6) $\int_C (dx/dt))\,dt$ for C any path of length 5.

7. (CGS 15.2.11, 13) Evaluate the "work" integral $W = \int_C \vec{F} \cdot d\vec{r}$ for the following cases. All curves go from $(1,0)$ to $(0,1)$

a) $\vec{F} = \vec{\imath} + y\vec{\jmath}$, C the line $\vec{r}(t) = \langle 1-t, t \rangle$.

b) $\vec{F}$ as above, C the arc $\vec{r}(t) = \langle \cos t, \sin t \rangle$.

c) $\vec{F} = xy^2\vec{\imath} + x^2y\vec{\jmath}$, C the line $\vec{r}(t) = \langle 1-t, t \rangle$.

d) $\vec{F}$ as above, C the arc $\vec{r}(t) = \langle \cos t, \sin t \rangle$.

8. (CGS 15.2.22(b)) Evaluate $\int_C \langle y, -x, z \rangle \cdot d\vec{r}$ along the space curve $x = t,\ y = t^2,\ z = t^3$

For further examples and exercises, look at "Calculus" by Gilbert Strang [CGS], Section 15.2 and TCMB Section 9.8.

4.5 (†) Directional Derivatives, Gradients, and Vector Fields

References: OSC-3 §§4.6, 6.1, and 6.2; CET §§14.1,6, 16.1; TCMB §§9.1, 10.3, 16.7,8

4.5.1 Directional Derivatives and the Gradient

Reference: OSC-3 §4.6

It is natural to ask about the rate of change of a function of several variables as the arguments change in any direction around a point — not just along the coordinate axes as given by the partial derivatives — and to ask questions like in which direction is change the fastest.

Directional Derivatives in the Plane. A direction of change in the plane can be specified by a unit vector $\vec{u} = \langle u_1, u_2 \rangle$, and we can consider how $f(x, y)$ changes in this direction near (x_0, y_0) looking at a "slice" of the function, along the line $\langle x_0, y_0 \rangle + t\vec{u}$. The value of the function along this line is $f(x_0 + tu_1, y_0 + tu_2)$ and its rate of change is given by the Chain Rule as

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

This is the **directional derivative of f at (x_0, y_0)**, denoted $D_{\vec{u}} f(x_0, y_0)$, and it does indeed depend only on the partial derivatives at the point. Note that its value is the same as if one used the linear approximation T at that point in place of f.

Directional Derivatives Defined With Limits. Equivalently, the directional derivative $D_{\vec{u}} f(x_0, y_0)$ can be defined in terms of limits as

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

This exists when f is differentiable at the point (x_0, y_0).

To summarize, for any unit vector $\vec{u} = \langle u_1, u_2 \rangle$ (or indeed any non-zero vector) and any function f differentiable at (x_0, y_0), the directional derivative of f at (x_0, y_0) in direction $\vec{u}$ is

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 = \left\langle f_x(x_0, y_0), f_y(x_0, y_0) \right\rangle \cdot \vec{u}$$

Example 1: Find the directional derivative $D_{\vec{u}} f(x, y)$ of $f(x, y) = x^3 - 3xy + 4y^2$ for $\vec{u}$ the unit vector in direction given by the polar angle $\pi/6$. Then evaluate $D_{\vec{u}} f(1,2)$.

The Gradient Vector. The vector $\langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle$ appearing in the formula for the directional derivative encapsulates all information about directional and partial derivatives of f at (x_0, y_0) in away that has a nice geometrical meaning.

It is called **the gradient vector of f at (x_0, y_0)**, and denoted "grad$f$", "∇f", or "$\vec{\nabla} f$", the latter two sometimes pronounced "del f". Considered as a function of position,

$$\operatorname{grad} f(x,y) = \vec{\nabla} f = \langle f_x(x,y), f_y(x,y)\rangle = \vec{\imath}\frac{\partial f}{\partial x} + \vec{\jmath}\frac{\partial f}{\partial y}.$$

Note: this is another example of a vector valued function of several variables, or vector field; see Subsection 4.5.2 below.

Example 2: Evaluate $\vec{\nabla} f$ for $f(x,y) = x^3 - 3xy + 4y^2$, and then also evaluate $\vec{\nabla} f(0,1)$.

Directional Derivatives in Terms of The Gradient Vector. The directional derivative above can be written as

$$D_{\vec{u}} f(x,y) = \vec{u} \cdot \vec{\nabla} f(x,y),$$

so the gradient contains all information about directional derivatives.

Example 3: Evaluate the directional derivative of function $f(x,y) = x^2y^3 - 4y$ in the direction of the vector $2\vec{\imath} + 5\vec{\jmath}$. Treat this as specifying a pure direction, so normalize the vector.

Directional Derivatives and Gradient of a Function of Three or More Variables. For a differentiable function f of three variables one can likewise define the directional derivative of in direction $\vec{u} = \langle u_1, u_2, u_3\rangle$

$$D_{\vec{u}} f(x,y,z) = \lim_{h\to 0} \frac{f(x+u_1h, y+u_2h, z+u_3h) - f(x,y,z)}{h}$$

and the gradient

$$\operatorname{grad} f(x,y,z) = \vec{\nabla} f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$$

Again

$$D_{\vec{u}} f(x,y,z) = \vec{u} \cdot \vec{\nabla} f(x,y,z).$$

More generally, for any function of n variables $f: D \to \mathbb{R}$ with domain $D \in \mathbb{R}^n$ with arguments $x_1, \ldots, x_n$ or vector argument $\vec{x} = \langle x_1, \ldots, x_n\rangle$,

$$\vec{\nabla} f = \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle \quad \text{and} \quad D_{\vec{u}} f(\vec{x}) = D_{\vec{u}} f(x_1, \ldots, x_n) = \vec{u} \cdot \vec{\nabla} f(x_1, \ldots, x_n).$$

Example 4: For $f(x,y,z) = x\sin(yz)$, evaluate (a) its gradient, and (b) its derivative in the direction of $\vec{v} = \langle 1, 2, -1\rangle$ at point $P(1,3,0)$.

Example 5: Suppose that the temperature at a point $P(x,y,z)$ in space is given by

$$T(x,y,z) = \frac{80}{1 + x^2 + 2y^2 + 3z^2} \quad \text{(Units are degrees Celsius and meters.)}$$

In what direction is the temperature increasing the fastest at the point $P(1,1,-2)$, and what is that maximum rate of increase?

4.5.2 Scalar and Vector Fields

References: OSC-3 §6.1, §6.2)

In physical contexts, a function defined over a region in space or in a plane is often called a *field*; for example electric fields and magnetic fields. The gradient introduces a new case of this, where a function like $\vec{\nabla} V(\vec{r})$ takes a vector value at each point in its domain. Such fields are called *vector fields*, and in contrast, fields with real ("scalar") values are sometimes called *scalar fields*. Two canonical examples are the gravitational potential $V(\vec{r})$, which is a scalar field, and the acceleration due to gravity, given by the vector field $\vec{F} = -\vec{\nabla} V(\vec{r})$.

Line Integrals of Vector Fields. The line integrals seen in Equations (4.26) and (4.32) in the previous section have the common compact vector form

$$\int_\alpha^\beta \vec{F}(r(\vec{t})) \cdot \frac{d\vec{r}}{dt}\, dt$$

and formally, we can use the vector-valued differential $d\vec{r} = \dfrac{d\vec{r}}{dt}\, dt$ and convert back to a line integral, so

$$\int_C \vec{F} \cdot d\vec{r} = \int_C F_1(\vec{r})\, dx_1 + F_2(\vec{r})\, dx_2 + \cdots + F_n(\vec{r})\, dx_n = \int_\alpha^\beta \vec{F}(r(\vec{t})) \cdot \frac{d\vec{r}}{dt}\, dt \tag{4.33}$$

This is called **the line integral of $\vec{F}$ along C**.

This integral $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ can instead be defined rigorously by the familiar process of approximation by a sum of terms $\vec{F}(\vec{r}(t_i^*))\Delta\vec{r}_i$ and taking a suitable limit: one again gets Eq. (4.33).

Reversing Orientation in Line Integrals of Vector Fields. When one reverses orientation to curve $-C$, an arc length integral does not change sign, but be careful: in the above formula (4.33) for the line integral of a vector field in terms of a line integral with respect to arc length, reversing orientation negates the unit tangent vector $\vec{T}$, so the integral is negated:

$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r} \tag{4.34}$$

This also follows from Eq. (4.33) combined with the sign changes in integrals with respect to coordinate variables, as in Eq. (4.27).

For the physical example where this integral gives the work done by force $\vec{F}$ on an object as it moves along the curve C, this says naturally that going in the opposite direction negates the amount of work done.

4.5 Review of Concepts

- Terms to know: scalar field, gradient, directional derivative, vector field, line integral of a vector field.
- Know how to compute gradients, directional derivatives, line integrals of vector fields.

4.5 Exercises

1. (CGS 13.4.6) Evaluate $\vec{\nabla} f$ for $f(x, y, x) = 1/\sqrt{x^2 + y^2 + z^2}$.

2. (CGS 13.4.11) For any smooth functions $f(x, y)$ and $g(x, y)$, which are the following are true?

a) There is a direction $\vec{u}$ such that at point P, $D_{\vec{u}} f = 0$.

b) There is a direction $\vec{u}$ in which $D_{\vec{u}} f = \vec{\nabla} f$.

c) There is a direction $\vec{u}$ in which $D_{\vec{u}} f = 1$.

d) The gradient of $f(x, y)g(x, y)$ is $f\vec{\nabla}g + g\vec{\nabla}f$.

3. (CGS 13.4.15) Find the direction $\vec{u}$ in which $f(x, y) = e^{x-y}$ increases fastest at $P(1, 2)$. How fast?

4. (CGS 13.4.16) As above, for $f(x, y) = \sqrt{5 - x^2 - y^2}$. (Be careful!)

5. (CGS 15.1.4) Find a potential function $U(x, y)$ for the vector field $\vec{F} = (1/y)\vec{\imath} - (x/y^2)\vec{\jmath}$.
(That is, solve $\vec{\nabla} U = \vec{F}$.)

6. (CGS 15.1.5) As above, but for the vector field $\vec{F} = \left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle$.

7. (CGS 15.1.7) As above, for the vector field $\vec{F} = \langle xy, ? \rangle$;
choose a suitable second component for $\vec{F}$ so that this is possible!

For further examples and exercises, look at “Calculus” by Gilbert Strang [CGS], Sections 13.4 and 15.1.

4.6 (†) The Divergence and Curl of a Vector Field, and the Laplacian

References: OSC-3 §6.5; CET §16.5; TCMB §16.9.1

4.6.1 The Divergence of a Vector Field

The Divergence of a Two Dimensional Vector Field. For two dimensional vector fields, the gradient can be considered as mimicking the product of a "fake vector" $\vec{\nabla} = \vec{\imath}\dfrac{\partial}{\partial x} + \vec{\jmath}\dfrac{\partial}{\partial y}$ with a scalar (field) F to get a vector (field):

$$\text{grad}F = \vec{\nabla}F = \left(\vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y}\right)F = \vec{\imath}\frac{\partial}{\partial x}F + \vec{\jmath}\frac{\partial}{\partial y}F = \vec{\imath}\frac{\partial F}{\partial x} + \vec{\jmath}\frac{\partial F}{\partial y} = \langle F_x, F_y\rangle$$

It is also useful to formally compute the dot product of ∇ with a vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$, giving the **divergence** of the vector field, which is a scalar field (just as a true dot product is a scalar):

$$\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y}\right)\cdot\left(\vec{\imath}P + \vec{\jmath}Q\right) = \frac{\partial}{\partial x}P + \frac{\partial}{\partial y}Q = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

This new object $\vec{\nabla}$ is sometimes called **del**, and is an example of an **operator**, as are grad and div. Back in Section 2.4.3 *operator* was introduced as a synonym for *transformation*: a function that takes input from a vector space and gives output in another vector space. This includes operators like del that input a *function* and output another *function*, because in Section 2.4.2 we defined vector spaces to include things like families of functions such as "all differentiable functions of two variables".

In practice, we typically use *transformation* when the input and output are normal vectors (as with matrix multiplication), and *operator* when the input and output are functions.

The Divergence of a Three Dimensional Vector Field. In three dimensions with coordinates (x, y, z), we define the operator del as

$$\vec{\nabla} = \vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

so that the gradient of a scalar field $F(x, y, z)$ again has the compact form

$$\text{grad}\ F(x,y,z) = \vec{\nabla}F(x,y,z)$$

For a vector field in three dimensions, $\vec{F} = \vec{\imath}P + \vec{\jmath}Q + \vec{k}R$, the **divergence** of $\vec{F}$ is

$$\text{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Example 1: If $\vec{F} = xy\vec{\imath} + xyz\vec{\jmath} + y^2\vec{k}$, compute $\vec{\nabla}\cdot\vec{F}$.

4.6.2 The Curl of a Vector Field

In three dimensions only, there is one more important combination of first order partial derivatives, the **curl**.

Definition: For a differentiable vector field $\vec{F} = P\vec{\imath} + Q\vec{\jmath} + R\vec{k}$ on $\mathbb{R}^3$, its **curl** is given by

$$\operatorname{curl}\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} \tag{4.35}$$

Example 2: For $\vec{F} = xy\vec{\imath} + xyz\vec{\jmath} + y^2\vec{k}$ as above, compute curl $\vec{F}$.

The fake cross product notation for curl $\vec{F}$ Just as the "del" notation $\vec{\nabla} = \vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ gives the short-hands grad $f = \vec{\nabla}f$ and div $\vec{F} = \vec{\nabla}\cdot\vec{F}$, the curl can be written in a cross-product notation:

$$\begin{aligned}
\vec{\nabla}\times\vec{F} &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&= \vec{\imath}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \vec{\jmath}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\
&= \operatorname{curl}\vec{F}.
\end{aligned}$$

Note that for a 2D vector field $\vec{F} = P(x,y)\vec{\imath} + Q(x,y)\vec{\jmath}$, this still makes sense with curl $\vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$.
So in that case, $\vec{\nabla}\times\ \vec{F} = \vec{0}$ is the mixed partials condition of Subsection 4.1.6 for the differential $du = P(x,y)\,dx + Q(x,y)\,dy$ to be exact, and as seen in Section 4.5 as a necessary condition for a vector field $\vec{F} = P\vec{\imath} + Q\vec{\jmath}$ to be a gradient, $\vec{F} = \vec{\nabla}U$, so that it is a **conservative vector field** (like electrostatic or gravitational force, but not magnetic force).

This also works in 3D: the three components are exactly the three conditions for a differential in 3D to be exact, leading to:

Theorem: The curl of a conservative vector field $\vec{F} = \vec{\nabla} f$ vanishes: that is,

$$\text{curl}(\text{grad } f) = \vec{\nabla} \times (\vec{\nabla} f) = \vec{0}.$$

Thus, $\vec{\nabla} \times \vec{F} = \vec{\nabla} \times \langle P, Q, R \rangle = \vec{0}$ is a necessary condition for the differential $P dx + Q dy + R dz$ to be exact. As a partial converse, if a vector field $\vec{F}$ is defined on all of $\mathbb{R}^3$ and $\vec{\nabla} \times \vec{F} = \vec{0}$, then $\vec{F}$ is conservative.

Proof: The verification of the first half is a straightforward calculation using Clairaut's Theorem on mixed partial derivatives: it is like three versions of the mixed partials condition for a conservative vector field in $\mathbb{R}^2$ seen in Section 4.1.

The main idea of proof of the converse is that one can:

1. integrate $\partial U / \partial x = P(x, y, z)$ in x getting a "constant" of integration $g(y, z)$ that depends on both y and z, then
2. put that into $\partial U / \partial y = Q(x, y, z)$ and integrate in y to solve for $g(y, z)$ new "constant" of integration $h(z)$, and finally
3. use integration of $\partial U / \partial z = R(x, y, z)$ to solve for $h(z)$ up to a true constant, giving $U(x, y, z)$.

Example 3: Show that the vector field $\vec{G} = \left\langle y^2 z^3, 2xyz^3, 3xy^2 z^2 \right\rangle$ is conservative, and find the corresponding potential.

Example 4: Show that the above vector field $\vec{F} = xy\vec{\imath} + xyz\vec{\jmath} + y^2\vec{k}$ is not exact.

One could also test this the hard way, by trying (unsuccessfully) to solve the three equations $\partial U / \partial x = xy$, $\partial U / \partial y = xyz$, $\partial U / \partial z = -y^2$ as suggested in the above sketch of a proof.

Curl and Rotation in a Fluid The name "curl" refers to a measure of rotation. For example, the vector field $\vec{F} = -y\vec{\imath} + x\vec{\jmath}$ describes velocity of a fluid (say) going anti-clockwise around the z-axis: it has $\vec{\nabla} \times \vec{F} = 2\vec{k}$ in which the direction $\vec{k}$ indicates the axis of rotation, the positive value indicates anti-clockwise direction as viewed from "above" down that z-axis, and the uniform magnitude indicating the uniform angular rate of rotation (which has period 2π). In fact, an older notation for $\vec{\nabla} \times \vec{F}$ is Rot $\vec{F}$.

4.6.3 The Laplacian of a Function and the Laplace Operator

There is a very important combination of the divergence and gradient:

$$\text{div grad } F = \vec{\nabla} \cdot \vec{\nabla} F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}$$

This is the **Laplacian of *f***, often abbreviated as $\vec{\nabla}^2 F$ or ΔF. For functions of two variables this is

$$\vec{\nabla}^2 F(x, y) = \vec{\nabla} \cdot \vec{\nabla} F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}$$

and in 3D,

$$\vec{\nabla}^2 F(x,y,z) = \vec{\nabla}\cdot\vec{\nabla}F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}.$$

This operator $\Delta = \vec{\nabla}^2 = \vec{\nabla}\cdot\vec{\nabla}$ is called the **Laplace operator**.

4.6.4 Some Fundamental Differential Equations of Physics

grad, div, curl and the Laplacian arise in many of the fundamental equations for physics:

$$\begin{aligned}
&\text{Laplace's Equation} && \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0\\
&\text{Poisson Equation} && \nabla^2 u = f\\
&\text{Heat Equation} && \frac{\partial u}{\partial t} = \nabla^2 u\\
&\text{Vacuum Maxwell's Equations} && \frac{\partial \vec{E}}{\partial t} = \frac{1}{\epsilon_0\mu_0}\vec{\nabla}\times\vec{B}, \quad \vec{\nabla}\cdot\vec{E} = 0\\
& && \frac{\partial \vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}, \quad \vec{\nabla}\cdot\vec{B} = 0\\
&\text{Wave Equation} && \frac{\partial^2 u}{\partial t^2} = \nabla^2 u\\
&\text{Schrödinger Equation} && i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\vec{x})\psi
\end{aligned}$$

used in the description of fluid motion, electric fields, heat conduction, waves in water and in electro-magnetic fields, and in quantum mechanics.

(This version of Maxwell's equations is for the simplest case of electric field $\vec{E}$ and magnetic field $\vec{B}$ in a region with no charge or electrical current; it leads in special cases to the following wave equation, as the simplest description of electro-magnetic waves.)

4.6 Review of Concepts

- Terms to know: divergence, curl, Laplacian, gradient field.
- Know how to compute the above three derivative-based quantities.
- The connection between $\vec{\nabla}\times\vec{F}$, gradient fields, and exact differentials.

4.6 Exercises

1. Compute the divergence and curl of the vector field $\vec{F} = \vec{r}\,/\,r^3$ in $\mathbb{R}^3$. (This is related to the electrostatic force around a charge at the origin.)

2. Compute the divergence and curl of the two "spin" vector fields $\vec{F} = \left\langle 0, -z, y \right\rangle / r$ and $\vec{G} = \left\langle 0, -z, y \right\rangle / r^2$.

3. Compute the divergence and curl of $\vec{F} = xye^z\vec{\imath} + yze^x\vec{k}$.

4. Verify that for any 3D vector field $\vec{F}(x, y, z) = \langle P, Q, R \rangle$, the divergence of its curl vanishes.
That is, $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$.

Appendices

Appendix A

Introductory Calculus Review

A.1 Review of Limits and Derivatives

Objectives and Concepts:

- Limits of functions can be found by substitution, algebraic manipulation, or L'Hospital's Rule.
- The derivative of a differentiable function $f(x)$ is another function that represents the rate of change of f with respect to x.
- The Chain Rule gives a method for computing the derivative of a composite function and is helpful in differentiating functions that are defined implicitly.

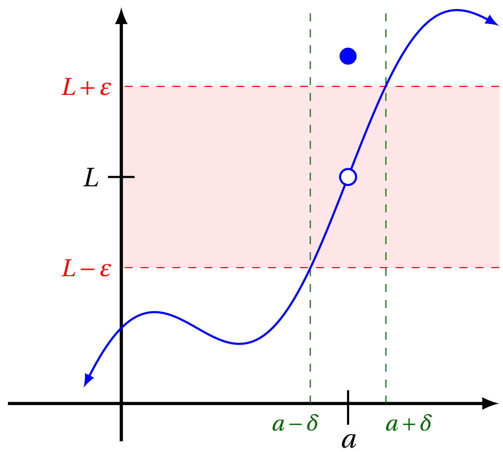
A.1.1 Review of Limits

In an introductory Calculus course, the concept of a **limit** of a function is presented via a formal definition that uses logical quantifiers. Informally, we say that "the limit of $f(x)$ as x approaches a equals L" and we write

$$\lim_{x\to a} f(x) = L,$$

provided the if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a (on either side of a) but not equal to a. The formal definition and its interpretation:

$$\underbrace{\text{Given any } \varepsilon > 0,}_{\text{For any arbitrary tolerance } \varepsilon} \quad \underbrace{\text{there is a } \delta > 0 \text{ such that if } 0 < |x-a| < \delta,}_{\text{how close to } a \text{ does } x \text{ have to be so that}} \quad \underbrace{\text{then } |f(x) - L| < \varepsilon.}_{f(x) \text{ differs from } L \text{ by less than } \varepsilon\text{?}}$$

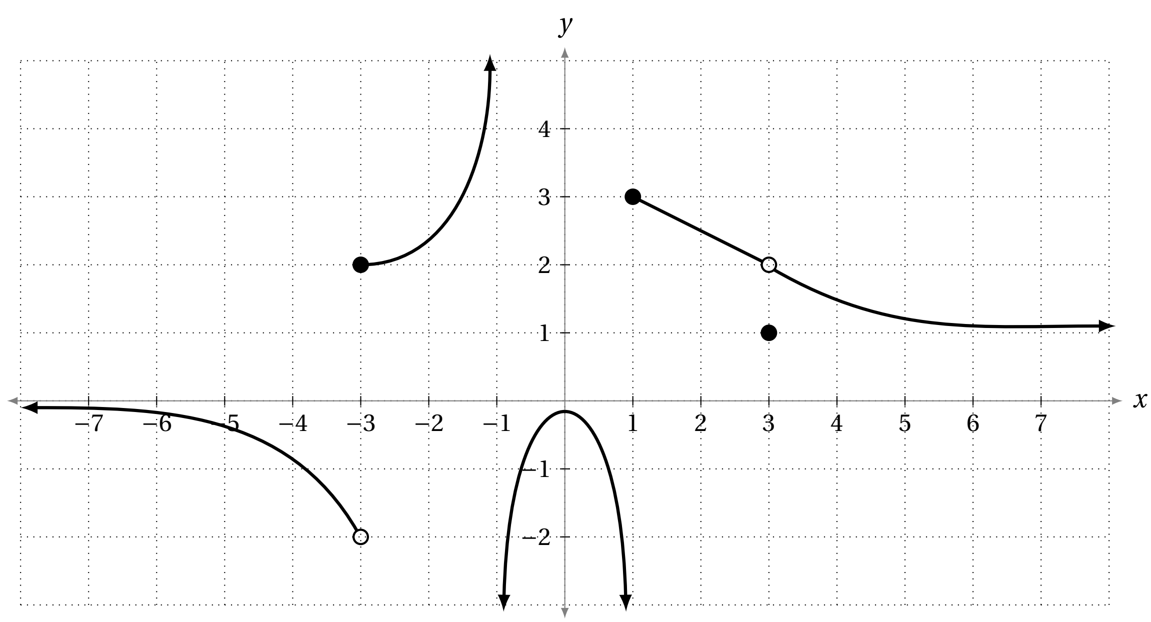


The symbols "$x \to a$" indicates that x is getting closer and closer to a (but x is not equal to a). If x is approaching a from the **right** side (that is, through numbers that are larger than a), then we write $x \to a^+$. Similarly, if x is approaching a from the **left** side (that is, through numbers that are smaller than a), then we write $x \to a^-$. If the limits from both directions are equal to the number L, we say $f(x) \to L$ as $x \to a$. **The function's value at a (if it is defined at a) does not affect the limit of the function as $x \to a$.**

Theorem: A function $f(x)$ has a limit as x approaches a if and only if it has a left-side limit, it has a ride-side limit, and these one-sided limits agree.

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L \;\text{ and }\; \lim_{x\to a^+} f(x) = L.$$

Example 1: Below is the graph of a function $y = f(x)$. Use the graph to answer the following questions.



a) $f(3) =$ ________

b) $f(-3) =$ ________

c) $\lim_{x\to 3^+} f(x) =$ ________

d) $\lim_{x\to 3} f(x) =$ ________

e) $\lim_{x\to -3^-} f(x) =$ ________

f) $\lim_{x\to -3^+} f(x) =$ ________

g) $\lim_{x\to -3} f(x) =$ ________

h) $\lim_{x\to \infty} f(x) =$ ________

i) $\lim_{x\to -\infty} f(x) =$ ________

j) $\lim_{x\to 1^-} f(x) =$ ______

k) $\lim_{x\to 1^+} f(x) =$ ______

l) $\lim_{x\to 1} f(x) =$ ______

m) $\lim_{x\to -1^-} f(x) =$ ______

n) $\lim_{x\to -1^+} f(x) =$ ______

o) $\lim_{x\to -1} f(x) =$ ______

Properties of Limits: Suppose that c is a constant, n is positive, and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
2. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$
3. $\lim_{x\to a} [f(x)g(x)] = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right)$
4. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \left(\lim_{x\to a} g(x) \neq 0\right)$
5. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$
6. $\lim_{x\to a} c = c$
7. $\lim_{x\to a} x = a$
8. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$

The Squeeze Theorem: If $g(x) \le f(x) \le h(x)$ for all x near a and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$.

Definition: A function $f(x)$ is **continuous** at a if $\lim_{x\to a} f(x) = f(a)$.

If f is continuous at a, then the limit can be evaluated by **direct substitution** (note that a must be in the domain of f). Otherwise, if we need to find a function g that behaves like f but whose limit can be found easily at $x = a$.

Theorem: If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, provided the limits exist.

Example 2: Evaluate the limits $\lim_{x\to 3} \dfrac{x^2-9}{x^2-2x-3}$, $\lim_{x\to 1^-} \dfrac{x^2-9}{x^2+2x-3}$, and $\lim_{x\to 1^+} \dfrac{x^2-9}{x^2+2x-3}$.

A.1.2 Derivatives and Differentiation Rules

The **derivative** $f'(x)$ of a function $f(x)$ with respect to x is formally defined as a limit:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

The derivative, when it exists, gives the slope $f'(a)$ of the line tangent to the curve $y = f(x)$ at the point $(a, f(a))$. The limit definition of the derivative can be used to derive several rules for differentiation.

Theorem: Let c, a, and n be (real) constants, and let f and g be differentiable functions. Then the following rules hold:

- $\frac{d}{dx}(c) = 0.$
- $\frac{d}{dx}\left(cf(x)\right) = c\,\frac{d}{dx}f(x).$
- $\frac{d}{dx}\left(x^n\right) = nx^{n-1}.$
- $\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- $\frac{d}{dx}(a^x) = a^x \ln a$
- $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

The following derivatives of trigonometric functions and inverse trigonometric functions are also important:

Derivatives of Trigonometric and Inverse Trigonometric Functions:

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{x\sqrt{x^2-1}}$$

Product, Quotient, and Chain Rules: Let f and g be differentiable functions, then

- **Product Rule:** $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$
- **Quotient Rule:** $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}, \quad (g(x) \neq 0)$
- **Chain Rule:** $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x)$

$$\underbrace{\frac{d}{dx}}_{\text{derivative}} \underbrace{f}_{\text{outer function}} \underbrace{(g(x))}_{\text{evaluated at inner function}} = \underbrace{f'}_{\text{derivative of outer function}} \underbrace{(g(x))}_{\text{evaluated at inner function}} \cdot \underbrace{g'(x)}_{\text{derivative of inner function}}$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then the Chain rule represents

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

In many applications, derivatives of "intermediate" functions (such as u above) may seem to appear out of nowhere. For example, if we are considering the pressure P of an ideal gas to be a function of the volume V, then

$$\frac{dP}{dV} = \frac{dP}{dT}\frac{dT}{dV}$$

if we can describe P as a function of T and T as a function of V.

Example 3: Find the derivatives (with respect to x) of the following functions:

a) $y = \dfrac{\sqrt{1-2x}}{\sqrt{1+2x}}$

b) $y = \left[\cos\left(e^{x^2} - 1\right)\right]^{1/2}$

c) $y = \ln(a + b\sin(x))$

Additionally, higher-order derivatives are used for many applications (recall that the sign of the second derivative gives the concavity of the function). Common notations are $f''(x)$ and $\dfrac{d^2y}{dx^2}$.

A.1.3 Implicit Differentiation

When an equation defines y is implicitly as a function of x, the derivative $\dfrac{dy}{dx}$ can still be found via implicit differentiation. We differentiate both sides of the equation using our standard rules, and whenever a derivative of a function of y is taken, the chain rule produces a factor of $\dfrac{dy}{dx}$. Then $\dfrac{dy}{dx}$ can be solved for algebraically.

Implicit Differentiation:

1. Take the derivative of each side of the equation, applying the Chain Rule to produce $\dfrac{dy}{dx}$ when necessary.
2. Solve the equation for dy/dx.

Example 4: Find $\dfrac{dy}{dx}$ if $x^2 + y = x \sin y$.

Example 5: Assume that for a particular ideal gas, the pressure P is a function of the volume V, with the amount of gas (n) and temperature (T) being held constant (let R denote the universal gas constant). Find the rate of change of the pressure with respect to the volume.

Logarithmic differentiation is a process that can be used to differentiate functions more easily by using the properties of logarithmic functions. This can be especially useful when the function to be differentiated involves a large number of terms in products or quotients. Logarithmic differentiation is also useful for finding the derivative of the form $y = f(x)^{g(x)}$.

Example 6: Find $\dfrac{dy}{dx}$ if $y = x^{\sqrt{x}}$.

A.1.4 Maximum and Minimum Values

One of the most important applications of derivatives is to find the local or global extreme values of a function. Recall that on any closed interval $[a, b]$, The Extreme Value Theorem says that a continuous function will attain a global maximum value and minimum value somewhere in the interval, including possibly at the endpoints.

Definition: A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist. The location $\big(c, f(c)\big)$ is called a **critical point** of $f(x)$.

Theorem: Let f be a function defined on a closed interval $[a, b]$ containing the point c. If $f(c)$ is an extreme value, then c must be an *endpoint of the interval* or a *critical number*; that is,

- c is an endpoint of $[a, b]$; or
- c is an interior point where $f' = 0$; or
- c is an interior point where f' is undefined.

If we are interested in finding local extreme values of a function defined on an open interval (a, b) or on all of $\mathbb{R}$, then we have several tools available.

First Derivative Test: Suppose c is a critical number of a continuous function f.

a) If f' changes from positive to negative at c, then f has a local maximum at c.

b) If f' changes from negative to positive at c, then f has a local minimum at c.

c) If f' does not change sign at c, then f has no local maximum or minimum at c.

We can also use the second derivative of a function to help us determine the behavior of the function at a critical number.

Concavity Test:

a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Second Derivative Test: Suppose f'' is continuous near c.

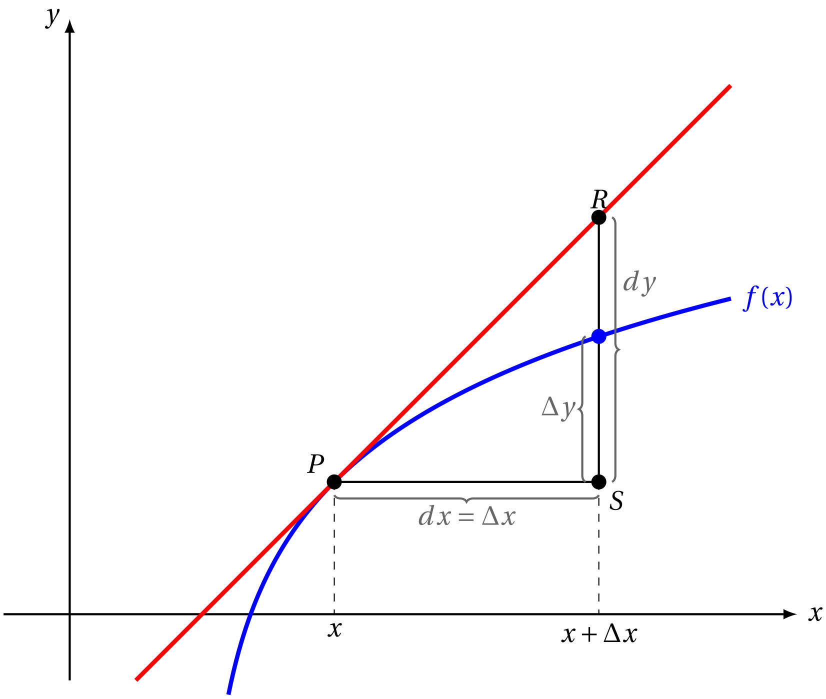
a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

A.1.5 Differentials

Let $y = f(x)$ be a differentiable function. Then the **differential** dx is an independent variable and the **differential** dy is $dy = f'(x)dx = \left(\frac{dy}{dx}\right)dx$. It is a dependent variable because it depends on the values of x and dx. Differentials have two relevant applications. In one sense, they provide us with a way to represent how *infinitesimal* changes in the input x (dx) influence changes in the output y (dy). This concept is useful in constructing many formulas involving rates of change and also in change of variables (which we will see later).

Another important use of differential is to **estimate** the how much a function's output could actually change based on a tolerance, or error, in the input. If dx is given a specific value (maybe an input tolerance or error in measurement) and x is taken to be some specific number (a measurement) in the domain of f, then the numerical value of dy (how far off $y = f(x)$ could be based on the error in the input) is determined. Now, if $dx = \Delta x$, the *actual* corresponding change in y is $\Delta y = f(x + \Delta x) - f(x)$ What is the difference between dy and Δy?



- Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by amount $dx = \Delta x$:

$$\Delta y = f(x + \Delta x) - f(x)$$

- The differential dy represents the amount that the tangent line rises or falls:

$$dy = f'(x)\,dx$$

which can serve as an approximation to the actual amount that the function changes from x to $x + \Delta x$.

Example 7: The edge of a cube is measure as 10 cm with a possible error of 1%. Estimate the possible error in the calculation of the cube's volume using that measurement.

The differential dx is 1% of the measurement, which is $0.01 \cdot 10 = 0.1$ cm. Now $V = x^3$, so $dV = 3x^2\,dx$. Thus an estimate for the possible error in the volume calculation is

$$dV = 3x^2\,dx = 3(100 \text{ cm}^2)(0.1 \text{ cm}) = 30 \text{ cm}^3.$$

Considering that the calculated volume itself is 1000 cm^3, the relative error in the volume calculation is estimated to be at most $30 \text{ cm}^3 / 1000 \text{ cm}^3 = 3\%$.

A.1 Review of Concepts

- Terms to know: limit, left-hand limit, right-hand limit, continuous, derivative, implicit differentiation, logarithmic differentiation, differential.
- Know how to find limits using substitution or algebraic means.
- Know how to find derivatives using derivative rules and implicit differentiation.

A.1 Practice Problems

1. Evaluate the following limits:

a) $\displaystyle\lim_{x\to\pi^+} \frac{\tan x}{1-\cos x}$

b) $\displaystyle\lim_{x\to\pi^-} \frac{1-\cos x}{\tan x}$

c) $\displaystyle\lim_{x\to\infty} e^{-x}\sin x$

d) $\displaystyle\lim_{x\to 1^+} \left(\frac{1}{x-1}+\frac{1}{x^2-3x+2}\right)$

2. Find the first derivatives of the following functions:

a) $y = e^{\cos 2x} + \cos(e^{2x})$

b) $y = (\arcsin 2x)^2$

c) $y = \dfrac{(x^2+1)^2}{(2x+1)^3(3x-1)^5}$

3. Find $\dfrac{dy}{dx}$:

a) $xe^y = y\sin x$

b) $y = (\cos x)^x$

A.1 Exercises

1. The van der Waals equation of state for a fluid composed of particles that have a non-zero volume and a pairwise attractive inter-particle force is

$$\left(P+\frac{n^2a}{V^2}\right)(V-nb) = nRT.$$

Find $\dfrac{dP}{dV}$ and $\dfrac{dV}{dP}$ assuming all other variables are constants.

2. The **radius of curvature** ρ of a function $f(x)$ is given by

$$\rho = \frac{\left(1+(f'(x))^2\right)^{3/2}}{f''(x)}$$

where $f'(x)$ and $f''(x)$ are the first and second derivatives of f, and it represents the radius of the circular arc which best approximates the curve at that point. The **curvature** K of f is the reciprocal of the radius of curvature. Find a formula for the radius of curvature of $P(V)$ (treat P as a function of V with all other variables constant) in the van der Waals equation.

3. The **Morse potential** is an equation that describes the potential energy V of a diatomic molecule. The potential energy as a function of the distance between the atoms r, is given by the equation

$$V(r) = D_e\left(1-e^{-\beta(r-r_e)}\right)^2,$$

where β is a force constant, r_e is the equilibrium bond length, and D_e is the dissociation energy. Find the minimum potential energy. What is the distance between the atoms when the attractive force $\dfrac{dV}{dr}$ is largest?

A.1 Answers to Practice Problems

1. a) Note that as $x \to \pi^+$, $\cos x \to -1$ and $\tan x \to 0$. Thus this limit can be evaluated by direct substitution:

$$\lim_{x\to\pi^+} \frac{\tan x}{1-\cos x} = \frac{0}{1+1} = 0.$$

b) Now as $x \to \pi^-$, $\cos x \to -1^+$ (as all values of $\cos x$ are greater than or equal to -1) so $1-\cos x \to 1-(-1^+) \to 2^+$. Note that as $x \to \pi^-$, $\tan x \to 0^-$ as the values of $\tan x$ for values of x close to π but less than π are in the second quadrant where $\tan x$ is negative. Thus we have

$$\lim_{x\to\pi^-} \frac{1-\cos x}{\tan x} = \frac{2}{0^-} = -\infty.$$

c) Now

$$\lim_{x\to\infty} e^{-x}\sin x = \lim_{x\to\infty} \frac{\sin x}{e^x}$$

and since $-1 \le \sin x \le 1$, we have

$$\frac{-1}{e^x} \le \frac{\sin x}{e^x} \le \frac{1}{e^x}.$$

Now

$$\lim_{x\to\infty} \frac{-1}{e^x} = \lim_{x\to\infty} \frac{-1}{e^x} = 0,$$

so by the Squeeze Theorem, $\lim_{x\to\infty} e^{-x}\sin x = 0$.

d) The denominator of the second term can be written as $x^2-3x+2 = (x-1)(x-2)$. Note that if we tried direct substitution we would get $\infty-\infty$ as $x-1 \to 0^+$ and $(x-1)(x-2) \to 0^-$ as $x \to 1^+$. Now we have

$$\begin{aligned}
\lim_{x\to1^+}\left(\frac{1}{x-1}+\frac{1}{x^2-3x+2}\right) &= \lim_{x\to1^+}\left(\frac{x-2}{(x-1)(x-2)}+\frac{1}{(x-1)(x-2)}\right)\\
&= \lim_{x\to1^+}\frac{x-2+1}{(x-1)(x-2)}\\
&= \lim_{x\to1^+}\frac{x-1}{(x-1)(x-2)}\\
&= \lim_{x\to1^+}\frac{1}{x-2} = \frac{1}{1-2} = -1.
\end{aligned}$$

2. a) $y = e^{\cos 2x} + \cos(e^{2x})$

$$y' = e^{\cos 2x}(-\sin 2x)(2) - \sin(e^{2x})(e^{2x})(2).$$

b) $y = (\arcsin 2x)^2$

$$y' = 2(\arcsin 2x)\left(\frac{1}{\sqrt{1-(2x)^2}}\right)(2) = \frac{4\arcsin 2x}{\sqrt{1-4x^2}}.$$

c) Note that if we try to apply the quotient rule directly to $y = \dfrac{(x^2+1)^2}{(2x+1)^3(3x-1)^5}$ we will have a very messy problem. To use logarithmic differentiation, first take the natural log of both sides to obtain

$$\ln y = \ln\left(\frac{(x^2+1)^2}{(2x+1)^3(3x-1)^5}\right) = 2\ln(x^2+1) - 3\ln(2x+1) - 5\ln(3x-1).$$

Then we have

$$\frac{1}{y}\frac{dy}{dx} = 2\frac{1}{x^2+1}(2x) - 3\frac{1}{2x+1}(2) - 5\frac{1}{3x-1}(3),$$

so, multiplying both sides by y and replacing y with its definition we have

$$\frac{dy}{dx} = \frac{(x^2+1)^2}{(2x+1)^3(3x-1)^5}\left(\frac{4x}{x^2+1} - \frac{6}{2x+1} - \frac{15}{3x-1}\right).$$

3. a) Using implicit differentiation we have

$$xe^y\frac{dy}{dx} + e^y = y\cos x + \sin x\frac{dy}{dx}.$$

Solving for $\dfrac{dy}{dx}$ we have

$$\frac{dy}{dx}(xe^y - \sin x) = y\cos x - e^y,$$

so

$$\frac{dy}{dx} = \frac{y\cos x - e^y}{xe^y - \sin x}.$$

b) To use logarithmic differentiation we take the ln of both sides:

$$\ln y = \ln\left((\cos x)^x\right) = x\ln(\cos x).$$

Then

$$\frac{1}{y}\frac{dy}{dx} = x\frac{1}{\cos x}(-\sin x) + \ln(\cos x),$$

so

$$\frac{dy}{dx} = y\left(\ln(\cos x) - \frac{x\sin x}{\cos x}\right) = (\cos x)^x\left(\ln(\cos x) - x\tan x\right).$$

A.2 Review of Integrals and Substitution

Objectives and Concepts:

- The indefinite integral of a function is the antiderivative of the function.
- The definite integral of a function is the net area between the curve and the x-axis over an interval.
- The Fundamental Theorem of Calculus connects the differentiation and integration operations as inverses of each other.
- Integrals can be evaluated using u-substitution, which mimics the chain rule in reverse.

A.2.1 Antiderivatives

Definition: A function $F(x)$ is an **antiderivative** of a function $f(x)$ means $F'(x) = f(x)$ for all x in the domain of f.

Example 1: An antiderivative of $\cos x$ is $\sin x$, and an antiderivative of $\sin x$ is $5 - \cos x$, as $\dfrac{d}{dx}(5 - \cos x) = 0 - (-\sin x)$.

Example 2: An antiderivative of $x^3 + x + 1$ is $\dfrac{x^4}{4} + \dfrac{x^2}{2} + x$. Another antiderivative is $\dfrac{x^4}{4} + \dfrac{x^2}{2} + x + 2$.

Theorem: If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$, where C is a constant.

Function	**Antiderivative**	**Function**	**Antiderivative**	**Function**	**Antiderivative**
x^n where $n \neq -1$	$\dfrac{x^{n+1}}{n+1}$	A	Ax	$\dfrac{1}{x}$	$\ln\lvert x\rvert$
$\sin(x)$	$-\cos(x)$	$\cos(x)$	$\sin(x)$	e^x	e^x
$\sec^2(x)$	$\tan(x)$	$\csc^2(x)$	$-\cot(x)$	$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin(x)$
$\sec(x)\tan(x)$	$\sec(x)$	$\csc(x)\cot(x)$	$-\csc(x)$	$\dfrac{1}{1+x^2}$	$\arctan(x)$

Example 3: Find the most general antiderivative for each function.

a) $f(x) = x^{-2} + \pi - 5e^x - \dfrac{7}{\sqrt{1-x^2}}$

b) $f(x) = \dfrac{2}{3}x^{1/3} + \dfrac{1}{x^{1/3}} - \dfrac{1}{x}$

c) $f(x) = 5\cos(x) - 3\csc(x)\cot(x) + \sec^2(x)$

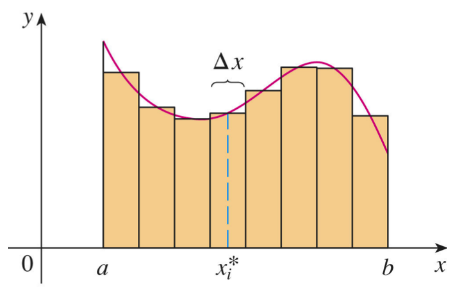
Example 4: Find an antiderivative F of f that satisfies the condition $F(1) = 2$ where $f(x) = 3x^4 + x^3 - x$.

A.2.2 The Definite Integral

Definition of the Definite Integral: If f is a function defined for $a \le x \le b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0 = a$, $x_1, \dots, x_n = b$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \dots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$\underbrace{\int_a^b}_{\text{integral sign}} \quad \underbrace{f(x)}_{\text{integrand}} \quad \underbrace{dx}_{\text{variable of integration}}$$



provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$. The sum on the right hand side above is called a **Riemann Sum**.

Theorem: If f is continuous on $[a, b]$, or if f has only finitely many jump discontinuities, then f is **integrable** on $[a, b]$; that is, the definite integral $\displaystyle\int_a^b f(x)\,dx$ exists.

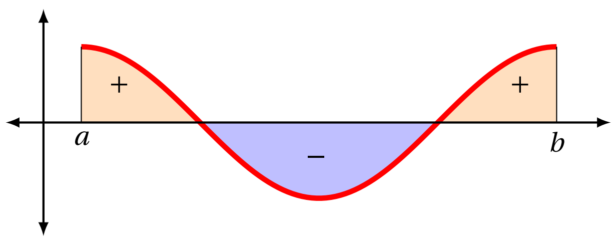
Suppose that $f(x)$ is continuous function on $[a, b]$ and $f(x) \ge 0$ on $[a, b]$. Then the **area of the region under the curve** is given by

$$A = \int_a^b f(x)\,dx$$

The **net area** is interpreted as the difference of the areas above and below the x-axis. A definite integral can be interpreted as the difference of areas:

$$\int_a^b f(x)\,dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis (and below the graph of f) and A_2 is the area of the region below the x-axis (and above the graph of f).



A.2.3 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus: differential calculus and integral calculus. It gives the precise inverse relationship between the derivative and the integral.

The Fundamental Theorem of Calculus, (Part I): If f is continuous on $[a,b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt \qquad a \le x \le b$$

is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$.

Using Leibniz notation for derivatives, we can write FTC1 as

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Example 5: Find the derivative of the function $f(x) = \displaystyle\int_1^{\sin(x)} 3t^2\,dt$.

The Fundamental Theorem of Calculus, (Part II): If f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Note that **any** antiderivative will do, because any constant associated with the antiderivative will be added and subtracted in the computation of $F(b)-F(a)$. Thus we generally omit the constant (i.e., choose $C=0$).

Example 6: Evaluate the integral $\displaystyle\int_0^{\pi/3} 2\sec(x)\tan(x)\,dx$.

Example 7: Evaluate the integral $\displaystyle\int_1^9 \frac{x-1}{\sqrt{x}}\,dx$.

Example 8: Evaluate the integral $\displaystyle\int_0^1 \frac{4}{t^2+1}\,dt$.

Notation: We write $\displaystyle\int f(x)\,dx$ for an antiderivative of f. It is called an **indefinite integral**.

$$\int f(x)\,dx = F(x) \quad \text{means } F'(x) = f(x).$$

Since the indefinite integral represents the most general antiderivative, we must include the "$+C$" when we evaluate indefinite integrals.

A.2.4 Properties of Integrals

Properties of the Definite Integral: Let f and g be continuous functions.

1. $\displaystyle\int_a^a f(x)\,dx = 0$
2. $\displaystyle\int_a^b c\,dx = c(b-a)$ for any constant c
3. $\displaystyle\int_a^b \left[f(x)+g(x)\right]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
4. $\displaystyle\int_a^b \left[f(x)-g(x)\right]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
5. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$, where k is any constant
6. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
7. $\displaystyle\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

Comparison Properties of the Definite Integral:

8. If $m \le f(x) \le M$ over $[a,b]$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.

9. If $f(x) \le g(x)$ over $[a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

Example 9: Explain why $\int_0^{\sqrt{\pi}} \sin(x^2)\,dx \le \sqrt{\pi}$.

Example 10: Given the following information, evaluate each integral below.

$$\int_1^9 f(x)\,dx = -1 \qquad \int_7^9 f(x)\,dx = 5 \qquad \int_7^9 h(x)\,dx = 4$$

a) $\int_1^9 2f(x)\,dx$

b) $\int_7^9 \big(f(x) + 3h(x)\big)\,dx$

c) $\int_1^9 \big(3 + f(x)\big)\,dx$

d) $\int_1^7 f(x)\,dx$

A.2.5 u-Substitution

The Substitution Rule: If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f\big(g(x)\big)\,g'(x)\,dx = \int f(u)\,du$$

The Substitution Rule says: It is permissible to operate with dx and du after integral signs as if they were differentials or symbols. To use the substitution rule, we must replace **everything** after the integral sign with expressions in the new variable u and the differential du. Once the integration is complete, we should write the antiderivative in terms of the original variable x.

When we need to use substitution in a definite integral, we must either (a) treat the integral as an indefinite one to find the antiderivative, or (b) convert the integral to a definite integral in u, convert the limits of integration to the new variable as well. The latter approach **almost always** works better, as the substitutions involved generally **make the problem easier**.

The Substitution Rule for Definite Integrals: If g' is continuous on $[a,b]$ and if f is continuous on the range of $u = g(x)$, then

$$\int_a^b f\left(g(x)\right) g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

One of the best ways to approach u-substitution problems is to identify a function (as part of the integrand) such that the derivative of that function is also present. Once that function is found, you want u to be that function, and then from that choice compute the differential du. If the integrand and differential $f(g(x))\,dx$ can be replaced entirely with $f(u)\,du$, then the substitution is successful. Of course, sometimes an additional constant appears as a product of the substitution, but that is easily handled.

Example 11: Evaluate $\int \sqrt[5]{10-7x}\,dx$.

Example 12: Evaluate $\int x^3(1-x^4)^5\,dx$.

Example 13: Evaluate $\int \frac{x}{x^2+1}\,dx$. Then $du = 2x\,dx$ so $(1/2)du = x\,dx$.

Example 14: Evaluate $\int_0^{\pi/4} \sin^3(2x)\cos(2x)\,dx$.

Example 15: Evaluate $\int_2^3 xe^{-x^2}\,dx$.

Some substitutions are not quite as obvious. In these cases, you may have to try several choices for u or manipulate your substitution equation before finding a substitution that yield something we can integrate.

Example 16: Evaluate $\int \frac{x}{\sqrt{x-4}}\,dx$.

A.2 Review of Concepts

- Terms to know: antiderivative, indefinite integral, definite integral, substitution.
- Know how to find aniderivatives, indefinite integrals, definite integrals, including with the use of substitution.

A.2 Practice Problems

1. Evaluate $\int e^x \sin\left(e^x\right)\,dx$.

2. Evaluate $\int \frac{y^2}{(y+1)^4}\,dy$.

3. Evaluate $\int_0^1 \frac{x+1}{(x^2+2x+6)^2}\,dx$.

4. Evaluate $\int_{1/2}^{1} \sin\left(\frac{\pi}{2}x - \frac{\pi}{4}\right)\,dx$.

A.2 Exercises

1. Evaluate the following indefinite integrals.

a) $\int (x^6 - 3x^2)^4 (x^5 - x)\,dx$

b) $\int \frac{(1+\sqrt{x})^3}{\sqrt{x}}\,dx$

c) $\int x\sqrt[3]{x+1}\,dx$

d) $\int x\sin(x^2)\cos^8(x^2)\,dx$

2. Evaluate the following definite integrals.

a) $\int_0^{1/2} \frac{1}{\sqrt{1-4x^2}}\,dx$

b) $\int_1^2 \frac{4}{9x^2+6x+1}\,dx$

c) $\int_1^{e^2} \frac{\ln t}{t}\,dt$

d) $\int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\,dx$

A.2 Answers to Practice Problems

1. $\int e^x \sin\left(e^x\right)\,dx$

Let $u = e^x$, then $du = e^x\,dx$. Then we have

$$\int e^x \sin\left(e^x\right)\,dx = \int \sin u\,du = -\cos u + C = -\cos(e^x) + C.$$

2. $\int \frac{y^2}{(y+1)^4}\,dy$

Let $u = y+1$. Then $u-1 = y$ so $(u-1)^2 = y^2$. Also, $du = dy$. Then

$$\int \frac{y^2}{(y+1)^4}\,dy = \int \frac{(u-1)^2}{u}\,du = \int \frac{u^2-2u+1}{u}\,du = \int \left(u - 2 + \frac{1}{u}\right)du$$
$$= \frac{u^2}{2} - 2u + \ln|u| + C = \frac{(y+1)^2}{2} - 2(y+1) + \ln|y+1| + C.$$

3. $\int_0^1 \frac{x+1}{(x^2+2x+6)^2}\,dx$

Let $u = x^2+2x+6$. Then $du = (2x+2)\,dx = 2(x+1)\,dx$ so $du/2 = (x+1)\,dx$. Now if $x = 0$, then $u = 0^2+2(0)+6 =$

6, and if $x = 1$, then $u = 1 + 2 + 6 = 9$. Then

$$\int_0^1 \frac{x+1}{(x^2+2x+6)^2}\,dx = \frac{1}{2}\int_6^9 \frac{1}{u^2}\,du = -\frac{1}{2}u^{-1}\Big|_6^9 = -\frac{1}{2}\left(\frac{1}{9} - \frac{1}{6}\right) = \frac{1}{36}.$$

4. $\displaystyle\int_{1/2}^1 \sin\left(\frac{\pi}{2}x - \frac{\pi}{4}\right)\,dx$

Let $u = \dfrac{\pi}{2}x - \dfrac{\pi}{4}$ and then $du = \dfrac{\pi}{2}\,dx$. If $x = 1/2$, then $u = \dfrac{\pi}{2}\left(\dfrac{1}{2}\right) - \dfrac{\pi}{4} = 0$. If $x = 1$, then $u = \dfrac{\pi}{2} - \dfrac{\pi}{4} = \dfrac{\pi}{4}$. Then we have

$$\int_{1/2}^1 \sin\left(\frac{\pi}{2}x - \frac{\pi}{4}\right)\,dx = \frac{2}{\pi}\int_0^{\pi/4} \sin u\,du = -\frac{2}{\pi}\cos(u)\Big|_0^{\pi/4} = -\frac{2}{\pi}\left(\cos\frac{\pi}{4} - \cos 0\right) = -\frac{2}{\pi}\left(\frac{\sqrt{2}}{2} - 1\right) = \frac{2-\sqrt{2}}{\pi}.$$

Appendix B

Solutions to Exercises

B.1 Section 1.1 Exercises

1. Determine if the following statements are true or false, giving a brief explanation or counterexample in each case.

a) $(1,0,0)_r = (1,0,0)_c = (1,0,0)_s$.

   False. If $z = 0$ then $\phi = \pi/2$ in spherical coordinates.

b) $\{(\rho,\theta,\phi) \mid \phi = \pi/2\} = \{(r,\theta,z) \mid z = 0\} = \{(x,y,z) \mid z = 0\}$

   This is true as all three sets represent all points in the xy-plane.

c) Any point on the z-axis has more than one representation both cylindrical and spherical coordinates.

   If a point is on the z-axis, then its polar/cylindrical radius r must be 0 because the projection of the point onto the xy-plane is the origin. Thus θ can be anything, so there is more than one representation of the point in cylindrical coordinates. In spherical, this means that the angle ϕ must be 0, so similarly any choice of θ can be used to represent the point. Thus the statement is true.

d) The surface defined by $z = x^2$ in Cartesian coordinates is the same as the surface $z = r^2\cos^2\theta$ in cylindrical coordinates.

   True. Using the conversion $x = r\cos\theta$, we have that $x^2 = r^2\cos^2\theta$.

e) In $\mathbb{R}^3$, if a point lies in both the xy-plane and the xz-plane, then the point lies on the x-axis.

   True. If the point is in the xy-plane, then $z = 0$. If the point is in the xz-plane, then $y = 0$. Thus the rectangular coordinates of the point must be $(x,0,0)$, which means the point is on the x-axis.

f) In cylindrical coordinates for a point, r also represents the distance from the point to the z-axis.

True, the distance from the point $(x, y, z)_r$ to the z-axis is the distance from the point to the point $(0, 0, z)_r$:

$$\sqrt{(x-0)^2 + (y-0)^2 + (z-z)^2} = \sqrt{x^2 + y^2} = r.$$

g) The surface $\theta = a$ (where a is a real number) in cylindrical coordinates is the same as the surface $\theta = a$ in spherical coordinates.

True, both describe the equation $a = \tan^{-1}(y/x)$, or $\tan a = y/x$.

h) If $r > 0$ and θ in the cylindrical coordinates for a point satisfies $\pi/2 < \theta < 3\pi/2$, then x in the rectangular coordinates for that point satisfies $x < 0$.

True, the condition $\pi/2 < \theta < 3\pi/2$ implies $x = r\cos\theta < 0$ as long as r is positive.

i) If $\rho > 0$ and ϕ in the spherical coordinates for a point satisfies $0 < \phi < \pi/2$, then y in the rectangular coordinates for that point satisfies $y > 0$.

False, the sign of y depends only upon θ and is independent of ϕ.

j) The spherical coordinates of any point in the sixth octant satisfy both $\pi/2 \le \theta \le \pi$ and $\pi/2 \le \phi \le \pi$.

True, if $\pi/2 \le \theta \le \pi$, then the point lies either in the second octant or the sixth octant, and the condition $\pi/2 \le \phi \le \pi$ implies the point is below the xy-plane, so it is in the sixth octant.

2. Give the spherical equation that represents the cylinder $x^2 + y^2 = a^2$ for some $a > 0$.

$$r = a.$$

3. Give the cylindrical equation that represents the Cartesian equation $y = mx + b$. What kind of surface is this?

$$r\sin\theta = mr\cos\theta + b,$$

this is a plane (note that $y = mx + b$ in the xy-plane is a line, so it is a plane in $\mathbb{R}^3$).

4. Convert the following Cartesian coordinates to cylindrical and spherical coordinates:

a) $(1, 0, 0)_r = (1, 0, 0)_c = (1, 0, \pi/2)_s$

b) $(0, 1, -1)_r = (1, \pi/2, -1)_c = (\sqrt{2}, \pi/2, 3\pi/4)_s$

c) $(-1, -\sqrt{3}, 2)_r = (2, 4\pi/3, 2)_c = (\sqrt{8}, 4\pi/3, \pi/4)_s$

5. Convert the following cylindrical coordinates to Cartesian and spherical coordinates:

a) $(\sqrt{6}, \pi/4, \sqrt{2})_c = (2\sqrt{2}, \pi/4, \pi/3)_s = (\sqrt{3}, \sqrt{3}, \sqrt{2})_r$

b) $(1, 0, -1)_c = (\sqrt{2}, 0, 3\pi/4)_s = (1, 0, -1)_r$

6. Convert the following spherical coordinates to Cartesian and cylindrical coordinates:

a) $(4, \pi/3, 2\pi/3)_s = (\sqrt{3}, 3, -2)_r = (\sqrt{12}, \pi/3, -2)_c$

b) $(1,\pi,\pi)_s = (0,0,-1)_r = (0,0,-1)_c$

7. Write the following equations in cylindrical and spherical coordinates:

a) $x^2 - 2x + y^2 + z^2 = 0$

$$x^2 - 2x + y^2 + z^2 = 0 \quad \Longrightarrow \quad x^2 - 2x + 1 + y^2 + z^2 = 1 \quad \Longrightarrow \quad (x-1)^2 + y^2 + z^2 = 1$$

(sphere centered at $(1,0,0)_r$ with radius 1)

cylindrical: $r^2 - 2r\cos\theta + z^2 = 0$

spherical: $\rho^2 - 2\rho\sin\phi\cos\theta = 0$

b) $x + 2y + 3z = 1$

cylindrical: $r\cos\theta + 2r\sin\theta + 3z = 1$

spherical: $\rho(\sin\phi\cos\theta + 2\sin\phi\sin\theta + 2\cos\phi) = 1$

c) $x^2 + y^2 = 2y$

cylindrical: $r^2 = 2r\sin\theta$ (or $r = 2\sin\theta$)

spherical: $\rho^2\sin^2\phi = 2\rho\sin\phi\sin\theta$ (or $\rho\sin\phi = 2\sin\theta$)

B.2 Section 1.2 Exercises

1. Determine if the following statements are true or false, giving a brief explanation or counterexample in each case.

a) In the complex plane, the complex conjugate of z is the reflection of z across the imaginary axis.

False, the complex conjugate of z is the reflection of z across the real (horizontal) axis.

b) For any complex number z, $|z| = \sqrt{\Re(z)^2 + \Im(z)^2}$.

True. If $z = a + bi$, then $\Re(z) = a$ and $\Im(z) = b$, so

$$|z| = \sqrt{a^2 + b^2} = \sqrt{\Re(z)^2 + \Im(z)^2}.$$

c) For any complex number z, the product $z\overline{z}$ is always real.

True. If $z = a + bi$ then $z\overline{z} = (a + bi)(a - bi) = a^2 - b^2 i^2 = a^2 + b^2$, which is real.

d) If $\pi < \arg(z) < 2\pi$, then the real part of z cannot be positive.

Since the argument is the polar angle, this condition means that z is in the bottom half of the complex plane. However, this only implies that the imaginary part of z is negative, so the statement is false. A specific counterexample is $z = 1 - i$, which has a polar representation of $z = \sqrt{2}\,(\cos(7\pi/4) + i\sin(7\pi/4))$.

e) The imaginary part of a complex number z is given by $\Im(z) = \dfrac{z - \overline{z}}{2i}$.

True:

$$\Im(z) = b = \frac{1}{2i}2bi = \frac{1}{2i}(a - a + bi + bi) = \frac{1}{2i}((a + bi) - (a - bi)) = \frac{z - \overline{z}}{2i}.$$

2. Which of the following is the conjugate of the standard form of the complex number $(\sqrt{3} + i)^4$?

A) $8 - 8\sqrt{3}i$ B) $-8 - 8\sqrt{3}i$ C) $-8 + 8\sqrt{3}i$ D) $8 + 8\sqrt{3}i$

The answer is B, $-8 - 8\sqrt{3}i$ as, using DeMoivre's Theorem,

$$(\sqrt{3} + i)^4 = (2(\cos(\pi/6) + i\sin(\pi/6))^4 = 2^4\,(\cos(4\pi/6) + i\sin(4\pi/6)) = 16\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = -8 + 8\sqrt{3}i,$$

so $\overline{(\sqrt{3} + i)^4} = -8 - 8\sqrt{3}i$.

3. Find wz, w/z, and $1/z$ for each:

a) $w = 1 + \sqrt{3}i, z = \sqrt{3} + i$

$$wz = (1 + \sqrt{3}i)(\sqrt{3} + i) = \sqrt{3} + i + 3i + \sqrt{3}i^2 = 4i$$

$$w/z = \frac{1+\sqrt{3}i}{\sqrt{3}+i}\cdot\frac{\sqrt{3}-i}{\sqrt{3}-i} = \frac{1}{4}\left(\sqrt{3}-i+3i-\sqrt{3}i^2\right) = \frac{1}{4}\left(2\sqrt{3}+2i\right) = \frac{\sqrt{3}}{2}+\frac{1}{2}i.$$

$$1/z = \frac{1}{\sqrt{3}+i}\cdot\frac{\sqrt{3}-i}{\sqrt{3}-i} = \frac{1}{4}\left(\sqrt{3}-i\right) = \frac{\sqrt{3}}{4}-\frac{1}{4}i.$$

b) $w = -3-3i, z = 4\sqrt{3}+4i$

$$wz = (-3-3i)(4\sqrt{3}+4i) = -12\sqrt{3}-12i-12\sqrt{3}i-12i^2 = 12(1-\sqrt{3})-12(1+\sqrt{3})i$$

$$w/z = \frac{-3-3i}{4\sqrt{3}+4i}\cdot\frac{4\sqrt{3}-4i}{4\sqrt{3}-4i} = \frac{1}{16\cdot 3+16}\left(-12\sqrt{3}+12i-12\sqrt{3}i+12i^2\right) = -\frac{3(1+\sqrt{3})}{16}+\frac{3(1-\sqrt{3})}{16}i$$

$$1/z = \overline{z}/|z|^2 = \frac{1}{64}\left(4\sqrt{3}-4i\right) = \frac{\sqrt{3}}{16}-\frac{1}{16}i.$$

4. Find the standard form of $z+w$ if $z = 4e^{3i}$ and $w = 5e^{2i}$.

$$z+w = 4e^{3i}+5e^{2i} = 4(\cos 3+i\sin 3)+5(\cos 2+i\sin 2) = (4\cos 3+5\cos 2)+(4\sin 3+5\sin 2)i.$$

5. Find the complex conjugate of $z = (x+iy)^2-4e^{ixy}$.

$$\overline{z} = \overline{(x+iy)^2-4e^{ixy}} = \overline{(x+iy)^2}-\overline{4e^{ixy}} = \left(\overline{x+iy}\right)^2-4(\overline{\cos xy+i\sin xy}) = (x-iy)^2-4(\cos xy-i\sin xy).$$

Note that the conjugate of $e^{i\theta}$ is

$$\overline{e^{i\theta}} = \overline{\cos\theta+i\sin\theta} = \cos\theta-i\sin\theta = \cos(-\theta)+i\sin(-\theta) = e^{-i\theta}.$$

6. Compute $(1-i)^8$.

The polar representation of $z = 1-i$ is $z = \sqrt{2}\,(\cos(7\pi/4)+i\sin(7\pi/4))$. Thus, using DeMoivre's theorem, we have

$$(1-i)^8 = \left(\sqrt{2}\,(\cos(7\pi/4)+i\sin(7\pi/4))\right)^8 = 2^4\,(\cos(56\pi/4)+i\sin(56\pi/4)) = 16\,(\cos 14\pi+i\sin 14\pi) = 16.$$

7. If $z = \left(\frac{\sqrt{3}-i}{2+2i}\right)^2$, find $\Re(z)$, $\Im(z)$, and the polar representation of z.

Let's first find the polar representation of z:

$$\sqrt{3}-i = 2(\cos(11\pi/6)+i\sin(11\pi/6)), \qquad 2+2i = \sqrt{8}(\cos(\pi/4)+\sin(\pi/4))$$

$$\frac{\sqrt{3}-i}{2+2i} = \frac{2}{2\sqrt{2}}\,(\cos(11\pi/6-\pi/4)+i\sin(11\pi/6-\pi/4)) = \frac{\sqrt{2}}{2}\,(\cos(19\pi/12)+i\sin(19\pi/12))$$

$$z = \left(\frac{\sqrt{3}-i}{2+2i}\right)^2 = \frac{2}{4}(\cos(19\pi/6) + i\sin(19\pi/6)) = \underbrace{\frac{1}{2}(\cos(7\pi/6) + i\sin(7\pi/6))}_{\text{polar representation}} = \frac{1}{2}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

$$\Re(z) = -\frac{\sqrt{3}}{4}, \qquad \Im(z) = -\frac{1}{4}$$

8. Give a general formula for the three cube roots of $z = e^{i\theta}$ and use it to find the three cube roots of $1-i$.

Note that since $z = e^{i\theta} = e^{i(\theta+2\pi)} = e^{i(\theta+4\pi)}$, we have that

$$\sqrt[3]{z} = e^{i\theta/3}, \quad e^{i(\theta+2\pi)/3}, \quad e^{i(\theta+4\pi)/3}$$

are the three cube roots of z. Since

$$1 - i = \sqrt{2}(\cos(7\pi/4) + i\sin(7\pi/4)) = \sqrt{2}e^{7i\pi/4},$$

we have that the cube roots of $1-i$ are

$$\sqrt[6]{2}e^{7i\pi/12}, \quad \sqrt[6]{2}e^{15i\pi/12}, \quad \sqrt[6]{2}e^{23i\pi/12}.$$

9. Use Euler's formula to derive the following formulas for $\sin x$ and $\cos x$:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

Using Euler's formula, note that

$$e^{ix} = \cos x + i\sin x, \qquad e^{-ix} = \cos(-x) + i\sin(-x) = \cos x - i\sin x.$$

Then we have

$$e^{ix} + e^{-ix} = \cos x + i\sin x + \cos x - i\sin x = 2\cos x \qquad \Longrightarrow \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2},$$

and

$$e^{ix} - e^{-ix} = \cos x + i\sin x - \cos x + i\sin x = 2i\sin x \qquad \Longrightarrow \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

B.3 Section 1.3 Exercises

1. Which of the following quantities are vectors and which are scalars?

a) The mass of a water molecule. (S)

b) The center of mass of a water molecule. (V)

c) The bond length of a water molecule. (S)

d) The dipole moment of a water molecule. (V)

e) The charge of an electron. (S)

f) The temperature of a gas. (S)

g) The pressure of a gas. (S)

h) The velocity of a gas. (V)

2. Suppose a molecule has three atoms. After establishing a coordinate system (in the xy-plane), you determine that atom 1 is located at the point $(1,1)$ and has a mass of 2 units, atom 2 is located at the point $(-1,1)$ and has a mass of 4 units, and atom 3 is located at the point $(1,-1)$ and has a mass of 8 units. Where do you expect the center of mass of this molecule to be?

A) Quadrant I B) Quadrant II C) Quadrant III D) Quadrant IV E) The Origin

The center of mass should be in Quadrant IV because (a) there is more mass below the x-axis than above it and there is more mass to the left of the y-axis than to the right.

3. Find two distinct unit vectors that are parallel to the vector $\vec{v} = 4\vec{\boldsymbol{\imath}} - 2\vec{\boldsymbol{\jmath}} + \frac{1}{2}\vec{\boldsymbol{k}}$.

First we find $|\vec{v}| = \sqrt{4^2+2^2+(1/2)^2} = \sqrt{16+4+1/4} = \sqrt{81/4} = 9/2$. Then two unit vectors parallel to $\vec{v}$ are

$$\frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{8}{9}, -\frac{4}{9}, \frac{1}{9} \right\rangle$$

and

$$-\frac{\vec{v}}{|\vec{v}|} = \left\langle -\frac{8}{9}, \frac{4}{9}, -\frac{1}{9} \right\rangle.$$

4. Find $4\vec{u} - \vec{v}$ and $|\vec{u} + 3\vec{v}|$ if $\vec{u} = \langle -2,-3,0 \rangle$ and $\vec{v} = \langle 1,2,1 \rangle$.

$$4\vec{u} - \vec{v} = 4\langle -2,-3,0\rangle - \langle 1,2,1\rangle = \langle -9,-14,-1\rangle$$

$$|\vec{u} + 3\vec{v}| = |\langle -2,-3,0\rangle + 3\langle 1,2,1\rangle| = |\langle 1,3,3\rangle| = \sqrt{1+9+9} = \sqrt{19}$$

5. Determine if the following statements are true or false, giving a brief explanation or counterexample in each case.

a) The magnitude of a sum of vectors is the sum of the magnitudes of the vectors.

False. The magnitude of a sum of vectors $\vec{u}$ and $\vec{v}$ would be $|\vec{u}+\vec{v}|$, while the sum of the magnitudes of $\vec{u}$ and $\vec{v}$ would be $|\vec{u}|+|\vec{v}|$. These are almost always not the same, for example let $\vec{u}=\langle 1,0\rangle$ and $\vec{v}=\langle -1,0\rangle$.

b) Given a nonzero scalar a and vectors $\vec{u}$ and $\vec{v}$, the vector equation $a\vec{w}+\vec{u}=\vec{v}$ always has a unique solution $\vec{w}$.

True, this is asking if it is possible to solve for $\vec{w}$ in the above equation. You can solve for $\vec{w}$ as

$$\vec{w}=\frac{1}{a}(\vec{v}-\vec{u})$$

since we can divide by a (as it is a nonzero scalar) and we can form $\vec{v}-\vec{u}$ (as subtraction of vectors is defined).

c) It is possible for two vectors that point in the same direction to have different magnitudes.

True, for example the vectors $\vec{v}$ and $2\vec{v}$ (where $\vec{v}$ is a nonzero vector) point in the same direction but have different magnitudes.

d) It is possible for two vectors that point in different directions to have the same magnitudes.

True, the unit vectors $\vec{\imath}$ and $\vec{\jmath}$ point in different directions and both have magnitude 1.

6. An object is acted upon by the forces $\vec{F}_1=\langle 10,6,3\rangle$ and $\vec{F}_2=\langle 0,4,9\rangle$. Find the force $\vec{F}_3$ that must act on the object so that the sum of their forces is zero (so the object is in static equilibrium).

We want to find $\vec{F}_3$ such that $\vec{F}_1+\vec{F}_2+\vec{F}_3=\vec{0}$. This means

$$\vec{F}_3=-\vec{F}_1-\vec{F}_2=-\langle 10,6,3\rangle-\langle 0,4,9\rangle=\langle -10,-10,-12\rangle.$$

7. Find a vector parallel to $\vec{v}=\langle 3,-2,6\rangle$ with length 10.

A vector parallel to $\vec{v}$ with magnitude 10 can be found by first finding the unit vector in the direction of $\vec{v}$ and then multiplying that vector by 10, i.e., $10\vec{v}/|\vec{v}|$. We have

$$10\frac{\vec{v}}{|\vec{v}|}=10\frac{\langle 3,-2,6\rangle}{\sqrt{9+4+36}}=\frac{10}{7}\langle 3,-2,6\rangle=\left\langle \frac{30}{7},-\frac{20}{7},\frac{60}{7}\right\rangle.$$

8. A molecule of carbon monoxide (CO) consists of a single carbon atom (with mass $m_{\mathrm{C}}=12.0107$amu) and a single oxygen atom (with mass $m_{\mathrm{O}}=15.9994$amu). The two atoms are $\ell=1.11\times 10^{-10}$m apart. Establish a coordinate system and find the center of mass. Explain your process.

The simplest coordinate system will be to set the molecule in the xy-plane, setting the carbon atom at the origin and the oxygen atom along the positive x-axis. This way, no parts of the molecule have any y ($\vec{\jmath}$) component. Then the position of the carbon atom is $\langle 0,0\rangle$ and the position of the oxygen atom is $\langle \ell,0\rangle$. Thus the center of mass $\vec{r}$ is given by

$$\vec{r}=\frac{1}{m_c+m_o}\left(m_c\langle 0,0\rangle+m_o\langle \ell,0\rangle\right)=\left\langle \frac{m_o\ell}{m_c+m_o},0\right\rangle\approx\left\langle 6.34\times 10^{-11},0\right\rangle.$$

9. Three charges $q_1 = -2$, $q_2 = 2$, and $q_3 = -1$ are located at points $(3,0,-2)_r$, $(1,1,1)_r$, and $(0,5,-2)_r$, respectively.

a) Find the dipole moment of the system of charges with respect to the origin.

$$\begin{aligned}\vec{\mu}(\vec{0}) &= q_1\vec{r}_1 + q_2\vec{r}_2 + q_3\vec{r}_3 \\ &= -2\langle 3,0,-2\rangle + 2\langle 1,1,1\rangle - 1\langle 0,5,-2\rangle \\ &= \langle -4,-3,8\rangle.\end{aligned}$$

b) Find the position of the point R such that the dipole moment with respect to R is the zero vector.

We want to find $\vec{R}$ such that $\vec{\mu}(\vec{R}) = \vec{0}$, or $\vec{\mu}(\vec{0}) - Q\vec{R} = \vec{0}$, where Q is the sum of the charges. This means that

$$\vec{R} = \frac{1}{Q}\vec{\mu}(\vec{0}) = \frac{1}{-2+2-1}\langle -4,-3,8\rangle = \langle 4,3,-8\rangle.$$

B.4 Section 1.4 Exercises

1. Determine if the following statements are true or false, giving a brief explanation or counterexample in each case.

a) If $\vec{u}$ and $\vec{v}$ are vectors, then the the product of the magnitudes of $\vec{u}$ and $\vec{v}$ is less than or equal to the absolute value of the dot product of $\vec{u}$ and $\vec{v}$.

This is asking if the inequality $|\vec{u}||\vec{v}| \le |\vec{u}\cdot\vec{v}|$ is true. Since $|\vec{u}\cdot\vec{v}| = |\vec{u}||\vec{v}||\cos\theta|$ and $|\cos\theta| \le 1$ for all θ, it is true that

$$|\vec{u}\cdot\vec{v}| = |\vec{u}||\vec{v}||\cos\theta| \le |\vec{u}||\vec{v}|,$$

so the original statement is false.

b) When $\vec{u}\cdot\vec{v} > 0$, the angle between $\vec{u}$ and $\vec{v}$ is less than $\pi/2$.

True, if the dot product of the two vectors is positive, then the cosine of the angle between the vectors is also positive, which means the angle must be less than $\pi/2$.

c) If $\vec{u} = \langle u_1, u_2, u_3\rangle$ is a vector, then the vector $\vec{w} = \vec{u} - u_2\vec{\boldsymbol{j}}$ is orthogonal to $\vec{\boldsymbol{j}}$.

True. This can be seen as

$$\vec{w}\cdot\vec{\boldsymbol{j}} = (\vec{u} - u_2\vec{\boldsymbol{j}})\cdot\vec{\boldsymbol{j}} = \vec{u}\cdot\vec{\boldsymbol{j}} - u_2\vec{\boldsymbol{j}}\cdot\vec{\boldsymbol{j}} = u_2 - u_2 = 0.$$

d) If $\vec{u} = \langle u_1, u_2, u_3\rangle$ is a vector in $\mathbb{R}^3$, then there are exactly two unit vectors that are orthogonal to $\vec{u}$.

False, there are infinitely many unit vectors that are orthogonal to $\vec{u}$. For example, any unit vector $\vec{v}/|\vec{v}|$ where $\vec{v} = \langle 0, a, b\rangle$ (for any nonzero real a, b) will be orthogonal to $\vec{\boldsymbol{\imath}}$.

e) Two vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $|\vec{u}+\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2$.

True. If $\vec{u}$ and $\vec{v}$ are orthogonal, then $\vec{u}\cdot\vec{v} = 0$. This implies that

$$|\vec{u}+\vec{v}|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} = |\vec{u}|^2 + 0 + |\vec{v}|^2.$$

It's easy to see that the converse is also true, i.e., if $|\vec{u}+\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2$, then $\vec{u}\cdot\vec{v}$ must be 0, so they are orthogonal.

2. Find all values of c such that the vectors $\vec{u} = \langle c, -6, 2\rangle$ and $\vec{v} = \langle c^2, c, c\rangle$ are orthogonal.

We want to find all c such that

$$0 = \vec{u}\cdot\vec{v} = \langle c, -6, 2\rangle\cdot\langle c^2, c, c\rangle = c^3 - 6c + 2c = c^3 - 4c.$$

This will be true when $c = 0, 2, -2$.

3. Find two vectors that are orthogonal to $\langle 0, 1, 1\rangle$ and to each other.

One obvious vector that is orthogonal to $\vec{v} = \langle 0,1,1\rangle$ is $\vec{u} = \langle 1,0,0\rangle$. To find a vector $\vec{w}$ that is orthogonal to both, we need to ensure that $\vec{w}\cdot\vec{u} = 0$ and $\vec{w}\cdot\vec{v} = 0$. Note that $\vec{w}\cdot\vec{u} = 0$ implies that $w_1 = 0$. Thus an easy choice is $\vec{w} = \langle 0,1,-1\rangle$ as it clearly satisfies $\vec{w}\cdot\vec{v} = 0$.

4. Find another vector that has the same projection onto $\vec{v} = \langle 1,1,1\rangle$ as $\vec{u} = \langle 1,2,3\rangle$.

We first compute $\text{proj}_{\vec{v}}\vec{u}$ as

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{6}{3}\vec{v} = \langle 1,1,1\rangle = \langle 2,2,2\rangle.$$

So we need to find a vector $\vec{w} = \langle w_1, w_2, w_3\rangle$ such that $\text{proj}_{\vec{v}}\vec{w} = \langle 2,2,2\rangle$. This would mean that

$$\text{proj}_{\vec{v}}\vec{w} = \frac{\vec{w}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{w_1+w_2+w_3}{3}\langle 1,1,1\rangle = \langle 2,2,2\rangle,$$

or $w_1 + w_2 + w_3 = 6$. For example, $\vec{w} = \langle 1,1,4\rangle$ would suffice.

5. Express $\vec{u} = \langle -1,2,3\rangle$ as the sum of a vector parallel to $\vec{v} = \langle 2,1,1\rangle$ and a vector orthogonal to $\vec{v}$.

We can write $\vec{u} = \text{proj}_{\vec{v}}\vec{u} + (\vec{u} - \text{proj}_{\vec{v}}\vec{u})$ where $\text{proj}_{\vec{v}}\vec{u}$ is parallel to $\vec{v}$ and $(\vec{u} - \text{proj}_{\vec{v}}\vec{u})$ is orthogonal to $\vec{v}$. We have

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{3}{6}\langle 2,1,1\rangle = \left\langle 1,\frac{1}{2},\frac{1}{2}\right\rangle,$$

so we have

$$\vec{u} = \left\langle 1,\frac{1}{2},\frac{1}{2}\right\rangle + \left(\vec{u} - \left\langle 1,\frac{1}{2},\frac{1}{2}\right\rangle\right) = \left\langle 1,\frac{1}{2},\frac{1}{2}\right\rangle + \left\langle -2,\frac{3}{2},\frac{5}{2}\right\rangle.$$

6. Find the direction angles of the vector $\vec{u} = \vec{\boldsymbol{\imath}} - \vec{\boldsymbol{\jmath}} + \sqrt{2}\vec{\boldsymbol{k}}$.

We have

$$\vec{u} = \langle 1,-1,\sqrt{2}\rangle = |\vec{u}|\langle\cos\alpha,\cos\beta,\cos\gamma\rangle = 2\left\langle\frac{1}{2},-\frac{1}{2},\frac{\sqrt{2}}{2}\right\rangle.$$

So $\cos\alpha = 1/2$ implies $\alpha = \pi/3$, $\cos\beta = -1/2$ implies $\beta = 2\pi/3$, and $\cos\gamma = \sqrt{2}/2$ implies $\gamma = \pi/4$.

7. If $(\vec{a} - \vec{b})\cdot\vec{c} = 0$, is it true that $\vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}$? Explain why or why not.

This is true as

$$0 = (\vec{a} - \vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} - \vec{b}\cdot\vec{c} \qquad \Longrightarrow \qquad \vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}.$$

8. Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors. Which of the following expressions are meaningful and which are not? Give the object that results (scalar, vector, etc.) or explain why it is not meaningful.

a) $(\vec{u}\cdot\vec{v})\vec{w}$ meaningful, result is a vector

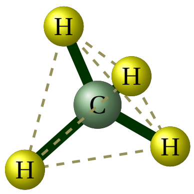
b) $|\vec{u}|(\vec{v}\cdot\vec{w})$ meaningful, result is a scalar

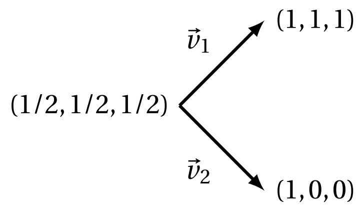
c) $\vec{u}\cdot\vec{v} + \vec{w}$ not meaningful, can't add a scalar and a vector

d) $\vec{u}\cdot(\vec{v} + \vec{w})$ meaningful, result is a scalar

e) $|\vec{u}|\cdot(\vec{v} + \vec{w})$ not meaningful, can't dot a scalar with a vector

9. A molecule of methane, CH_4, is structured with the four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid. The **bond angle** is the angle formed by the H-C-H combination - it is the angle between the lines that join the carbon atom to two of the hydrogen atoms. Find the bond angle (in degrees) of methane. (Hint: set the hydrogen atoms at the vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,1)$, then the centroid is at $(1/2,1/2,1/2)$.)





We can determine the bond angle by finding the angle between the two vectors $\vec{v}_1$ and $\vec{v}_2$ that originate from the carbon atom and terminate at two hydrogen atoms. In the figure above, $\vec{v}_1 = \langle 1/2,1/2,1/2\rangle$ and $\vec{v}_2 = \langle 1/2,-1/2,-1/2\rangle$. Then the angle θ between $\vec{v}_1$ and $\vec{v}_2$ satisfies

$$\cos\theta = \frac{\vec{v}_1\cdot\vec{v}_2}{|\vec{v}_1||\vec{v}_2|} = \frac{\frac{1}{4}-\frac{1}{4}-\frac{1}{4}}{\sqrt{\frac{3}{4}}\sqrt{\frac{3}{4}}} = -\frac{1}{4}\cdot\frac{4}{3} = -\frac{1}{3}.$$

Thus $\theta = \cos^{-1}(-1/3) \approx 109.47°$.

B.5 Section 1.5 Exercises

1. Let $\vec{t}$, $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^3$. Which of the following expressions are meaningful and which are not? Give the object that results (scalar, vector, etc.) or explain why it is not meaningful.

a) $(\vec{u}\cdot\vec{v})\times\vec{w}$ is not meaningful as you cannot cross a scalar $(\vec{u}\cdot\vec{v})$ with a vector.

b) $(\vec{u}\times\vec{v})\vec{w}$ is not meaningful because $\vec{u}\times\vec{v}$ is not a scalar.

c) $|\vec{u}|(\vec{v}\times\vec{w})$ is meaningful and results in a vector.

d) $(\vec{u}\cdot\vec{v})\vec{t}+\vec{w}$ is meaningful and results in a vector.

e) $(\vec{u}+\vec{t})\times(\vec{v}+\vec{w})$ is meaningful and results in a vector.

f) $\vec{t}\times(\vec{u}\cdot\vec{v})\vec{w}$ is meaningful and results in a vector.

2. Suppose that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^3$ and that $\vec{u}\neq\vec{0}$. Determine if the statements below are true or false, giving a brief explanation or counterexample.

a) If $\vec{u}\cdot\vec{v}=\vec{u}\cdot\vec{w}$, then $\vec{v}=\vec{w}$.

False, a counterexample is given with $\vec{u}=\vec{\boldsymbol{\imath}}$, $\vec{v}=\vec{\boldsymbol{\jmath}}$, $\vec{w}=\vec{\boldsymbol{k}}$.

b) If $\vec{u}\times\vec{v}=\vec{u}\times\vec{w}$, then $\vec{v}=\vec{w}$.

False, a counterexample is given with $\vec{v}=\vec{u}$ and $\vec{w}=-\vec{u}$.

c) If $\vec{u}\times\vec{v}=\vec{0}$ then $\vec{u}$ and $\vec{v}$ must be the same vector.

False, $\vec{u}\times 2\vec{u}=\vec{0}$ as well.

3. Determine if the following are true or false, giving a brief explanation or counterexample.

a) If $\vec{u}\times\vec{v}=\vec{0}$ and $\vec{u}\cdot\vec{v}=0$, then either $\vec{u}=\vec{0}$ or $\vec{v}=\vec{0}$.

This is true. Note that if $\vec{u}\times\vec{v}=\vec{0}$, then either $|\vec{u}|=0$ (which means $\vec{u}=\vec{0}$), $|\vec{v}|=0$ ($\vec{v}=\vec{0}$), or $\sin\theta=0$. Suppose neither $|\vec{u}|=0$ or $|\vec{v}|=0$. If $\sin\theta=0$, then the angle θ between $\vec{u}$ and $\vec{v}$ must either be 0 or π. If $\theta=0$, then $\vec{u}=\vec{v}$ so $0=\vec{u}\cdot\vec{v}=|\vec{u}|^2$, implying $\vec{u}=\vec{v}=\vec{0}$. If $\theta=\pi$ then $\vec{v}=-\vec{u}$ so $0=\vec{u}\cdot\vec{v}=-|\vec{u}|^2$, again implying that $\vec{u}=\vec{v}=\vec{0}$.

b) $\vec{u}\times(\vec{u}\times\vec{v})=\vec{0}$.

False. Let $\vec{u}=\vec{\boldsymbol{\imath}}$ and $\vec{v}=\vec{\boldsymbol{\jmath}}$. Then

$$\vec{u}\times(\vec{u}\times\vec{v})=\vec{\boldsymbol{\imath}}\times(\vec{\boldsymbol{\imath}}\times\vec{\boldsymbol{\jmath}})=\vec{\boldsymbol{\imath}}\times\vec{\boldsymbol{k}}=-\vec{\boldsymbol{\jmath}}\neq\vec{0}.$$

c) $|\vec{u} \times \vec{v}|$ is less than both $|\vec{u}|$ and $|\vec{v}|$.

False. While we know that $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin\theta \le |\vec{u}||\vec{v}|$, it is certainly possible that $|\vec{u} \times \vec{v}| \ge |\vec{u}|$. For example, let $\vec{u} = 0.1\vec{\imath}$ and $\vec{v} = 10\vec{\jmath}$. Then $\vec{u} \times \vec{v} = (0.1\vec{\imath}) \times (10\vec{\jmath}) = \vec{k}$ has a magnitude of 1, while $\vec{u}$ has a magnitude of 0.1.

d) For any vector $\vec{v} \in \mathbb{R}^3$, $\vec{v} \cdot (\vec{v} \times \vec{v}) = 0$.

True. $\vec{v} \times \vec{v} = \vec{0}$ and every vector is orthogonal to $\vec{0}$.

e) The magnitude of $\vec{u} \times \vec{v}$ is equal to the absolute value of $\vec{u} \cdot \vec{v}$.

False. We know $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin\theta$ and $|\vec{u} \cdot \vec{v}| = |\vec{u}||\vec{v}||\cos\theta|$, so in fact these two quantities will be equal for nonzero $\vec{u}$ and $\vec{v}$ only when $\theta = \pi/4$. The choices $\vec{u} = \vec{\imath}$ and $\vec{v} = \vec{\jmath}$ produce an easy counterexample.

4. If $\vec{u}$ and $\vec{v}$ are unit vectors with an angle between them of $\pi/3$, what is $|\vec{u} \times \vec{v}|$?

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin\theta = 1 \cdot 1 \cdot \sin(\pi/3) = \sqrt{3}/2.$$

5. Find the area of the triangle with vertices $A(1,2,3)$, $B(5,1,5)$, and $C(2,3,3)$.

We can find the area by taking $\frac{1}{2}|\vec{v}_1 \times \vec{v}_2|$ where $\vec{v}_1$ and $\vec{v}_2$ are two vectors formed by two edges of the triangle. Let $\vec{v}_1 = \vec{AB} = \langle 4,-1,2\rangle$ and $\vec{v}_2 = \vec{AC} = \langle 1,1,0\rangle$. Then

$$\vec{v}_1 \times \vec{v}_2 = \langle 0-2, 2-0, 4+1\rangle = \langle -2,2,5\rangle,$$

and so the area is

$$\frac{1}{2}|\langle -2,2,5\rangle| = \frac{1}{2}\sqrt{4+4+25} = \frac{\sqrt{33}}{2}.$$

6. Find two unit vectors orthogonal to both $\vec{\imath} + \vec{\jmath} + \vec{k}$ and $2\vec{\imath} + \vec{k}$.

Let $\vec{u} = \langle 1,1,1\rangle$ and $\vec{v} = \langle 2,0,1\rangle$. To find a unit vector that is orthogonal to both $\vec{u}$ and $\vec{v}$, we first find a vector $\vec{w}$ orthogonal to both and then divide it by its magnitude:

$$\vec{w} = \vec{u} \times \vec{v} = \langle 1,1,1\rangle \times \langle 2,0,1\rangle = \langle 1,1,-2\rangle.$$

Then

$$\frac{\vec{w}}{|\vec{w}|} = \frac{\langle 1,1,-2\rangle}{\sqrt{6}} = \left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right\rangle$$

is a unit vector orthogonal to both $\vec{u}$ and $\vec{v}$, and $-\vec{w}/|\vec{w}|$ is another.

7. A particle with a positive unit charge ($q = 1$) enters a constant magnetic field $\vec{B} = \vec{\imath} + \vec{\jmath}$ with a velocity $\vec{v} = 20\vec{k}$. Find the magnitude and direction of the magnetic force on the particle.

$$\vec{F} = q(\vec{v} \times \vec{B}) = 1(\langle 0,0,20\rangle \times \langle 1,1,0\rangle) = \langle -20,20,0\rangle.$$

$$|\vec{F}| = \sqrt{800} = 20\sqrt{2}.$$

8. A charge q moving with velocity $\vec{v}$ in the presence of both a magnetic field $\vec{B}$ and an electric field $\vec{E}$ experiences a total force of $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$ called the **Lorentz force**. Calculate the force acting on the charge $q = 3$ moving with velocity $\vec{v} = \langle 2, 3, 1 \rangle$ in the presence of an electric field $\vec{E} = 2\vec{\boldsymbol{\imath}}$ and magnetic field $\vec{B} = 3\vec{\boldsymbol{\jmath}}$.

$$\begin{aligned}
\vec{F} &= q\vec{E} + q\vec{v} \times \vec{B} \\
&= 3\langle 2,0,0\rangle + 3(\langle 2,3,1\rangle \times \langle 0,3,0\rangle) \\
&= \langle 6,0,0\rangle + 3\langle -3,0,6\rangle \\
&= \langle -3,0,18\rangle.
\end{aligned}$$

B.6 Section 1.6 Exercises

1. Give a parametrization of the line segment from the point $(2,4,8)$ and $(7,5,3)$.

$$\vec{r}_0 = \langle 2,4,8\rangle, \qquad \vec{r}_1 = \langle 7,5,3\rangle$$

$$\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1 = (1-t)\langle 2,4,8\rangle + t\langle 7,5,3\rangle = \langle 2+5t, 4+t, 8-5t\rangle, \qquad 0 \le t \le 1$$

2. Find an equation of the line through the points $(-3,4,6)$ and $(5,-1,0)$.

The direction of the line is $\vec{v} = \langle 5-(-3), -1-4, 0-6\rangle = \langle 8,-5,-6\rangle$, so the vector equation of the line is

$$\vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle -3+8t, 4-5t, 6-6t\rangle.$$

3. Find an equation of the line through $(-3,4,2)$ that is perpendicular to both $\vec{u} = \langle 1,1,-5\rangle$ and $\vec{v} = \langle 0,4,0\rangle$.

The direction of the line is $\vec{u} \times \vec{v} = \langle 20,0,4\rangle$, so the vector equation of the line is

$$\vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle -3+20t, 4, 2+4t\rangle.$$

4. Is the line through $(4,1,-1)$ and $(2,5,3)$ perpendicular to the line through $(-3,2,0)$ and $(5,1,4)$?

We determine this by finding the directions of both lines and checking for orthogonality. The two direction vectors are $\vec{v}_1 = \langle -2,4,4\rangle$ and $\vec{v}_2 = \langle 8,-1,4\rangle$, and we have

$$\vec{v}_1 \cdot \vec{v}_2 = \langle -2,4,4\rangle \cdot \langle 8,-1,4\rangle = -16-4+16 = -4 \neq 0,$$

so the lines are not perpendicular.

5. Determine whether the line $x = 3+8t$, $y = 4+5t$, $z = -3-t$ is parallel to the plane $x-3y+5z = 12$.

To determine this, we check to see if the direction of the line is orthogonal to the normal vector of the plane. The direction of the line is $\vec{v} = \langle 8,5,-1\rangle$ and the normal vector is $\vec{n} = \langle 1,-3,5\rangle$, and we have

$$\vec{v} \cdot \vec{n} = \langle 8,5,-1\rangle \cdot \langle 1,-3,5\rangle = 8-15-5 = -12 \neq 0,$$

so the line is not parallel to the plane.

6. Find an equation of the line of intersection of the planes $x-y-2z = 1$ and $x+y+z = -1$.

We can find the direction of the line of intersection by finding a vector that is orthogonal to both normal vectors:

$$\vec{n}_1 \times \vec{n}_2 = \langle 1,-1,-2\rangle \times \langle 1,1,1\rangle = \langle 1,-3,2\rangle.$$

To find a point on the line, we need to find a point that lies in both planes, i.e., a point that satisfies both plane equations. This would imply that $x-y-2z = 1$ and $x+y+z = -1$. If we set $z = 0$, we get the two equations

$x-y=1$ and $x+y=-1$, and a solution to these two equations is $x=0$ and $y=-1$. Thus $(0,-1,0)$ is a point in both planes and therefore is on the line of intersection. The vector equation of the line of intersection is then

$$\vec{r}(t)=\vec{r}_0+t\vec{v}=\langle t,-1-3t,2t\rangle.$$

7. Find the point at which the line $x=1+2t$, $y=4t$, $z=2-3t$ intersects the plane $x+2y-z+1=0$.

We need to find the value of t where the x, y, and z coordinates of points on the line satisfy the plane equation. Substituting in the parametric equations into the plane equation gives

$$(1+2t)+2(4t)-(2-3t)+1=0 \qquad \Longrightarrow \qquad 13t=0,$$

or $t=0$. Thus the point on the line that is also in the plane is $(1,0,2)$.

8. Find an equation of the plane containing $(3,0,-2)$ that is parallel to both $\langle 1,-3,1\rangle$ and $\langle 4,2,0\rangle$.

We can find the normal vector of the plane by taking the cross product of the above vectors:

$$\vec{n}=\langle 1,-3,1\rangle\times\langle 4,2,0\rangle=\langle -2,4,14\rangle.$$

Thus the equation of the plane is

$$-2(x-3)+4(y-0)+14(z+2)=0,$$

or

$$-2x+4y+14z=-34.$$

9. Find an equation of the plane containing the points $(2,-1,4)$, $(1,1,-1)$, and $(-4,1,1)$.

We need two vectors in the plane to find the normal vector. Two vectors in the plane are

$$\vec{v}_1=\langle 1-2,1+1,-1-4\rangle=\langle -1,2,-5\rangle, \qquad \vec{v}_2=\langle -4-2,1+1,1-4\rangle=\langle -6,2,-3\rangle.$$

$$\vec{n}=\vec{v}_1\times\vec{v}_2=\langle -1,2,-5\rangle\times\langle -6,2,-3\rangle=\langle 4,27,10\rangle.$$

Thus an equation of the plane is

$$4(x-2)+27(y+1)+10(z-4)=0,$$

or

$$4x+27y+10z=21.$$

10. Determine if the statements below are true or false, giving a brief explanation or counterexample.

a) The line $\vec{r}(t)=\langle 3,-1,4\rangle+t\langle 6,-2,8\rangle$ passes through the origin.

True. Note that if $t=-1/2$ we have

$$\vec{r}(-1/2)=\langle 3,-1,4\rangle-\langle 3,-1,4\rangle=\langle 0,0,0\rangle.$$

b) The plane containing the point $(1,1,1)$ with normal vector $\vec{n}=\langle 1,2,-3\rangle$ is the same as the plane containing the point $(3,0,1)$ with normal vector $\vec{n} = \langle -2,-4,6\rangle$.

We form the plane equations and see if they represent the same plane. The first equation is $x-1+2(y-1)-3(z-1)=0$, or $x+2y-3z=0$. The second equation is $-2(x-3)-4(y-0)+6(z-1)=0$, or $-2x-4y+6z=0$. These represent the same plane, so the statement is true.

c) If plane p_1 is orthogonal to plane p_2, and plane p_2 is orthogonal to plane p_3, then planes p_1 and p_3 are parallel.

False. Consider the three coordinate planes (p_1 is the xy-plane, p_2 is the yz-plane, and p_3 is the xz-plane). All three planes are pairwise orthogonal.

d) Any two distinct lines in $\mathbb{R}^3$ determine a plane.

False. A plane cannot contain two skew lines.

e) Given any plane p, there is exactly one plane that is orthogonal to p.

False. There are infinitely many planes that are orthogonal to any given plane. For example, the plane $x=k$ for any real number k is orthogonal to the the xy-plane.

f) Given a plane p and a point P_0 not in the plane, there is exactly one plane that is orthogonal to p and contains P_0.

False. Consider the xy-plane and let P_0 be the origin. Then there are two planes (the xz-plane and the yz-plane) that are both orthogonal to the xy-plane that also contain the origin.

g) Three noncollinear points give enough information to determine a plane.

True. Two vectors that lie in the plane can be determined from these three points, so the normal vector can be determined from those, and any of the three points can be used to form the plane equation.

h) Two nonparallel vectors give enough information to determine a plane.

False. While this is enough information to form the normal vector, it is not enough for the plane equation, as we need a point in the plane.

i) A line and a point not on the line give enough information to determine a plane.

True. The direction vector of the line and a vector from any point on the line to the other point will give the normal vector, so there will be enough information to find the plane equation.

11. Suppose you were given the equations of three planes

$$\begin{aligned} p_1: \quad a_1x+b_1y+c_1z &= d_1 \\ p_2: \quad a_2x+b_2y+c_2z &= d_2 \\ p_3: \quad a_3x+b_3y+c_3z &= d_3 \end{aligned}$$

and you were told that there is a single point where all three planes intersect. Describe, in words, the process you would use to find this point of intersection.

First, find the line of intersection L_1 of planes p_1 and p_2 by finding the direction $\vec{n}_1 \times \vec{n}_2$ and a point satisfying both p_1 and p_2. Then, find the line of intersection L_2 of planes p_2 and p_3 by finding the direction $\vec{n}_2 \times \vec{n}_3$ and a point satisfying both p_2 and p_3. Finally, the point of intersection of all three planes can be found by finding the point of intersection of the lines L_1 and L_2.

B.7 Section 2.1 Exercises

1. Determine if the statements below are true or false, giving a brief explanation or counterexample.

a) A system of 3 linear equations in 4 unknowns can have a unique solution.

False. There is not enough information to completely specify all 4 unknowns.

b) A system of 3 linear equations in 2 unknowns can have a unique solution.

True. An example is the system $x_1 + x_2 = 1$, $x_1 - x_2 = 4$, $2x_1 + 2x_2 = 2$.

c) A system of 3 linear equations in 4 unknowns can have infinitely many solutions.

True. If the system is consistent, it will necessarily have infinitely many solutions. An example is $x_1 + x_2 = 1$, $2x_1 + 2x_2 = 2$, $x_3 + x_4 = 1$, $2x_3 + 2x_4 = 2$.

d) A system of 3 linear equations in 2 unknowns can have infinitely many solutions.

True. This would require all three equations to be the same equation: $x_1 + x_2 = 1$, $2x_1 + 2x_2 = 2$, $-x_1 - x_2 = -1$.

e) A system of 2 linear equations in 3 unknowns can have no solution.

True. Geometrically, this would be represented by two parallel planes. An example: $x + y + z = 1$, $x + y + z = 2$.

2. Find the solution set (if it exists) of the linear system

$$\begin{aligned} x_1 + 3x_2 + 4x_3 &= 7 \\ 3x_1 + 9x_2 + 7x_3 &= 6 \end{aligned}$$

Subtracting 3 times the first equation from the second gives $-5x_3 = -15$, so $x_3 = 3$. Then note that the first equation implies $x_1 + 3x_2 = -5$, and the second equation implies $3x_1 + 9x_2 = -15$. These two equations are consistent so we can describe the solution set as

$$x_1 = -5 - 3x_2, \quad x_2 \text{ free}, \quad x_3 = 3.$$

3. Find the solution set (if it exists) of the linear system

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 5 \\ 2x_1 + 4x_2 + 5x_3 &= 4 \\ 4x_1 + 5x_2 + 4x_3 &= 2 \end{aligned}$$

Subtracting the second equation from 2 times the first equation gives $3x_3 = 6$, so $x_3 = 2$. The third equation then gives $4x_1 + 5x_2 = -6$, and the second equation is $2x_1 + 4x_2 = -6$. Subtracting 2 times the second equation from the third gives $-3x_2 = 6$, so $x_2 = -2$. Then $x_1 = 5 - 2x_2 - 4x_3 = 5 + 4 - 8 = 1$.

4. Find the solution set (if it exists) of the linear system

$$\begin{aligned} 2x_1 \qquad\qquad\qquad - 4x_4 &= -10 \\ 3x_2 + 3x_3 \qquad\quad &= 0 \\ x_3 + 4x_4 &= -1 \\ -3x_1 + 2x_2 + 3x_3 + \ \ x_4 &= 5 \end{aligned}$$

The second equation says that $x_2 = -x_3$ and the third equation implies $x_3 = -1 - 4x_4$. So $x_2 = 1 + 4x_4$ and the fourth equation is then

$$-3x_1 + 2(1 + 4x_4) + 3(-1 - 4x_4) + x_4 = 5$$

or $-3x_1 - 3x_4 = 6$, or $x_1 + x_4 = -2$. This, together with the first equation states that $2(-2 - x_4) - 4x_4 = -10$, or $-6x_4 = -6$, so $x_4 = 1$. Then $x_1 = -3$, $x_2 = 5$ and $x_3 = -5$.

5. Find an equation involving a, b, and c such that the following linear system is consistent:

$$\begin{aligned} x_1 - 4x_2 + 7x_3 &= a \\ -3x_2 + 5x_3 &= b \\ -2x_1 + 5x_2 - 9x_3 &= c \end{aligned}$$

Note that multiplying the first equation by -2 and adding this to the second equation actually gives you the third equation. Thus, in order for the system to be consistent, we must have $-2a + b = c$, or $2a - b + c = 0$.

6. Balance the equation $Cr_2O_7{}^{-2} + H^+ \xrightarrow{+6e} Cr^{+3} + H_2O$.

Using $x_1 Cr_2O_7{}^{-2} + x_2 H^+ \xrightarrow{+6e} x_3 Cr^{+3} + x_4 H_2O$ we arrive at the system of equations:

$$\begin{aligned} 2x_1 \qquad - x_3 \qquad\quad &= 0 \\ 7x_1 \qquad\qquad\quad - \ \ x_4 &= 0 \\ x_2 \qquad - 2x_4 &= 0 \end{aligned}$$

A solution to this system is $x_3 = 2x_1$, $x_4 = 7x_1$, $x_2 = 2x_4 = 14x_1$ with x_1 free. Setting $x_1 = 1$ we have

$$Cr_2O_7{}^{-2} + 14\,H^+ \xrightarrow{+6e} 2\,Cr^{+3} + 7\,H_2O.$$

7. Balance the equation $PhCH_3 + KMnO_4 + H_2SO_4 \longrightarrow PhCOOH + K_2SO_4 + MnSO_4 + H_2O$.

Using

$$x_1 PhCH_3 + x_2 KMnO_4 + x_3 H_2SO_4 \longrightarrow x_4 PhCOOH + x_5 K_2SO_4 + x_6 MnSO_4 + x_7 H_2O$$

we arrive at the system of equations:

$$\begin{array}{rcl}
x_1 - x_4 &= 0 & \text{(Ph)}\\
x_1 - x_4 &= 0 & \text{(C)}\\
3x_1 + 2x_3 - x_4 - 2x_7 &= 0 & \text{(H)}\\
x_2 - 2x_5 &= 0 & \text{(K)}\\
x_2 - x_6 &= 0 & \text{(Mn)}\\
4x_2 + 4x_3 - 2x_4 - 4x_5 - 4x_6 - x_7 &= 0 & \text{(O)}\\
x_3 - x_5 - x_6 &= 0 & \text{(S)}
\end{array}$$

Using the equations for K, Mn, and S we have $x_3 = (3/2)x_2$. Using the equations for Ph, H and S we have

$$3x_1 + 3x_2 = x_1 + 2x_7 \qquad \Longrightarrow \qquad x_7 = \frac{1}{2}(2x_1 + 3x_2).$$

Collectively using the information for H, K, Mn, S, O we have

$$4x_2 + 6x_2 = 2x_1 + 2x_2 + 4x_2 + \frac{1}{2}(2x_1 + 3x_2)$$

which implies $x_1 = (5/6)x_2$. To ensure an integer solution for all variables, let $x_2 = 6$. Then the balanced equation is

$$5\,PhCH_3 + 6\,KMnO_4 + 9\,H_2SO_4 \longrightarrow 5\,PhCOOH + 3\,K_2SO_4 + 6\,MnSO_4 + 14\,H_2O.$$

8. How many of each coin and how much money in total do you have if all of the following are true:

- You have only nickels, dimes, and quarters. (n, d, q)
- You have twice as many dimes as you have quarters. ($2q = d$)
- You have 13 coins that are nickels or quarters. ($n + q = 13$)
- You have \$1.25 in nickels and dimes. ($5n + 10d = 125$)

$$n + q = 13 \qquad \Longrightarrow \qquad q = 13 - n$$
$$2q = d \qquad \Longrightarrow \qquad d = 26 - 2n$$
$$5n + 10d = 125 \qquad \Longrightarrow \qquad 5n + 260 - 20n = 125 \qquad \Longrightarrow \qquad 15n = 135 \qquad \Longrightarrow \qquad n = 9$$
$$\Longrightarrow \qquad d = 26 - 18 = 8 \qquad \Longrightarrow \qquad q = 4.$$

The total amount of money is

$$9 \cdot \$0.05 + 8 \cdot \$0.10 + 4 \cdot \$0.25 = \$2.25.$$

B.8 Section 2.2 Exercises

1. Let $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ be arbitrary matrices for which the given sums and products are defined. Determine if the following statements are true or false, giving a brief explanation or counterexample in each case.

a) $\boldsymbol{AB} + \boldsymbol{AC} = \boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C})$.

This is true, it is the left distributive law for matrix multiplication.

b) If $\boldsymbol{A} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix}$ are both 2×2, then $\boldsymbol{AB} = \begin{bmatrix} \vec{a}_1\vec{b}_1 & \vec{a}_2\vec{b}_2 \end{bmatrix}$.

False. The description of $\boldsymbol{AB}$ is meaningless - it makes no sense to "multiply" two vectors $\vec{a}$ and $\vec{b}$ together.

c) $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{CB})$.

False, matrix multiplication is not commutative. Any example where $\boldsymbol{BC} \neq \boldsymbol{CB}$ is a counterexample.

d) If $\boldsymbol{A}$ and $\boldsymbol{B} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix}$ are both 3×3, then $\boldsymbol{AB} = \begin{bmatrix} \boldsymbol{A}\vec{b}_1 + \boldsymbol{A}\vec{b}_2 + \boldsymbol{A}\vec{b}_3 \end{bmatrix}$.

False, the expression on the far right would be the sum of three 3×1 vectors, but $\boldsymbol{AB}$ should be a 3×3 matrix.

2. Suppose the second column of $\boldsymbol{B}$ is all zeros. What can you say about the second column of $\boldsymbol{AB}$ (if $\boldsymbol{AB}$ is defined)?

The second column of $\boldsymbol{AB}$ would also be all zeros, as all entries in the second column of $\boldsymbol{AB}$ are dot products of the second column of $\boldsymbol{B}$ with rows of $\boldsymbol{A}$.

3. Let $\boldsymbol{A} = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$. Construct a matrix $\boldsymbol{B}$ (with two different nonzero columns) such that $\boldsymbol{AB}$ is the zero matrix.

We need a matrix $\boldsymbol{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$\boldsymbol{AB} = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3a - 6c & 3b - 6d \\ -a + 2c & -b + 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

One such $\boldsymbol{B}$ with two different nonzero columns is $\boldsymbol{B} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$.

4. If $\vec{u}$ and $\vec{v}$ are in $\mathbb{R}^n$ (column vectors), how are $\vec{u}^T\vec{v}$ and $\vec{v}^T\vec{u}$ related? How are $\vec{u}\vec{v}^T$ and $\vec{v}\vec{u}^T$ related?

Note that $\vec{u}^T\vec{v}$ is a scalar value, and since the transpose of a scalar is itself, we have that $\vec{u}^T\vec{v} = \vec{v}^T\vec{u}$. Now $\vec{u}\vec{v}^T$ is a 3×3 matrix, and we have

$$(\vec{u}\vec{v}^T)^T = (\vec{v}^T)^T\vec{u}^T = \vec{v}\vec{u}^T.$$

5. Assume that the matrices are partitioned conformably for block multiplication. Compute the following products.

a) $\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{D} & \boldsymbol{0} \\ \boldsymbol{E} & \vec{F} \end{bmatrix} = \begin{bmatrix} \boldsymbol{AD}+\boldsymbol{BE} & \boldsymbol{B}\vec{F} \\ \boldsymbol{CE} & \boldsymbol{C}\vec{F} \end{bmatrix}$

b) $\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{0} & -\boldsymbol{D} \\ \boldsymbol{D} & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}^T\boldsymbol{D} & -\boldsymbol{AD} \\ \boldsymbol{CD} & -\boldsymbol{BD} \end{bmatrix}$

6. Find the symmetric and skew parts of the matrix $\boldsymbol{A} = \begin{bmatrix} 4 & 0 & -1 \\ 2 & -2 & 1 \\ -3 & 2 & -6 \end{bmatrix}$. (See practice problems.)

Symmetric part:

$$\frac{1}{2}\left(\boldsymbol{A}+\boldsymbol{A}^T\right) = \frac{1}{2}\begin{bmatrix} 8 & 2 & -4 \\ 2 & -4 & 3 \\ -4 & 3 & -12 \end{bmatrix} = \begin{bmatrix} 4 & 1 & -2 \\ 1 & -2 & 3/2 \\ -2 & 3/2 & -6 \end{bmatrix}.$$

Skew part:

$$\frac{1}{2}\left(\boldsymbol{A}-\boldsymbol{A}^T\right)=\frac{1}{2}\begin{bmatrix} 0 & -2 & 2 \\ 2 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}=\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1/2 \\ -1 & 1/2 & 0 \end{bmatrix}.$$

7. Let $\boldsymbol{A}=\begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}$ and $\boldsymbol{B}=\begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$. Compute $\boldsymbol{AB}$. Then write each column of $\boldsymbol{AB}$ as a linear combination of the columns of $\boldsymbol{A}$.

$$\boldsymbol{AB}=\begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}\begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}=\begin{bmatrix} 67 & 41 & 41 \\ 64 & 21 & 59 \\ 63 & 67 & 57 \end{bmatrix}.$$

We have that

$$\begin{bmatrix} 67 \\ 64 \\ 63 \end{bmatrix}=6\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}+0\begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}+7\begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix},$$

$$\begin{bmatrix} 41 \\ 21 \\ 67 \end{bmatrix}=-2\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}+1\begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}+7\begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix},$$

$$\begin{bmatrix} 41 \\ 59 \\ 57 \end{bmatrix} = 4 \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} + 5 \begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix},$$

8. Suppose that the matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E}$ all have the following sizes:

$$\boldsymbol{A} : 4 \times 5 \qquad \boldsymbol{B} : 4 \times 5 \qquad \boldsymbol{C} : 5 \times 2 \qquad \boldsymbol{D} : 4 \times 2 \qquad \boldsymbol{E} : 5 \times 4$$

Determine which of the following matrix expressions are defined. For those that are defined, give the size of the resulting matrix, and for those that are not defined, indicate why.

1. $\boldsymbol{B}\boldsymbol{A}$ is not defined
2. $\boldsymbol{AC} + \boldsymbol{D}$ is a 4×2 matrix
3. $\boldsymbol{AE} + \boldsymbol{B}$ is not defined
4. $\boldsymbol{AB} + \boldsymbol{B}$ is not defined
5. $\boldsymbol{E}(\boldsymbol{A} + \boldsymbol{B})$ is a 5×5 matrix
6. $\boldsymbol{E}(\boldsymbol{AC})$ is a 5×2 matrix
7. $\boldsymbol{E}^T \boldsymbol{A}$ is not defined
8. $(\boldsymbol{A}^T + \boldsymbol{E})\boldsymbol{D}$ is a 5×2 matrix

B.9 Section 2.3 Exercises

1. Determine if the statements below are true or false, giving a brief explanation or counterexample.

a) For an $n \times n$ matrix $\boldsymbol{A}$, $\det(\boldsymbol{A}^2) = (\det(\boldsymbol{A}))^2$.

True, since $\det(\boldsymbol{A}^2) = \det(\boldsymbol{A})\det(\boldsymbol{A})$.

b) For $n \times n$ matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, $\det(\boldsymbol{ABC}) = \det(\boldsymbol{C})\det(\boldsymbol{A})\det(\boldsymbol{B})$.

True, while the matrices themselves do not necessarily commute, their determinants will, as they are just real numbers and we know multiplication of real numbers is commutative.

c) For $n \times n$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, $\det(\boldsymbol{A}+\boldsymbol{B}) = \det(\boldsymbol{A}) + \det(\boldsymbol{B})$.

False. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\det(\boldsymbol{A}) = 0 = \det(\boldsymbol{B})$ but $\det(\boldsymbol{A}+\boldsymbol{B}) = \det(\boldsymbol{I}) = 1$.

d) The product of invertible matrices is always invertible.

True, if both matrices are invertible, then their determinants are nonzero, so the determinant of the product will also be nonzero.

e) The product of singular matrices is always singular.

True, same argument as above, with the exception of both matrices having a determinant of 0.

f) For invertible $n \times n$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, $\left((\boldsymbol{AB})^T\right)^{-1} = \left(\boldsymbol{A}^{-1}\right)^T\left(\boldsymbol{B}^{-1}\right)^T$.

True.

$$\left((\boldsymbol{AB})^T\right)^{-1} = \left(\boldsymbol{B}^T\boldsymbol{A}^T\right)^{-1} = \left(\boldsymbol{A}^T\right)^{-1}\left(\boldsymbol{B}^T\right)^{-1} = \left(\boldsymbol{A}^{-1}\right)^T\left(\boldsymbol{B}^{-1}\right)^T.$$

2. Find the determinants of the following matrices:

a) $\boldsymbol{A} = \begin{bmatrix} 0 & 3 & 2 \\ 2 & 0 & 1 \\ 2 & 6 & 0 \end{bmatrix}$

$$\det(\boldsymbol{A}) = \begin{vmatrix} 0 & 3 & 2 \\ 2 & 0 & 1 \\ 2 & 6 & 0 \end{vmatrix} = -3\begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} + 2\begin{vmatrix} 2 & 0 \\ 2 & 6 \end{vmatrix} = -3(0-2) + 2(12-0) = 30.$$

b) $\boldsymbol{B} = \begin{bmatrix} 3 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 4 & 2 \end{bmatrix}$

$$\det(\boldsymbol{B}) = \begin{vmatrix} 3 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 4 & 2 \end{vmatrix} = 3\begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 4 & 2 \end{vmatrix} - 4\begin{vmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 4 & 2 \end{vmatrix} = 3 \cdot 2\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} - 4 \cdot 1\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = 6(6-4) - 4(6-4) = 4.$$

c) $\boldsymbol{C} = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$

$$\det(\boldsymbol{C}) = \begin{vmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{vmatrix} = 3 \begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} = 3 \cdot 2 \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = 3 \cdot 2 \cdot (-1)^{3+2}(-2) \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = -12.$$

3. Find the inverse of the matrix $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ 0 & \boldsymbol{A}_{22} \end{bmatrix}$ where $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22}$ are $m \times m$ and $n \times n$ invertible matrices, respectively. You should check that your inverse works on both the left and the right of $\boldsymbol{A}$.

We need to find a matrix $\boldsymbol{B}$ of the form $\begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{bmatrix}$ such that $\boldsymbol{AB} = \boldsymbol{I}$. This would imply

$$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ 0 & \boldsymbol{A}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{11}\boldsymbol{B}_{11} + \boldsymbol{A}_{12}\boldsymbol{B}_{21} & \boldsymbol{A}_{11}\boldsymbol{B}_{12} + \boldsymbol{A}_{12}\boldsymbol{B}_{22} \\ \boldsymbol{A}_{22}\boldsymbol{B}_{21} & \boldsymbol{A}_{22}\boldsymbol{B}_{22} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{I} \end{bmatrix}.$$

The equation $\boldsymbol{A}_{22}\boldsymbol{B}_{22} = \boldsymbol{I}$ would imply that $\boldsymbol{B}_{22} = \boldsymbol{A}_{22}^{-1}$. Also, the equation $\boldsymbol{A}_{22}\boldsymbol{B}_{21} = 0$ would imply that $\boldsymbol{B}_{21} = 0$. These two results would then imply that

$$\boldsymbol{A}_{11}\boldsymbol{B}_{12} + \boldsymbol{A}_{12}\boldsymbol{B}_{22} = 0 \quad \Longrightarrow \quad \boldsymbol{B}_{12} = -\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}$$

and

$$\boldsymbol{A}_{11}\boldsymbol{B}_{11} + \boldsymbol{A}_{12}\boldsymbol{B}_{21} = \boldsymbol{I} \quad \Longrightarrow \quad \boldsymbol{B}_{11} = \boldsymbol{A}_{11}^{-1}.$$

Thus $\boldsymbol{A}^{-1} = \begin{bmatrix} \boldsymbol{A}_{11}^{-1} & -\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1} \\ 0 & \boldsymbol{A}_{22}^{-1} \end{bmatrix}$ and it is easy to check that this also satisfies $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$.

4. How is $\det \boldsymbol{A}^{-1}$ related to $\det \boldsymbol{A}$? Explain.

We know that $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$, and that $\det(\boldsymbol{I}) = 1$ (since the determinant of a diagonal matrix is the product of the entries on the diagonal), so we must have that

$$\det(\boldsymbol{A}\boldsymbol{A}^{-1}) = 1,$$

and since the determinant of a product is the product of the determinants, we have

$$1 = \det(\boldsymbol{A}\boldsymbol{A}^{-1}) = \det(\boldsymbol{A})\det(\boldsymbol{A}^{-1}),$$

so

$$\det(\boldsymbol{A}^{-1}) = \frac{1}{\det(\boldsymbol{A})}.$$

5. Explain why the matrix $\boldsymbol{A} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is invertible for any value of θ. Give the inverse matrix (hint: use blocks).

First note that the determinant of this matrix is $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$, so we know it is invertible.

We can use the result of problem 2 to find its inverse. Let $\boldsymbol{A}_{11} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, $\boldsymbol{A}_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $\boldsymbol{A}_{22} = [1]$.

Then we know

$$\boldsymbol{A}_{11}^{-1} = \frac{1}{\det(\boldsymbol{A}_{11})}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

so

$$\boldsymbol{A}^{-1} = \begin{bmatrix} \boldsymbol{A}_{11}^{-1} & -\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1} \\ 0 & \boldsymbol{A}_{22}^{-1} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

6. Let $\boldsymbol{A} = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$. Compute $\boldsymbol{A}^3$, $\boldsymbol{A}^{-3}$, and $\boldsymbol{A}^2 - 2\boldsymbol{A} + \boldsymbol{I}$.

$$\boldsymbol{A}^2 = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix}, \qquad \boldsymbol{A}^3 = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 28 & 1 \end{bmatrix},$$

$$\boldsymbol{A}^{-3} = (\boldsymbol{A}^3)^{-1} = \frac{1}{8}\begin{bmatrix} 1 & 0 \\ -28 & 8 \end{bmatrix} = \begin{bmatrix} 1/8 & 0 \\ -28/8 & 1 \end{bmatrix},$$

and

$$\boldsymbol{A}^2 - 2\boldsymbol{A} + \boldsymbol{I} = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 8 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}.$$

7. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Find the inverse of $\boldsymbol{A}$ by solving the matrix equation $\boldsymbol{AX} = \boldsymbol{I}$ for $\boldsymbol{X}$.

We have

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This results in the equations

$$\begin{aligned} x_{11} + x_{31} &= 1 \\ x_{12} + x_{32} &= 0 \\ x_{13} + x_{33} &= 0 \end{aligned} \qquad \begin{aligned} x_{11} + x_{21} &= 0 \\ x_{12} + x_{22} &= 1 \\ x_{13} + x_{23} &= 0 \end{aligned} \qquad \begin{aligned} x_{21} + x_{31} &= 0 \\ x_{22} + x_{32} &= 0 \\ x_{23} + x_{33} &= 1 \end{aligned}$$

Solving this system gives

$$X = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}.$$

8. Find 2×2 matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ such that $(\boldsymbol{A}+\boldsymbol{B})^2 \neq \boldsymbol{A}^2 + 2\boldsymbol{AB} + \boldsymbol{B}^2$. What relationship would $\boldsymbol{A}$ and $\boldsymbol{B}$ need to satisfy in order to ensure $(\boldsymbol{A}+\boldsymbol{B})^2 = \boldsymbol{A}^2 + 2\boldsymbol{AB} + \boldsymbol{B}^2$?

Since $(\boldsymbol{A}+\boldsymbol{B})^2 = \boldsymbol{A}^2 + \boldsymbol{AB} + \boldsymbol{BA} + \boldsymbol{B}^2$, we must have that $\boldsymbol{AB} = \boldsymbol{BA}$, i.e., that $\boldsymbol{A}$ and $\boldsymbol{B}$ commute. This is satisfied if $\boldsymbol{A} = \boldsymbol{I}$ and $\boldsymbol{B} = 2\boldsymbol{I}$, for example.

9. The **trace** of a square matrix $\boldsymbol{A}$, denoted $\operatorname{tr}(\boldsymbol{A})$, is the sum of the entries on the diagonal of $\boldsymbol{A}$:

$$\operatorname{tr}(\boldsymbol{A}) = \sum_{i=1}^{n} a_{ii}.$$

Explain why $\operatorname{tr}(\boldsymbol{AB}) = \operatorname{tr}(\boldsymbol{BA})$. Is it possible for any square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ to satisfy $\boldsymbol{AB} - \boldsymbol{BA} = \boldsymbol{I}$?

Note, using the definition of matrix multiplication, that the sum of the diagonal elements of $\boldsymbol{AB}$ is given by

$$\operatorname{tr}(\boldsymbol{AB}) = \sum_{i=1}^{n} (\boldsymbol{AB})_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} b_{ji},$$

and the sum of the diagonal entries of $\boldsymbol{BA}$ are

$$\operatorname{tr}(\boldsymbol{BA}) = \sum_{i=1}^{n} (\boldsymbol{BA})_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} a_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji} b_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ji} b_{ij} = \operatorname{tr}(\boldsymbol{AB}).$$

It is not too hard to see that the trace of a sum is the sum of the traces:

$$\operatorname{tr}(\boldsymbol{A}+\boldsymbol{B}) = \operatorname{tr}(\boldsymbol{A}) + \operatorname{tr}(\boldsymbol{B}).$$

With this in mind, we see that

$$\operatorname{tr}(\boldsymbol{AB} - \boldsymbol{BA}) = \operatorname{tr}(\boldsymbol{AB}) - \operatorname{tr}(\boldsymbol{BA}) = 0,$$

and since $\operatorname{tr}(\boldsymbol{I}) = n > 0$, it is impossible for two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ to satisfy $\boldsymbol{AB} - \boldsymbol{BA} = \boldsymbol{I}$.

B.10 Section 2.4 Exercises

1. Determine if the following statements are true or false, giving a brief explanation or counterexample in each case.

a) The columns of a matrix $\boldsymbol{A}$ are linearly independent if and only if the equation $\boldsymbol{A}\vec{x} = \vec{0}$ has the trivial solution.

False, the columns of $\boldsymbol{A}$ are linearly independent if and only if the equation $\boldsymbol{A}\vec{x} = \vec{0}$ has **only** the trivial solution.

b) If S is a linearly dependent set of vectors, then each vector in S is a linear combination of the other vectors in S.

False. The set of vectors $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$ is a dependent set, but there is no way to write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in terms of the other two.

c) If the vectors $\vec{x}$ and $\vec{y}$ are independent and $\vec{z}$ is in span$\{\vec{x}, \vec{y}\}$, then the set $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly independent.

False, if $\vec{z}$ is in the span of the other two vectors, then it is a linear combination of the other two, so the set $\{\vec{x}, \vec{y}, \vec{z}\}$ must be linearly dependent.

d) If a set of vectors contains fewer vectors than there are entries in the vectors, then the set is linearly independent.

False, the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a dependent set.

e) The columns of a 4×5 matrix must be linearly dependent.

True, if there are five column vectors with 4 entries in each vector, the set of vectors must be a dependent set.

2. The vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ -8 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$ span $\mathbb{R}^2$ but do not form a basis. Find two different ways to express $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

We essentially want to find two different solutions to the vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus we want two solutions to the system

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 1 \\ -3x_1 - 8x_2 + 7x_3 &= 1 \end{aligned}$$

Three times the first equation plus the second gives $-2x_2 - 2x_3 = 4$, so $x_2 = -2 - x_3$. Then $x_1 = 1 + 3x_3 - 2x_2 = 1 + 3x_3 + 4 + 2x_3 = 5 + 5x_3$. Choosing two different values for x_3 gives us the two relations

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ -3 \end{bmatrix} - 2\begin{bmatrix} 2 \\ -8 \end{bmatrix} + 0\begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 10\begin{bmatrix} 1 \\ -3 \end{bmatrix} - 4\begin{bmatrix} 2 \\ -8 \end{bmatrix} + 1\begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

3. Find the coordinates of $\vec{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ in the basis $\mathscr{B} = \left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \end{bmatrix}\right\}$.

We want to find $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathscr{B}}$ such that

$$x_1\begin{bmatrix} 1 \\ -3 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

This means

$$\begin{aligned} x_1 + 2x_2 &= -2 \\ -3x_1 - 5x_2 &= 1 \end{aligned}$$

The solution to this system is $x_1 = 8$, $x_2 = -5$, so the coordinate vector of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ in $\mathscr{B}$ is $\begin{bmatrix} 8 \\ -5 \end{bmatrix}_{\mathscr{B}}$.

4. Find the values of λ for which the following system of equations has a **nontrivial** (not all zeros) solution, and find the solution for these values of λ.

$$\begin{aligned} 2x_1 + x_2 &= \lambda x_1 \\ x_1 + 2x_2 &= \lambda x_2 \end{aligned}$$

We are looking for a nontrivial solution to the system

$$\begin{aligned} (2-\lambda)x_1 + \qquad\quad x_2 &= 0 \\ x_1 + (2-\lambda)x_2 &= 0 \end{aligned}$$

We know this system will have a nontrivial solution if and only if the columns of the matrix $\boldsymbol{A} = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$ are linearly dependent, so

$$\det(\boldsymbol{A}) = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 4 - 1 = \lambda^2 - 4\lambda + 3 = 0.$$

This is true when $(\lambda - 3)(\lambda - 1) = 0$, or when $\lambda = 1$ or 3. A solution when $\lambda = 1$ is $x_1 = -x_2$, and when $\lambda = 3$, $x_1 = x_2$ is a solution.

B.11 Section 2.5 Exercises

1. Determine if the following statements are true or false, giving a brief explanation or counterexample in each case.

a) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by its effect on the columns of the $n \times n$ identity matrix.

   True, the columns of the identity matrix are the elementary basis vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$.

b) When two linear transformations are performed one after another, the combined effect may not always be a linear transformation.

   False, it will always be a linear transformation. Let S and T both be linear transformations and let α and β be scalars, and let $\vec{u}$ and $\vec{v}$ be vectors. Then we have

$$(S \circ T)(\alpha\vec{u} + \beta\vec{v}) = S(T(\alpha\vec{u} + \beta\vec{v})) = S(\alpha T(\vec{u}) + \beta T(\vec{v})) = \alpha S(T(\vec{u})) + \beta S(T(\vec{v})) = \alpha(S \circ T)(\vec{u}) + \beta(S \circ T)(\vec{v}).$$

c) If $\boldsymbol{A}$ is a 3×2 matrix, then the transformation $T(\vec{x}) = \boldsymbol{A}\vec{x}$ cannot be one-to-one.

   False, the transformation T with matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ is one-to-one because it has linearly independent columns.

d) If $\boldsymbol{A}$ is a 3×2 matrix, then the transformation $T(\vec{x}) = \boldsymbol{A}\vec{x}$ cannot be onto.

   True, even if the 2 columns of $\boldsymbol{A}$ are linearly independent, they do not span all of $\mathbb{R}^3$, so there will always be a vector in $\mathbb{R}^3$ that $\boldsymbol{A}$ does not map to.

e) A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each vector in $\mathbb{R}^n$ maps onto a unique vector in $\mathbb{R}^m$.

   This statement is somewhat ambiguous. For example, if you interpret the statement to mean that for every input, there is exactly one output, then the statement is false, as it just means that a transformation is a function and may not necessarily map every vector to a **different** vector in $\mathbb{R}^m$. However, if you interpret the word “unique” in the statement to mean that each output is different, then this is really another way to state the definition of one-to-one, so it would be true.

f) Let V be the vector space of all twice-differentiable functions of the variable x. The transformation A defined by, if $f \in V$, then $A(f) = \dfrac{d^2}{dx^2} f(x) + f(x)$ is a linear transformation.

Let's see. Let α, β be real numbers and let $f, g \in V$. Then we have

$$\begin{aligned}
A(\alpha f + \beta g) &= \frac{d^2}{dx^2}\big(\alpha f(x) + \beta g(x)\big) + \alpha f(x) + \beta g(x) \\
&= \frac{d^2}{dx^2}\big(\alpha f(x)\big) + \frac{d^2}{dx^2}\big(\beta g(x)\big) + \alpha f(x) + \beta g(x) = \alpha \frac{d^2}{dx^2} f(x) + \beta \frac{d^2}{dx^2} g(x) + \alpha f(x) + \beta g(x) \\
&= \alpha \frac{d^2}{dx^2} f(x) + \alpha f(x) + \beta \frac{d^2}{dx^2} g(x) + \beta g(x) \\
&= \alpha\left(\frac{d^2}{dx^2} f(x) + f(x)\right) + \beta\left(\frac{d^2}{dx^2} g(x) + g(x)\right) = \alpha A(f) + \beta A(g).
\end{aligned}$$

Thus this statement is true.

2. Find the (single) standard matrix of the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that first reflects points across the x_1-axis and then reflects points across the line $x_2 = x_1$. Show that this is a rotation transformation by finding the angle of rotation.

Reflection across the x_1-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Reflection across the line $x_1 = x_2$: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Combined: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, which is a rotation of $\pi/2$

3. Find the (single) standard matrix of the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that first rotates points an angle of θ about the x_1-axis, then reflects across the $x_1 x_2$-plane, and finally rotates an angle of ϕ about the x_3-axis. Then give the matrix for $\theta = \pi/3$ and $\phi = -\pi/6$.

Rotation of θ about the x_1-axis: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

Reflection across the x_1x_2-plane: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Rotation of ϕ about the x_3-axis: $\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Single matrix:

$$\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi\cos\theta & \sin\phi\sin\theta \\ \sin\phi & \cos\phi\cos\theta & -\cos\phi\sin\theta \\ 0 & -\sin\theta & -\cos\theta \end{bmatrix}$$

Matrix for $\theta = \pi/3$ and $\phi = -\pi/6$:

$$\begin{bmatrix} \cos(-\pi/6) & -\sin(-\pi/6)\cos(\pi/3) & \sin(-\pi/6)\sin(\pi/3) \\ \sin(-\pi/6) & \cos(-\pi/6)\cos(\pi/3) & -\cos(-\pi/6)\sin(\pi/3) \\ 0 & -\sin(\pi/3) & -\cos(\pi/3) \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/4 & -\sqrt{3}/4 \\ -1/2 & \sqrt{3}/4 & -3/4 \\ 0 & -\sqrt{3}/2 & -1/2 \end{bmatrix}$$

4. The spin matrices for a nucleus with spin quantum number 1 are

$$\boldsymbol{I}_x = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \boldsymbol{I}_y = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \qquad \boldsymbol{I}_z = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Find the commutators $[\boldsymbol{I}_x, \boldsymbol{I}_y]$, $[\boldsymbol{I}_y, \boldsymbol{I}_z]$, and $[\boldsymbol{I}_z, \boldsymbol{I}_x]$.

$$[\boldsymbol{I}_x, \boldsymbol{I}_y] = \boldsymbol{I}_x\boldsymbol{I}_y - \boldsymbol{I}_y\boldsymbol{I}_x = \frac{\hbar^2}{2}\left(\begin{bmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{bmatrix} \right) = \hbar^2 \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{bmatrix}$$

$$[\boldsymbol{I}_y, \boldsymbol{I}_z] = \boldsymbol{I}_y\boldsymbol{I}_z - \boldsymbol{I}_z\boldsymbol{I}_y = \frac{\hbar^2}{2}\left(\begin{bmatrix} 0 & 0 & 0 \\ i & 0 & i \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & -i & 0 \end{bmatrix} \right) = \hbar^2 \begin{bmatrix} 0 & i & 0 \\ i & 0 & i \\ 0 & i & 0 \end{bmatrix}$$

$$[\boldsymbol{I}_z, \boldsymbol{I}_x] = \boldsymbol{I}_z\boldsymbol{I}_x - \boldsymbol{I}_x\boldsymbol{I}_z = \frac{\hbar^2}{2}\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \right) = \frac{\hbar^2}{2} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

5. Describe (in words) the geometric action of the following matrices on a vector $\vec{v} \in \mathbb{R}^2$.

a) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ is a reflection through the origin.

b) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is a reflection across the x_1-axis.

c) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is a reflection across the x_2-axis.

6. It is true that the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ represents a particular rotation transformation on a vector in $\mathbb{R}^2$. Can you find two non-rotational transformations that, when subsequently applied, will result in the same transformation matrix? In what order should they be applied? (In essence you are finding a **factorization** of the above rotation matrix.)

Note that if we reflect across the x_2-axis and subsequently reflect across the line $x_1 = x_2$, we will have the matrix transformation

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

which is the same matrix as a rotation of $3\pi/2$.

B.12 Section 2.6 Exercises

1. Determine if the following statements are true or false, giving a brief explanation or counterexample in each case.

a) $\boldsymbol{A}$ and $\boldsymbol{A}^T$ have the same eigenvalues.

True, this can be seen by noting that $\boldsymbol{A}$ and $\boldsymbol{A}^T$ have the same characteristic polynomial:

$$\begin{aligned}\det(\boldsymbol{A}^T - \lambda\boldsymbol{I}) &= \det(\boldsymbol{A}^T - \lambda\boldsymbol{I}^T) && \text{(because } \lambda\boldsymbol{I} = \lambda\boldsymbol{I}^T\text{)}\\ &= \det((\boldsymbol{A} - \lambda\boldsymbol{I})^T) && \text{(because } \boldsymbol{A}^T + \boldsymbol{B}^T = (\boldsymbol{A}+\boldsymbol{B})^T\text{)}\\ &= \det(\boldsymbol{A} - \lambda\boldsymbol{I}) && \text{(because } \det(\boldsymbol{A}^T) = \det(\boldsymbol{A})\text{)}\end{aligned}$$

b) The eigenvalues of $\boldsymbol{M} = \boldsymbol{AB}$ are the products of the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$.

False. It's hard to pin down what exactly we mean by "products of the eigenvalues" of $\boldsymbol{A}$ and $\boldsymbol{B}$. For example, both $\boldsymbol{A}$ and $\boldsymbol{B}$ (if they are of size $n \times n$) each have n eigenvalues, so there are n^2 possible ways to multiply and eigenvalue from each together. Note that $\boldsymbol{M}$ only has n eigenvalues. Several counterexamples are also easy to come up with, for example $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Both of these have the eigenvalues of 0 and 1, but $\boldsymbol{AB}$ has only 0 as an eigenvalue.

c) If $\boldsymbol{A}$ is singular, then 0 is an eigenvalue of $\boldsymbol{A}$.

True, because if $\boldsymbol{A}$ is singular, then $\det(\boldsymbol{A}) = 0$, and since the determinant is the product of all of the eigenvalues, at least one must be 0.

d) The sum of two eigenvectors of $\boldsymbol{A}$ is also an eigenvector of $\boldsymbol{A}$.

False in general, but it is true if the eigenvectors are associated with the same eigenvalue. Let $\vec{v}_1$ and $\vec{v}_2$ both be eigenvectors of $\boldsymbol{A}$ associated with the eigenvalue λ. Then $\boldsymbol{A}\vec{v}_1 = \lambda\vec{v}_1$ and $\boldsymbol{A}\vec{v}_2 = \lambda\vec{v}_2$. Now

$$\boldsymbol{A}(\vec{v}_1 + \vec{v}_2) = \boldsymbol{A}\vec{v}_1 + \boldsymbol{A}\vec{v}_2 = \lambda\vec{v}_1 + \lambda\vec{v}_2 = \lambda(\vec{v}_1 + \vec{v}_2),$$

which means that $(\vec{v}_1 + \vec{v}_2)$ is also an eigenvector for λ. But if $\vec{v}_1$ was associated with λ_1 and $\vec{v}_2$ with λ_2 where $\lambda_1 \neq \lambda_2$, then we would have

$$\boldsymbol{A}(\vec{v}_1 + \vec{v}_2) = \boldsymbol{A}\vec{v}_1 + \boldsymbol{A}\vec{v}_2 = \lambda_1\vec{v}_1 + \lambda_2\vec{v}_2,$$

which doesn't really mean anything.

2. Find the eigenvalues and eigenvectors of the matrix $\boldsymbol{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Given your knowledge of the action of this matrix as a linear transformation, give a geometric interpretation of the eigenvectors.

Eigenvalues: $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{vmatrix} \lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$, so the eigenvalues are $\lambda = \pm i$. For $\lambda = i$, an eigenvector found by solving the two equations

$$-ix_1 - x_2 = 0, \quad x_1 + ix_2 = 0,$$

so $\vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ is an eigenvector. For $\lambda = -i$, we find $\vec{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ to be an eigenvector. Since the eigenvectors of a matrix are the *only* vectors that don't change direction when multiplied by the matrix, and both of these eigenvectors are complex, this means that **all real vectors** change direction under this transformation. Since this is really a matrix that rotates vectors by $\pi/2$ radians, this interpretation is in line with our understanding of rotations.

3. Find a basis for the eigenspace of each eigenvalue of the matrix $\boldsymbol{A} = \begin{bmatrix} -4 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & 3 & -2 \end{bmatrix}$.

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{vmatrix} -4-\lambda & 1 & 1 \\ 2 & -3-\lambda & 2 \\ 3 & 3 & -2-\lambda \end{vmatrix} = (-4-\lambda) \begin{vmatrix} -3-\lambda & 2 \\ 3 & -2-\lambda \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ 3 & -2-\lambda \end{vmatrix} + \begin{vmatrix} 2 & -3-\lambda \\ 3 & 3 \end{vmatrix}$$

$$= (-4-\lambda)\,[(-3-\lambda)(-2-\lambda) - 6] - [2(-2-\lambda) - 6] + [6 - 3(-3-\lambda)]$$

$$= -\lambda^3 - 9\lambda^2 - 15\lambda + 25.$$

Note that $\lambda = 1$ is a root: $-1 - 9 - 15 + 25 = 0$. To find the others, first divide out the factor $\lambda - 1$ from the polynomial:

$$\begin{array}{r} -\lambda^2 - 10\lambda - 25 \\ \lambda - 1 \overline{\smash{\big)} -\lambda^3 \quad -9\lambda^2 - 15\lambda + 25} \\ \underline{\lambda^3 \quad -\lambda^2 \qquad\qquad\quad} \\ -10\lambda^2 - 15\lambda \qquad \\ \underline{10\lambda^2 - 10\lambda \qquad} \\ -25\lambda + 25 \\ \underline{25\lambda - 25} \\ 0 \end{array}$$

Then the remaining eigenvalues are found by factoring:

$$-\lambda^2 - 10\lambda - 25 = -1(lambda + 5)^2.$$

For the eigenvalue $\lambda = 1$, we have

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\vec{x} = \begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \vec{x} = \begin{bmatrix} x_3/3 \\ 2x_3/3 \\ x_3 \end{bmatrix},$$

so $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is an eigenvector. For $\lambda = -5$ we have

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\vec{x} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies x_1 + x_2 + x_3 = 0.$$

So there are two free variables, let them be x_1 and x_2. Then a basis for the eigenspace is given by $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$.

4. Find the eigenvalues and eigenvectors of the matrices $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$. Use these results to make a conjecture about the eigenvalues and eigenvectors of $\begin{bmatrix} a & a+1 \\ 1 & 0 \end{bmatrix}$ where a is any real number.

For the matrix $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$, eigenvalues are $\lambda = 2, -1$ with corresponding eigenvectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

For the matrix $\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, eigenvalues are $\lambda = 3, -1$ with corresponding eigenvectors $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

For the matrix $\begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$, eigenvalues are $\lambda = 4, -1$ with corresponding eigenvectors $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Thus it seems that the eigenvalues of the matrix $\begin{bmatrix} a & a+1 \\ 1 & 0 \end{bmatrix}$ are going to be $\lambda = a+1, -1$ and the eigenvectors will be $\begin{bmatrix} a+1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Note that the characteristic polynomial of this matrix is $\lambda^2 - a\lambda - (a+1)$, which factors into the product $(\lambda+1)(\lambda-(a+1))$.

5. Find the eigenvalues and associated eigenvectors of the following matrices:

a) $\boldsymbol{A} = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$

$$\det(\boldsymbol{A}-\lambda\boldsymbol{I}) = \begin{vmatrix} 7-\lambda & 2 \\ -4 & 1-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 15 = (\lambda-5)(\lambda-3) \quad \Rightarrow \quad \lambda = 3, 5.$$

For $\lambda = 3$, an eigenvector is $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and for $\lambda = 5$, an eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

b) $\boldsymbol{A} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

$$\det(\boldsymbol{A}-\lambda\boldsymbol{I}) = \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = -\lambda(2-\lambda)(3-\lambda) - 2(-(2-\lambda)) = (2-\lambda)\,(-\lambda(3-\lambda)+2)$$

$$= (2-\lambda)(\lambda^2 - 3\lambda + 2) = -(\lambda-2)^2(\lambda-1).$$

So $\lambda = 2$ (repeated) and $\lambda = 1$ are the eigenvalues. For $\lambda = 1$, an eigenvector is $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$. For $\lambda = 2$, we have that x_2 is free and that $x_1 = x_3$. Thus two linearly independent eigenvectors are $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

c) $\boldsymbol{A} = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}$

Eigenvalues are $\lambda = 3, 3, 4$. An eigenvector for $\lambda = 4$ is $\begin{bmatrix} 6 \\ -8 \\ -3 \end{bmatrix}$. There is only one linearly independent eigenvector, $\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$, for $\lambda = 3$.

6. Suppose you knew the eigenvalues of the product $\boldsymbol{AB}$. What would you be able to say about the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$?

Some things you could say:

- If 0 is an eigenvalue of $\boldsymbol{AB}$, then 0 is an eigenvalue of $\boldsymbol{A}$ or $\boldsymbol{B}$.
- If all of the eigenvalues of $\boldsymbol{AB}$ are nonzero, then so are all of the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$.

7. The Hamiltonian operator for a one-dimensional harmonic oscillator moving in the x direction is

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{kx^2}{2}.$$

Find the value of the coefficient a such that the function e^{-ax^2} is an eigenfunction of the Hamiltonian operator. The quantity k is the force constant, m is the mass of the oscillating particle, and $\hbar$ is Planck's constant divided by 2π.

So we want to find a such that $\hat{H}e^{-ax^2} = \lambda e^{-ax^2}$. Now

$$\begin{aligned}\hat{H}e^{-ax^2} = \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{kx^2}{2}\right)e^{-ax^2} &= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}e^{-ax^2} + \frac{kx^2}{2}e^{-ax^2} \\ &= -\frac{\hbar^2}{2m}\left(\frac{d}{dx}\left(-2axe^{-ax^2}\right)\right) + \frac{kx^2e^{-ax^2}}{2} \\ &= -\frac{\hbar^2}{2m}\left((-2ax)(-2axe^{-ax^2}) + (-2a)(-2axe^{-ax^2})\right) + \frac{kx^2e^{-ax^2}}{2} \\ &= e^{-ax^2}\left(\frac{a\hbar^2}{m} + x^2\left(\frac{k}{2} - \frac{2a^2\hbar^2}{m}\right)\right).\end{aligned}$$

Now, in order for e^{-ax^2} to be an eigenfunction, we must have that

$$\frac{k}{2} - \frac{2a^2\hbar^2}{m} = 0,$$

or

$$a = \frac{\sqrt{km}}{\hbar}.$$

In this case, we have

$$\hat{H}e^{-ax^2} = \frac{a\hbar^2}{m}e^{-ax^2} = \hbar\sqrt{\frac{k}{m}}e^{-x^2\sqrt{km}/\hbar},$$

so the eigenvalue associated with this eigenfunction is $\hbar\sqrt{k/m}$.

B.13 Section 2.7 Exercises

1. Consider the set of positive integers $\mathbb{Z}_7^* = \{1,2,3,4,5,6\}$. Let the binary operation $*$ be defined by, for $a, b \in \mathbb{Z}_7^*$, $a * b = ab \bmod 7$, i.e., the remainder when the product of a and b is divided by 7. Verify that this is a group under this operation. What is the identity element? Give the group table for this group. Comment on whether or not this represents the same group as D_3 discussed in the text.

The identity element is 1. The group table is given below:

$*$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

This is not the same group as D_3, as this group is abelian (commutative) whereas D_3 is not.

2. List the symmetry elements of the water molecule and give the table of the corresponding point group.

The water molecule is V-shaped, all lying in one plane. The symmetry elements are an axis of rotation C_2, a plane of reflection σ (through the oxygen nucleus) and another plane of reflection (the plane the molecule lies in) σ'.

The group table is given below:

	e	C_2	σ	σ'
e	e	C_2	σ	σ'
C_2	C_2	e	σ'	σ
σ	σ	σ'	e	C_2
σ'	σ'	σ	C_2	e

3. Find 3×3 matrices that represent the following symmetry operations.

a) $C_8(x)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\pi/4 & -\sin\pi/4 \\ 0 & \sin\pi/4 & \cos\pi/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

b) $S_4(z)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\pi/2 & -\sin\pi/2 & 0 \\ \sin\pi/2 & \cos\pi/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

c) σ_{xz}

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

d) $C_6(y)$

$$\begin{bmatrix} \cos\pi/3 & 0 & \sin\pi/3 \\ 0 & 1 & 0 \\ -\sin\pi/3 & 0 & \cos\pi/3 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ -\sqrt{3}/2 & 0 & 1/2 \end{bmatrix}$$

B.14 Section 3.1 Exercises

1. Evaluate the following integrals:

a) $\displaystyle\int_0^1 \ln(x^2+1)\,dx$

Let $u = \ln(x^2+1)$ and $dv = dx$. Then $du = (2x/x^2+1)dx$ and $v = x$. Then we have

$$\begin{aligned}\int_0^1 \ln(x^2+1)\,dx = x\ln(x^2+1)\Big|_0^1 - \int_0^1 \frac{2x^2}{x^2+1}\,dx = x\ln(x^2+1)\Big|_0^1 - 2\int_0^1 \frac{x^2+1-1}{x^2+1}\,dx \\ = x\ln(x^2+1)\Big|_0^1 - 2\int_0^1\left(1-\frac{1}{x^2+1}\right)dx = \left(x\ln(x^2+1)-2x+2\tan^{-1}x\right)\Big|_0^1 \\ = \ln 2 - 2 + 2\tan^{-1}1 - (0-0+2\tan^{-1}0 = \ln 2 + \frac{\pi}{4} - 2.\end{aligned}$$

b) $\displaystyle\int \arcsin(x)\,dx$

Let $u = \arcsin x$ and $dv = dx$. Then $du = \dfrac{1}{\sqrt{1-x^2}}dx$ and $v = x$. Then

$$\int \arcsin(x)\,dx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}du = x\arcsin x + \sqrt{1-x^2} + C,$$

where the last integral is computed using $u = 1-x^2$.

c) $\displaystyle\int 9t^2\sin(3t)\,dt$

Let $u = 9t^2$ and $dv = \sin(3t)dt$. Then $du = 18t$ and $v = -\cos(3t)/3$ and we have

$$\int 9t^2\sin(3t)\,dt = -\frac{9t^2\cos(3t)}{3} + \frac{18}{3}\int t\cos(3t)\,dt.$$

For the second integral, use $u = t$ and $dv = \cos(3t)dt$, so $du = dt$ and $v = \sin(3t)/3$. Then we have

$$\begin{aligned}\int 9t^2\sin(3t)\,dt = -\frac{9t^2\cos(3t)}{3} + \frac{18}{3}\int t\cos(3t)\,dt = -3t^2\cos(3t) + 6\left(\frac{t\sin(3t)}{3} - \frac{1}{3}\int \sin(3t)\,dt\right) \\ = -\frac{9t^2\cos(3t)}{3} + 2t\sin(3t) + \frac{2}{3}\cos(3t) + C\end{aligned}$$

d) $\displaystyle\int (\ln x)^3\,dx$

Let $u = (\ln x)^3$ and $dv = dx$. Then $du = \dfrac{3(\ln x)^2}{x} dx$ and $v = x$. Then we have

$$\int (\ln x)^3\, dx = x(\ln x)^3 - 3\int (\ln x)^2\, dx.$$

Using the same idea but now $u = (\ln x)^2$, we have

$$\int (\ln x)^3\, dx = x(\ln x)^3 - 3\left(x(\ln x)^2 - 2\int \ln x\, dx\right).$$

Finally, using $u = \ln x$ and $dv = dx$ we have

$$\int (\ln x)^3\, dx = x(\ln x)^3 - 3\left(x(\ln x)^2 - 2\,(x\ln x - x)\right) + C.$$

e) $\displaystyle\int (t+1)e^{5t}\, dt$

Let $u = t+1$ and $dv = e^{5t}\, dt$, then $du = 1$ and $v = \dfrac{1}{5}e^{5t}$. Thus

$$\int (t+1)e^{5t}\, dt = \frac{(t+1)e^{5t}}{5} - \frac{1}{5}\int e^{5t}\, dt = \frac{(t+1)e^{5t}}{5} - \frac{1}{25}e^{5t} + C.$$

f) $\displaystyle\int_1^e \frac{\ln x^2}{x^2}\, dx$

Let $u = \ln x^2 = 2\ln x$ and $dv = dx/x^2$. Then $du = 2dx/x$ and $v = -1/x$. Then we have

$$\int_1^e \frac{\ln x^2}{x^2}\, dx = -\frac{2\ln x}{x}\Big|_1^e + \int_1^e \frac{2}{x^2} dx = \left(-\frac{2\ln x}{x} - \frac{2}{x}\right)\Big|_1^e = \left(-\frac{2\ln e}{e} - \frac{2}{e}\right) - \left(-\frac{2\ln 1}{1} - \frac{2}{1}\right) = 2 - \frac{4}{e}.$$

g) $\displaystyle\int \theta \sec^2\theta\, d\theta$

Let $u = \theta$ and $dv = \sec^2\theta\, d\theta$. Then $du = d\theta$ and $v = \tan\theta$. Then we have

$$\int \theta\sec^2\theta\, d\theta = \theta\tan\theta - \int \tan\theta\, d\theta.$$

Note that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ has an antiderivative that can be found easily by u-substitution. If $u = \cos\theta$ then $du = -\sin\theta\, d\theta$ so

$$\int \tan\theta\, d\theta = \int \frac{\sin\theta}{\cos\theta}\, d\theta = -\ln|\cos\theta| + C,$$

thus

$$\int \theta\sec^2\theta\, d\theta = \theta\tan\theta + \ln|\cos\theta| + C.$$

h) $\displaystyle\int_0^{\sqrt[4]{3}} y\tan^{-1} y^2\, dy$

Let $u = \tan^{-1} y^2$ and $dv = y\,dy$. Then

$$du = \frac{2y}{1+y^4}, \qquad v = \frac{1}{2}y^2.$$

So we have

$$\int_0^{\sqrt[4]{3}} y \tan^{-1} y^2 \, dy = \frac{y^2 \tan^{-1} y^2}{2}\bigg|_0^{\sqrt[4]{3}} - \int_0^{\sqrt[4]{3}} \frac{y^3}{1+y^4}\, dy.$$

The integral on the right can be evaluated using $u = 1 + y^4$, so $du = 4y^3\,dy$ and the new limits of integration are from 1 to $1 + (\sqrt[4]{3})^4 = 1 + 3 = 4$:

$$\begin{aligned}\int_0^{\sqrt[4]{3}} y \tan^{-1} y^2 \, dy &= \frac{y^2 \tan^{-1} y^2}{2}\bigg|_0^{\sqrt[4]{3}} - \int_0^{\sqrt[4]{3}} \frac{y^3}{1+y^4}\, dy \\ &= \frac{y^2 \tan^{-1} y^2}{2}\bigg|_0^{\sqrt[4]{3}} - \frac{1}{4}\int_1^4 \frac{1}{u}\, du = \frac{\sqrt{3}\tan^{-1}(\sqrt{3})}{2} - \frac{1}{4}\ln|u|\bigg|_1^4 = \frac{\sqrt{3}}{2}\cdot\frac{\pi}{3} - \frac{1}{4}\ln 4.\end{aligned}$$

2. In the study of periodic (wave-like) functions, definite integrals of the form

$$I = \int_0^\pi \theta \cos(n\theta)\, d\theta$$

must often be evaluated for some positive integer n. Find a formula for I in terms of n.

Let $u = \theta$ and $dv = \cos(n\theta)\,d\theta$. Then $du = d\theta$ and $v = \frac{1}{n}\sin(n\theta)$ and we have

$$I = \int_0^\pi \theta\cos(n\theta)\,d\theta = \frac{\theta\sin(n\theta)}{n}\bigg|_0^\pi - \int_0^\pi \sin(n\theta)\,d\theta = \left(\frac{\theta\sin(n\theta)}{n} + \frac{\cos(n\theta)}{n}\right)\bigg|_0^\pi = \begin{cases} \dfrac{1}{n}, & n \text{ is even,} \\[2ex] \dfrac{-1}{n}, & n \text{ is odd.}\end{cases}$$

3. The **hyperbolic sine** and **hyperbolic cosine** functions $\sinh x$ and $\cosh x$ are defined by the following formulas:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Using these definitions, find $\frac{d}{dx}(\sinh x)$ and $\frac{d}{dx}(\cosh x)$ in terms of each other. Use this information to evaluate the integral

$$\int \cosh x \sinh x \, dx.$$

$$\begin{aligned}\frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x \\ \frac{d}{dx}(\cosh x) &= \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x\end{aligned}$$

Then, using $u = \cosh x$, $du = \sinh x\,dx$, we have

$$\int \cosh x \sinh x\,dx = \frac{1}{2}\cosh^2 x + C.$$

B.15 Section 3.2 Exercises

1. Evaluate the following integrals:

a) $\int e^{-3x}\cos(4x)\,dx$

Using #96 in the table, we have

$$\int e^{-3x}\cos(4x)\,dx = \frac{1}{9+16}e^{-3x}\left(4\sin(4x)-3\cos(4x)\right)+C.$$

b) $\int \frac{1}{x^2\sqrt{4x-9}}\,dx$

Using #32 in the table with $a=-9$ and $b=4$, we can reduce this integral to

$$\int \frac{1}{x^2\sqrt{4x-9}}\,dx = -\frac{\sqrt{4x-9}}{-9x} - \frac{4}{-18}\int \frac{1}{x\sqrt{4x-9}}\,dx = \frac{\sqrt{4x-9}}{9x} + \frac{2}{9}\int \frac{1}{x\sqrt{4x-9}}\,dx.$$

Now using #27 for the remaining integral, we have

$$\int \frac{1}{x^2\sqrt{4x-9}}\,dx = \frac{\sqrt{4x-9}}{9x} + \frac{2}{9}\int \frac{1}{x\sqrt{4x-9}}\,dx = \frac{\sqrt{4x-9}}{9x} + \frac{2}{9}\left(\frac{2}{3}\tan^{-1}\sqrt{\frac{4x-9}{9}}\right)+C.$$

c) $\int \frac{\sqrt{9-4x}}{x^2}\,dx$

Using #29 in the table, we have

$$\int \frac{\sqrt{9-4x}}{x^2}\,dx = -\frac{\sqrt{9-4x}}{x} - \frac{4}{2}\int \frac{1}{x\sqrt{9-4x}}\,dx,$$

and again using #27 we have

$$\int \frac{\sqrt{9-4x}}{x^2}\,dx = -\frac{\sqrt{9-4x}}{x} - \frac{2}{3}\ln\left|\frac{\sqrt{9-4x}-3}{\sqrt{9-4x}+3}\right|+C.$$

d) $\int \frac{\cos^{-1}\sqrt{x}}{\sqrt{x}}\,dx$

First let $u=\sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}dx$, so $2du = dx/\sqrt{x}$. Then this integral can be found by #101 in the table:

$$\int \frac{\cos^{-1}\sqrt{x}}{\sqrt{x}}\,dx = 2\int \cos^{-1}u\,du = 2u\cos^{-1}u - 2\sqrt{1-u^2}+C = 2\sqrt{x}\cos^{-1}\sqrt{x} - 2\sqrt{1-x}+C.$$

e) $\displaystyle\int \frac{\sqrt{x-x^2}}{x}\,dx$
This is #68 with $a = 1/2$:

$$\int \frac{\sqrt{x-x^2}}{x}\,dx = \sqrt{x-x^2} + \frac{1}{2}\sin^{-1}(2x-1) + C.$$

f) $\displaystyle\int \frac{1}{t\sqrt{4+(\ln t)^2}}\,dt$
Let $u = \ln t$, then $du = dt/t$. This integral becomes

$$\int \frac{1}{t\sqrt{4+(\ln t)^2}}\,dt = \int \frac{1}{\sqrt{4+u^2}}\,du.$$

This is #38 in the table, so we have

$$\int \frac{1}{t\sqrt{4+(\ln t)^2}}\,dt = \int \frac{1}{\sqrt{4+u^2}}\,du = \ln\left(u+\sqrt{4+u^2}\right) + C = \ln\left(\ln t + \sqrt{4+(\ln t)^2}\right) + C.$$

g) $\displaystyle\int \frac{x^2}{\sqrt{x^2-4x+5}}\,dx$
First, we complete the square on the quadratic inside the radical:

$$x^2 - 4x + 5 = x^2 - 4x + 4 - 4 + 5 = (x-2)^2 + 1.$$

Then, letting $u = x-2$, so $x = u+2$, we have

$$\int \frac{x^2}{\sqrt{x^2-4x+5}}\,dx = \int \frac{x^2}{\sqrt{(x-2)^2+1}}\,dx = \int \frac{(u+2)^2}{\sqrt{u^2+1}}\,du = \int \frac{u^2+4u+4}{\sqrt{u^2+1}}\,du.$$

Splitting the last one into three different integrals, we have, using #38, #39, and a substitution of $v = u^2+1$ so $dv = 2u\,du$,

$$\begin{aligned}
\int \frac{x^2}{\sqrt{x^2-4x+5}}\,dx &= \int \frac{u^2}{\sqrt{u^2+1}}\,du + \int \frac{4u}{\sqrt{u^2+1}}\,du + \int \frac{4}{\sqrt{u^2+1}}\,du \\
&= \frac{u}{2}\sqrt{u^2+1} - \frac{1}{2}\ln\left(u+\sqrt{u^2+1}\right) + 4\sqrt{u^2+1} + 4\ln\left(u+\sqrt{u^2+1}\right) + C \\
&= \left(\frac{u}{2}+4\right)\sqrt{u^2+1} + \frac{7}{2}\ln\left(u+\sqrt{u^2+1}\right) + C \\
&= \left(\frac{x-2}{2}+4\right)\sqrt{x^2-4x+5} + \frac{7}{2}\ln\left((x-2)+\sqrt{x^2-4x+5}\right) + C.
\end{aligned}$$

h) $\displaystyle\int \frac{3x^2+7x-2}{x^3-x^2-2x}\,dx$
First we can factor the denominator:

$$x^3 - x^2 - 2x = x(x^2-x-2) = x(x-2)(x+1).$$

Using partial fraction decomposition, we have that

$$\frac{3x^2+7x-2}{x^3-x^2-2x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1} = \frac{A(x^2-x-2)+B(x^2+x)+C(x^2-2x)}{x^3-x^2-2x}$$

which gives the three equations

$$\begin{aligned} A+B+C &= 3 \\ -A+B-2C &= 7 \\ -2A &= -2 \end{aligned}$$

Since $A = 1$ we have that $B + C = 2$ and $B - 2C = 8$, so $3C = -6$, or $C = -2$, which means $B = 4$. Thus we have

$$\int \frac{3x^2+7x-2}{x^3-x^2-2x}\,dx = \int \left(\frac{1}{x} + \frac{4}{x-2} - \frac{2}{x+1}\right) dx = \ln|x| + 4\ln|x-2| - 2\ln|x+1| + C.$$

2. The work w done by a gas as its volume V changes is given by $w = \int P\,dV$ where P is the pressure. Calculate the work done in increasing the volume of a van der Waals gas where

$$\left(P - \frac{an^2}{V^2}\right)(V - nb) = nRT,$$

assuming all other variables are constant with respect to P and V. (Hint: solve for P in terms of V.)

First, solving for P, we have

$$P = \frac{nRT}{V-nb} + \frac{an^2}{V^2}.$$

Then

$$\int P\,dV = \int \left(\frac{nRT}{V-nb} + \frac{an^2}{V^2}\right) dV = nRT\ln|V-nb| - \frac{an^2}{V} + C.$$

3. The square-well potential for the interaction of two spherically symmetric molecules separated by a distance r is given by

$$u(r) = \begin{cases} \infty & 0 < r < \sigma \\ -\varepsilon & \sigma < r < \lambda\sigma \\ 0 & r > \lambda\sigma \end{cases},$$

where σ, λ, and ε are constants that are characteristic of the molecule. The second virial coefficient of imperfect gas theory is given by

$$B(T) = -2\pi \int_0^\infty \left(e^{-u(r)/(k_B T)} - 1\right) r^2\,dr,$$

where k_B is the Boltzmann constant and T is the kelvin temperature. Derive an expression for $B(T)$ (i.e., evaluate the integral).

Substituting in the values for $u(r)$, we have that

$$\begin{aligned}
B(T) &= -2\pi\left(\int_0^{\sigma}\left(e^{-\infty/(k_B T)}-1\right)r^2\,dr+\int_{\sigma}^{\lambda\sigma}\left(e^{\varepsilon/(k_B T)}-1\right)r^2\,dr+\int_{\lambda\sigma}^{\infty}\left(e^{0}-1\right)r^2\,dr\right)\\
&= -2\pi\left(\int_0^{\sigma}(0-1)\,r^2\,dr+\int_{\sigma}^{\lambda\sigma}\left(e^{\varepsilon/(k_B T)}-1\right)r^2\,dr+\int_{\lambda\sigma}^{\infty}0\,dr\right)\\
&= -2\pi\left(-\int_0^{\sigma}r^2\,dr+\int_{\sigma}^{\lambda\sigma}\left(e^{\varepsilon/(k_B T)}-1\right)r^2\,dr\right) = -2\pi\left(-\frac{r^3}{3}\bigg|_0^{\sigma}+\frac{\left(e^{\varepsilon/(k_B T)}-1\right)r^3}{3}\bigg|_{\sigma}^{\lambda\sigma}\right)\\
&= -2\pi\left(-\frac{\sigma^3}{3}+\frac{\left(e^{\varepsilon/(k_B T)}-1\right)\left((\lambda\sigma)^3-\sigma^3\right)}{3}\right).
\end{aligned}$$

B.16 Section 3.3 Exercises

1. Evaluate the following limits:

a) $\lim\limits_{x\to 0^+} x\ln(x)$

This gives an indeterminate form of the type $0\cdot\infty$. We can rewrite this by moving the x into the denominator:

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} \overset{0/0}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+}(-x) = 0.$$

b) $\lim\limits_{x\to(\pi/2)^-} \left(x-\frac{\pi}{2}\right)\tan(x)$

This is also of the type $0\cdot\infty$ as $\tan x \to +\infty$ as $x \to (\pi/2)^-$. We have

$$\lim_{x\to(\pi/2)^-} \left(x-\frac{\pi}{2}\right)\tan(x) = \lim_{x\to(\pi/2)^-} \frac{x-\pi/2}{\cot(x)} \overset{0/0}{=} \lim_{x\to(\pi/2)^-} \frac{1}{-\csc^2(x)} = \frac{1}{-1} = -1.$$

c) $\lim\limits_{x\to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

This is of the form $\infty-\infty$. We first get a common denominator, then apply L'Hospital's Rule:

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) = \lim_{x\to 1}\left(\frac{x-1}{(x-1)\ln x} - \frac{\ln x}{(x-1)\ln x}\right) = \lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x} \overset{0/0}{=} \lim_{x\to 1}\frac{1-\frac{1}{x}}{\frac{x-1}{x}+\ln x}$$
$$= \lim_{x\to 1}\frac{x-1}{x-1+x\ln x} \overset{0/0}{=} \lim_{x\to 1}\frac{1}{1+1+\ln x} = \frac{1}{2}.$$

d) $\lim\limits_{x\to\infty}\left(xe^{1/x}-x\right)$

This is another $\infty-\infty$. Here we will first factor the x out, then see that it becomes an $\infty\cdot 0$. Then we will rewrite and use L'Hospital's Rule.

$$\lim_{x\to\infty}\left(xe^{1/x}-x\right) = \lim_{x\to\infty} x\left(e^{1/x}-1\right) = \lim_{x\to\infty}\frac{e^{1/x}-1}{\frac{1}{x}} \overset{0/0}{=} \lim_{x\to\infty}\frac{e^{1/x}\frac{-1}{x^2}}{\frac{-1}{x^2}} \lim_{x\to\infty} e^{1/x} = e^0 = 1.$$

2. Evaluate the following integrals or explain why they are divergent:

a) $\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx$

Make the substitution $u = \ln x$. Then $du = \dfrac{1}{x}\,dx$ and if $x = 2$, then $u = \ln 2$, and if $x \to \infty$, we have

$u \to \ln(\infty) \to \infty$.

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x \ln x}\, dx &= \int_{\ln 2}^{\infty} \frac{1}{u}\, du \\
&= \lim_{b\to\infty} \int_{\ln 2}^{b} \frac{1}{u}\, du \\
&= \lim_{b\to\infty} \ln|u| \Big|_{\ln 2}^{b} \\
&= \lim_{b\to\infty} \ln|b| - \ln|\ln 2| = \infty.
\end{aligned}$$

Thus the integral diverges.

b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+5}$

Note that, since

$$x^2+2x+5 = x^2+2x+1-1+5 = (x+1)^2+4,$$

that the integrand is always defined (the denominator is never zero). Also note that we can use table entry #33 to find the antiderivative. Then we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+5} &= \int_{-\infty}^{0} \frac{dx}{(x+1)^2+4} + \int_{0}^{\infty} \frac{dx}{(x+1)^2+4} \\
&= \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{(x+1)^2+4} + \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{(x+1)^2+4} \\
&= \lim_{a\to-\infty} \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)\Big|_a^0 + \lim_{b\to\infty} \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)\Big|_0^b \\
&= \frac{1}{2}\left(\lim_{a\to-\infty}\left(\tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{a+1}{2}\right)\right) + \lim_{b\to\infty}\left(\tan^{-1}\left(\frac{b+1}{2}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right)\right) \\
&= \frac{1}{2}\left(\lim_{a\to-\infty}\left(-\tan^{-1}\left(\frac{a+1}{2}\right)\right) + \lim_{b\to\infty}\left(\tan^{-1}\left(\frac{b+1}{2}\right)\right)\right) \\
&= \frac{1}{2}\left(-\left(\frac{-\pi}{2}\right) + \frac{\pi}{2}\right) = \frac{\pi}{2}.
\end{aligned}$$

c) $\displaystyle\int_0^{\ln 3} \frac{e^y}{(e^y-1)^{2/3}}\, dy$

First, a substitution is found by letting $u = e^y - 1$, then $du = e^y\, dy$. If $y = 0$, then $u = e^0 - 1 = 0$, and if $y = \ln 3$, then $u = e^{\ln 3} - 1 = 3 - 1 = 2$. Then we have

$$\int_0^{\ln 3} \frac{e^y}{(e^y-1)^{2/3}}\, dy = \int_0^2 \frac{1}{u^{2/3}}\, du.$$

This integrand is undefined at the lower limit of integration. Thus we have

$$\begin{aligned}
\int_0^{\ln 3} \frac{e^y}{(e^y-1)^{2/3}}\,dy &= \int_0^2 \frac{1}{u^{2/3}}\,du \\
&= \lim_{a\to 0^+} \int_a^2 \frac{1}{u^{2/3}}\,du \\
&= \lim_{a\to 0^+} \frac{u^{1/3}}{1/3}\bigg|_a^2 \\
&= \lim_{a\to 0^+} \left(3(2^{1/3}) - 3a^{1/3}\right) \\
&= 3\sqrt[3]{2} - 0 = 3\sqrt[3]{2}.
\end{aligned}$$

d) $\displaystyle\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$

Note that the integrand is undefined at the upper limit. So we have, using #49 in the table,

$$\begin{aligned}
\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx &= \lim_{b\to 3^-} \int_0^b \frac{1}{\sqrt{9-x^2}}\,dx \\
&= \lim_{b\to 3^-} \sin^{-1}\left(\frac{x}{3}\right)\bigg|_0^b \\
&= \lim_{b\to 3^-} \left(\sin^{-1}\left(\frac{b}{3}\right) - \sin^{-1}\left(\frac{0}{3}\right)\right) \\
&= \sin^{-1}(1) - \sin^{-1}(0) \\
&= \frac{\pi}{2} - 0 = \frac{\pi}{2}.
\end{aligned}$$

3. Use the Direct Comparison Test to determine if the integral $\displaystyle\int_0^\infty \frac{e^{-x}}{1+x^2}\,dx$ converges or diverges.

Note that for all $x \ge 1$, $e^{-x} \le 1$, and $\frac{1}{1+x^2} \le \frac{1}{x^2}$, so together we have

$$\frac{e^{-x}}{1+x^2} \le \frac{1}{1+x^2} \le \frac{1}{x^2}.$$

Now we know that $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges $(p=2)$, so the Direct Comparison Test implies that $\displaystyle\int_1^\infty \frac{e^{-x}}{1+x^2}\,dx$ also converges. Since

$$\int_0^\infty \frac{e^{-x}}{1+x^2}\,dx = \int_0^1 \frac{e^{-x}}{1+x^2}\,dx + \int_1^\infty \frac{e^{-x}}{1+x^2}\,dx,$$

and the integral $\displaystyle\int_0^1 \frac{e^{-x}}{1+x^2}\,dx$ is finite because $\displaystyle\frac{e^{-x}}{1+x^2} \le 1$ for $0 \le x \le 1$, we have that $\displaystyle\int_0^\infty \frac{e^{-x}}{1+x^2}\,dx$ converges.

4. In gas kinetic theory, the probability that a molecule of mass m in a gas at temperature T has speed v is given by

$$f(v) = 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2k_B T}\right),$$

where k_B is the Boltzmann constant. The mean speed is given by

$$\overline{v} = \int_0^\infty v f(v)\, dv = 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_0^\infty v^3 e^{-mv^2/(2k_B T)}\, dv.$$

Calculate $\overline{v}$. (**Hint:** let $u = -mv^2/(2k_B T)$, pull as many constants out of the integral as possible, and you should be left with something like $\int ue^u\, du$.)

Using the hint, we have that if $u = -mv^2/(2k_B T)$, then $du = -mv/(k_B T)\, dv$, or

$$v\, dv = \frac{-k_B T}{m}\, du.$$

Also we have that $v^2 = (-2k_B T/m)u$. Note that if $v = 0$, then $u = 0$, and if $v \to \infty$, then $u \to infty$. Then we have

$$\begin{aligned}
\overline{v} &= 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_0^\infty v^3 e^{-mv^2/(2k_B T)}\, dv \\
&= 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_0^\infty v^2 \left(e^{-mv^2/(2k_B T)} v\right) dv \\
&= 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_0^{-\infty} \left(\frac{-2k_B T}{m} u\right) e^u \left(\frac{-k_B T}{m}\right) du \\
&= 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} \left(\frac{2k_B^2 T^2}{m^2}\right) \int_0^{-\infty} ue^u\, du \\
&= 4\sqrt{\frac{2k_B T}{\pi m}} \lim_{b \to -\infty} \left(ue^u - e^u\right)\Big|_0^b \\
&= 4\sqrt{\frac{2k_B T}{\pi m}} \left(\lim_{b \to -\infty} \left(be^b - e^b\right) - 0 + e^0\right) \\
&= 4\sqrt{\frac{2k_B T}{\pi m}} \left(1 + \lim_{b \to -\infty} be^b - 0\right).
\end{aligned}$$

Now as $b \to -\infty$, $ue^u \to -\infty \cdot 0$, so we rewrite and use L'Hospital's Rule:

$$\lim_{b \to -\infty} be^b = \lim_{b \to -\infty} \frac{b}{e^{-b}} \overset{\frac{\infty}{\infty}}{=} \lim_{b \to -\infty} \frac{1}{-e^{-b}} = \lim_{b \to -\infty} \left(-e^b\right) = 0.$$

Thus we have

$$\overline{v} = 4\sqrt{\frac{2k_B T}{\pi m}} \left(1 + \lim_{b \to -\infty} be^b - 0\right) = 4\sqrt{\frac{2k_B T}{\pi m}}.$$

B.17 Section 3.4 Exercises

1. Determine whether each sequence converges or diverges. If it converges, find the limit.

a) $a_n = n - \sqrt{n^2 - 1}$

Let's check $\lim_{x\to\infty} f(x)$ where $f(n) = a_n$. We have

$$\lim_{x\to\infty}\left(x - \sqrt{x^2-1}\right) = \lim_{x\to\infty}\left(x - \sqrt{x^2-1}\right)\cdot\frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = \lim_{x\to\infty}\frac{x^2-(x^2-1)}{x+\sqrt{x^2-1}} = \lim_{x\to\infty}\frac{1}{x+\sqrt{x^2-1}} = 0.$$

So the sequence converges to a limit of 0.

b) $a_n = \dfrac{(-1)^n}{0.9^n}$

Note that

$$a_n = \frac{(-1)^n}{0.9^n} = \left(-\frac{1}{0.9}\right)^n$$

and since $1/0.9 > 1$, this sequence will diverge.

c) $a_n = \dfrac{n^2+4n+1}{\sqrt{4n^4+1}}$

Again, let's examine $\lim_{x\to\infty} f(x)$ where $f(n) = a_n$. We have

$$\lim_{x\to\infty}\frac{x^2+4x+1}{\sqrt{4x^4+1}} = \lim_{x\to\infty}\frac{x^2+4x+1}{\sqrt{4x^4+1}}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty}\frac{x^2+4x+1}{\sqrt{4x^4+1}}\cdot\frac{\frac{1}{x^2}}{\frac{1}{\sqrt{x^4}}} = \lim_{x\to\infty}\frac{1+\frac{4}{x}+\frac{1}{x^2}}{\sqrt{4+\frac{1}{x^4}}} = \frac{1}{\sqrt{4}} = \frac{1}{2}.$$

Thus the sequence converges to a limit of $1/2$.

2. Determine whether each series converges or diverges. If it converges and it is possible, find its sum.

a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{3^{n+1}}$

We need to get this into the form of a geometric series. First, we want to re-index so the index starts at 0. Let $k = n-1$, then $n = k+1$ and we have

$$\sum_{n=1}^{\infty}\frac{(-2)^n}{3^{n+1}} = \sum_{k=0}^{\infty}\frac{(-2)^{k+1}}{3^{k+2}} = \sum_{k=0}^{\infty}\frac{(-2)(-2)^k}{9\cdot 3^k} = \sum_{k=0}^{\infty}\frac{-2}{9}\left(\frac{-2}{3}\right)^k = \left(\frac{-2}{9}\right)\frac{1}{1-\frac{-2}{3}} = \left(\frac{-2}{9}\right)\frac{1}{\frac{5}{3}} = \frac{-2}{9}\cdot\frac{3}{5} = \frac{-2}{15}.$$

b) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n 5^{3-n}$

We have

$$\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n 5^{3-n} = \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n\frac{5^3}{5^n} = \sum_{n=0}^{\infty}5^3\left(\frac{1}{20}\right)^n = \frac{5^3}{1-\frac{1}{20}} = \frac{5^3}{\frac{19}{20}} = \frac{20\cdot 125}{19} = \frac{2500}{19}.$$

c) $\displaystyle\sum_{n=1}^{\infty}\left(\arcsin\left(\frac{1}{n}\right)-\arcsin\left(\frac{1}{n+1}\right)\right)$

This is a telescoping series:

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\arcsin\left(\frac{1}{n}\right)-\arcsin\left(\frac{1}{n+1}\right)\right)\\
&\qquad=\left(\arcsin\left(\frac{1}{1}\right)-\arcsin\left(\frac{1}{2}\right)\right)+\left(\arcsin\left(\frac{1}{2}\right)-\arcsin\left(\frac{1}{3}\right)\right)+\left(\arcsin\left(\frac{1}{3}\right)-\arcsin\left(\frac{1}{4}\right)\right)+\cdots\\
&\qquad=\arcsin\left(\frac{1}{1}\right)-\cancel{\arcsin\left(\frac{1}{2}\right)}+\cancel{\arcsin\left(\frac{1}{2}\right)}-\cancel{\arcsin\left(\frac{1}{3}\right)}+\cancel{\arcsin\left(\frac{1}{3}\right)}-\cancel{\arcsin\left(\frac{1}{4}\right)}+\cdots\\
&\qquad=\arcsin 1=\frac{\pi}{2}.
\end{aligned}$$

d) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{(3n+1)(3n+4)}$

Let's decompose $\dfrac{1}{(3n+1)(3n+4)}$ using partial fractions first. We have

$$\frac{1}{(3n+1)(3n+4)}=\frac{A}{3n+1}+\frac{B}{3n+4}=\frac{3An+4A+3Bn+B}{(3n+1)(3n+4)}.$$

Thus $3A+3B=0$ (i.e., $A=-B$) and $4A+B=1$. Then $-3B=1$ so $B=-1/3$ and $A=1/3$. Thus we have, since $3n+4=3(n+1)+1$

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{(3n+1)(3n+4)} &= \sum_{n=0}^{\infty}\left(\frac{1/3}{3n+1}-\frac{1/3}{3n+4}\right)\\
&= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{3n+1}-\frac{1}{3(n+1)+1}\right)\\
&= \frac{1}{3}\left(\frac{1}{1}-\frac{1}{4}+\frac{1}{4}-\frac{1}{7}+\frac{1}{7}-\frac{1}{10}+\cdots\right)\\
&= \frac{1}{3}.
\end{aligned}$$

e) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{16n^2+8n-3}$

Again let's find the partial fraction decomposition of the summand:

$$\frac{1}{16n^2+8n-3}=\frac{1}{(4n-1)(4n+3)}=\frac{A}{4n-1}+\frac{B}{4n+3}=\frac{4An+3A+4Bn-B}{(4n-1)(4n+3)}.$$

So again $A = -B$ and then $-4B = 1$ so $B = -1/4$ and $A = 1/4$. Thus we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{16n^2+8n-3} &= \sum_{n=0}^{\infty} \left(\frac{1/4}{4n-1} - \frac{1/4}{4n+3}\right) \\
&= \frac{1}{4}\sum_{n=0}^{\infty} \left(\frac{1}{4n-1} - \frac{1}{4(n+1)-1}\right) \\
&= \frac{1}{4}\left(\frac{1}{-1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{7} + \frac{1}{7} - \frac{1}{11} + \cdots\right) \\
&= -\frac{1}{4}.
\end{aligned}$$

f) $1 + \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi^3} + \cdots$
This is a geometric series:

$$1 + \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi^3} + \cdots = \sum_{n=0}^{\infty} \left(\frac{1}{\pi}\right)^n = \frac{1}{1-\frac{1}{\pi}} = \frac{\pi}{\pi - 1}.$$

3. Give an example of a sequence that is bounded but that does not converge. Give an example of a sequence that is monotonic but that does not converge. Can you give an example of a sequence that converges but is not monotonic or bounded?

A sequence that is bounded but does not converge is

$$\{\cos(\pi n)\}_{n=1}^{\infty} = \begin{cases} 1 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$$

A sequence that is monotonic but does not converge is $\{n+1\}_{n=1}^{\infty} = 2, 3, 4, 5, \ldots$.

While it is possible for a non-monotonic sequence to converge (an example is $\{(\cos(\pi n))/n\}_{n=1}^{\infty}$), it is not possible for an unbounded sequence to converge.

4. Let the Fibonacci sequence be denoted by $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. Show that

$$\frac{1}{f_{n-1}f_{n+1}} = \frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}}.$$

Then use this fact to prove that

$$\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1}f_{n+1}} = 2.$$

We use the definition $f_{n+1} = f_n + f_{n-1}$ for $n \geq 3$ to simplify the right hand side:

$$\begin{aligned}
\frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}} &= \frac{f_{n+1}}{f_{n-1}f_n f_{n+1}} - \frac{f_{n-1}}{f_{n-1}f_n f_{n+1}} \\
&= \frac{f_{n+1} - f_{n-1}}{f_{n-1}f_n f_{n+1}} \\
&= \frac{\left(f_n + f_{n-1}\right) - f_{n-1}}{f_{n-1}f_n f_{n+1}} \\
&= \frac{f_n}{f_{n-1}f_n f_{n+1}} \\
&= \frac{1}{f_{n-1}f_{n+1}}.
\end{aligned}$$

Now we have

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1}f_{n+1}} &= \frac{f_2}{f_1 f_3} + \frac{f_3}{f_2 f_4} + \frac{f_4}{f_3 f_5} + \frac{f_5}{f_4 f_6} + \cdots + \frac{f_n}{f_{n-1}f_{n+1}} + \cdots \\
&= \left(\frac{f_2}{f_1 f_2} - \frac{f_2}{f_2 f_3}\right) + \left(\frac{f_3}{f_2 f_3} - \frac{f_3}{f_3 f_4}\right) + \left(\frac{f_4}{f_3 f_4} - \frac{f_4}{f_4 f_5}\right) + \left(\frac{f_5}{f_4 f_5} - \frac{f_5}{f_5 f_6}\right) + \cdots \\
&= \left(\frac{1}{f_1} - \frac{1}{f_3}\right) + \left(\frac{1}{f_2} - \frac{1}{f_4}\right) + \left(\frac{1}{f_3} - \frac{1}{f_5}\right) + \left(\frac{1}{f_4} - \frac{1}{f_6}\right) + \cdots \\
&= \frac{1}{f_1} - \cancel{\frac{1}{f_3}} + \frac{1}{f_2} - \cancel{\frac{1}{f_4}} + \cancel{\frac{1}{f_3}} - \cancel{\frac{1}{f_5}} + \cancel{\frac{1}{f_4}} - \cancel{\frac{1}{f_6}} + \cdots \\
&= \frac{1}{f_1} + \frac{1}{f_2} = 1 + 1 = 2.
\end{aligned}$$

5. The energy of a quantum-mechanical harmonic oscillator is given by $\varepsilon_n = \left(n + \frac{1}{2}\right) h\nu$, $n = 0, 1, 2, \ldots$, where h is the Planck constant and ν is the fundamental frequency of the oscillator. The average vibrational energy of a harmonic oscillator is given by

$$\varepsilon_{\text{vib}} = \left(1 - e^{-h\nu/(k_B T)}\right) \sum_{n=0}^{\infty} \varepsilon_n e^{-nh\nu/(k_B T)},$$

where k_B is the Boltzmann constant and T is the kelvin temperature. Show that

$$\varepsilon_{\text{vib}} = \frac{h\nu}{2} + \frac{h\nu e^{-h\nu/(k_B T)}}{1 - e^{-h\nu/(k_B T)}}.$$

First we split the series into two parts based on the definition of ε_n:

$$\begin{aligned}
\varepsilon_{\text{vib}} &= \left(1-e^{-h\nu/(k_B T)}\right)\sum_{n=0}^{\infty}\varepsilon_n e^{-nh\nu/(k_B T)} \\
&= \left(1-e^{-h\nu/(k_B T)}\right)\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)h\nu e^{-nh\nu/(k_B T)} \\
&= \left(1-e^{-h\nu/(k_B T)}\right)\left(\sum_{n=0}^{\infty}nh\nu e^{-nh\nu/(k_B T)}+\sum_{n=0}^{\infty}\frac{1}{2}h\nu e^{-nh\nu/(k_B T)}\right) \\
&= \left(1-e^{-h\nu/(k_B T)}\right)\sum_{n=0}^{\infty}nh\nu e^{-nh\nu/(k_B T)}+\left(1-e^{-h\nu/(k_B T)}\right)\sum_{n=0}^{\infty}\frac{1}{2}h\nu e^{-nh\nu/(k_B T)}
\end{aligned}$$

Note that since $h\nu/(k_B T)$ is a positive number, the series on the right above is a convergent geometric series as $e^{-nh\nu/(k_B T)} < 1$ if $n > 0$. Thus we have, with $a = \frac{1}{2}h\nu$ and $r = e^{-h\nu/(k_B T)}$,

$$\begin{aligned}
\left(1-e^{-h\nu/(k_B T)}\right)\sum_{n=0}^{\infty}\frac{1}{2}h\nu e^{-nh\nu/(k_B T)} &= \left(1-e^{-h\nu/(k_B T)}\right)\sum_{n=0}^{\infty}\frac{1}{2}h\nu\left(e^{-h\nu/(k_B T)}\right)^n \\
&= \left(1-e^{-h\nu/(k_B T)}\right)\left(\frac{\frac{1}{2}h\nu}{1-e^{-h\nu/(k_B T)}}\right) \\
&= \frac{h\nu}{2}.
\end{aligned}$$

For the first series, we need a new formula:

$$\sum_{n=0}^{\infty} nar^n = \frac{ar}{(1-r)^2}, \qquad |r| < 1.$$

(In two sections we will see how this is found.) Then we have

$$\begin{aligned}
\left(1-e^{-h\nu/(k_B T)}\right)\sum_{n=0}^{\infty}nh\nu e^{-nh\nu/(k_B T)} &= \left(1-e^{-h\nu/(k_B T)}\right)\sum_{n=0}^{\infty}nh\nu\left(e^{-h\nu/(k_B T)}\right)^n \\
&= \left(1-e^{-h\nu/(k_B T)}\right)\left(\frac{h\nu e^{-h\nu/(k_B T)}}{\left(1-e^{-h\nu/(k_B T)}\right)^2}\right) \\
&= \frac{h\nu e^{-h\nu/(k_B T)}}{1-e^{-h\nu/(k_B T)}},
\end{aligned}$$

which, when combined with the result above, gives the value of ε_{vib}.

B.18 Section 3.5 Exercises

1. The vibrational partition function of a diatomic molecule is given by the series

$$q(T) = \sum_{n=0}^{\infty} e^{-\left(n+\frac{1}{2}\right)h\nu/(k_B T)},$$

where h is the Planck constant, ν is the frequency of the oscillator, k_B is the Boltzmann constant, and T is the kelvin temperature. Does this geometric series converge, and if it converges, what is its sum?

It does converge, as $\left(n+\frac{1}{2}\right)h\nu/(k_B T)$ is always a positive number for $n \geq 0$. We have

$$\begin{aligned} q(T) &= \sum_{n=0}^{\infty} e^{-\left(n+\frac{1}{2}\right)h\nu/(k_B T)} \\ &= \sum_{n=0}^{\infty} e^{-h\nu/(2k_B T)} \left(e^{-h\nu/(k_B T)}\right)^n \\ &= e^{-h\nu/(2k_B T)} \left(\frac{1}{1-e^{-h\nu/(k_B T)}}\right) \\ &= \frac{e^{-h\nu/(2k_B T)}}{1-e^{-h\nu/(k_B T)}}. \end{aligned}$$

2. Determine if the following series are absolutely convergent, conditionally convergent, or divergent:

a) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

All terms are nonnegative, so convergence = absolute convergence. This is a candidate for the Integral Test, as $f(x) = xe^{-x^2} > 0$ and $f'(x) = (1-x^2)e^{-x^2} < 0$ for $x > 1$. We have

$$\int_1^{\infty} xe^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b xe^{-x^2}\,dx = \lim_{b\to\infty} -\frac{1}{2}e^{-x^2}\Big|_1^b = \lim_{b\to\infty} -\frac{1}{2}e^{-b^2} + \frac{1}{2}e^{-1} = \frac{1}{2e}.$$

Since the integral converges, so does the series.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

All terms are nonnegative, so convergence = absolute convergence. This is a good candidate for the Ratio Test:

$$\begin{aligned} \lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n} \\ &= \lim_{n\to\infty}\frac{2\cdot 2^n\cdot n!}{(n+1)\cdot n!\cdot 2^n} \\ &= \lim_{n\to\infty}\frac{2}{n+1} = 0. \end{aligned}$$

Since the limit is less than 1, the series converges.

c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(4n)}{n^{3/2}}$

This series has some negative terms, so we need to first check for absolute convergence. Note that

$$\left|\frac{\sin(4n)}{n^{3/2}}\right| \le \frac{1}{n^{3/2}}$$

for all n, and since $\sum \frac{1}{n^{3/2}}$ converges (p-series with $p > 1$), we have by the Direct Comparison Test, that $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\sin(4n)}{n^{3/2}}\right|$ converges, so $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(4n)}{n^{3/2}}$ is absolutely convergent.

d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3}$

This series has some negative terms, so we need to first check for absolute convergence, which means we analyze $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^3}$. Now if $n \ge 1$, note that $e^{1/n} \le e^1 = e$, so we have

$$\frac{e^{1/n}}{n^3} \le \frac{e}{n^3}.$$

Now $\displaystyle\sum_{n=1}^{\infty} \frac{e}{n^3} = e \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p-series, $p > 1$), so by the Direct Comparison Test we have that $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^3}$ converges, which implies that the original series converges absolutely.

e) $\displaystyle\sum_{n=4}^{\infty} \frac{2}{n^2 - 10}$

All terms are nonnegative, so convergence = absolute convergence. Note that the Direct Comparison Test with $2/n^2$ doesn't work as $n^2 - 10 < n^2$. However, we can use the Limit Comparison Test with $b_n = 2/n^2$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2}{n^2-10}}{\frac{2}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2 - 10} = 1.$$

Since the limit is finite and positive, both series behave the same way, and since $\sum \frac{2}{n^2}$ converges ($p > 1$) we have that $\displaystyle\sum_{n=4}^{\infty} \frac{2}{n^2 - 10}$ converges.

f) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^3}$

All terms are nonnegative, so convergence = absolute convergence. Note that if $n \ge 1$, $\sqrt[n]{n} \le n$, so

$$\frac{\sqrt[n]{n}}{n^3} \le \frac{n}{n^3} = \frac{1}{n^2} \qquad \text{for } n \ge 1.$$

Now $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p = 2$), so by the Direct Comparison Test, we have that $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^3}$ converges.

g) $\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$

All terms are nonnegative, so convergence = absolute convergence. We can use the Limit Comparison Test with $b_n = 1/n$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{1+\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{1+\ln n} \overset{\infty/\infty}{=} \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1.$$

Since the limit is finite and positive, both series behave in the same way, and since $\sum \frac{1}{n}$ diverges (harmonic series), then $\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$ diverges as well.

h) $\sum_{n=1}^{\infty} \frac{\ln(n^2)}{n^2}$

All terms are nonnegative, so convergence = absolute convergence. This appears to be a candidate for the Integral Test. Note that, if $f(x) = \ln(x^2)/x^2 = 2\ln(x)/x^2$, we have $f(x) > 0$ for $x > 1$ and

$$f'(x) = \frac{2x - 4x\ln x}{x^4} = \frac{2x(1-2\ln x)}{x^4} < 0 \qquad \text{for } x > e^{1/2}.$$

Now, using $u = \ln x$ and $dv = dx/x^2$, integration by parts yields

$$\begin{aligned}
\int_1^{\infty} f(x)\,dx &= 2\lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx \\
&= 2\lim_{b\to\infty} \left(-\frac{\ln x}{x}\Big|_1^b + \int_1^b \frac{1}{x^2}\,dx \right) \\
&= 2\lim_{b\to\infty} \left(-\frac{\ln b}{b} + \int_1^b \frac{1}{x^2}\,dx \right).
\end{aligned}$$

Now we know the integral on the right is convergent ($p = 2$), so the series converges if $\lim_{b\to\infty} -\frac{\ln b}{b}$ is finite. We have

$$-\lim_{b\to\infty} \frac{\ln b}{b} \overset{\infty/\infty}{=} -\lim_{b\to\infty} \frac{1}{b} = 0.$$

Thus by the Integral Test, the series converges.

B.19 Section 3.6 Exercises

1. Find the radius and interval of convergence for each of the following power series.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

Using the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2}\cdot\frac{n^2}{x^n}\right| = \lim_{n\to\infty}\left|\frac{xn^2}{n^2+2n+1}\right| = |x|\lim_{n\to\infty}\frac{n^2}{n^2+2n+1} = |x|.$$

So the power series converges whenever $|x| < 1$, i.e., whenever $-1 < x < 1$ (radius of convergence is 1). To check the left endpoint, we see that if $x = -1$, then

$$\sum_{n=1}^{\infty}\frac{x^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2},$$

which converges by the Alternating Series Test. If $x = 1$, then

$$\sum_{n=1}^{\infty}\frac{x^n}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2},$$

which converges (p-series, $p > 1$). Thus the interval of convergence is $[-1, 1]$.

b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-2x)^n}{\sqrt{n+3}}$

Using the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2x)^{n+1}}{\sqrt{n+4}}\cdot\frac{\sqrt{n+3}}{(-2x)^n}\right| = \lim_{n\to\infty}\left|-2x\sqrt{\frac{n+3}{n+4}}\right| = |2x|\lim_{n\to\infty}\sqrt{\frac{n+3}{n+4}} = |2x|.$$

Thus the series converges if $|2x| < 1$, or if $x < 1/2$, so the radius of convergence is $1/2$. To check the left endpoint ($x = -1/2$) we have

$$\sum_{n=0}^{\infty}\frac{(-2x)^n}{\sqrt{n+3}} = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n+3}},$$

which diverges by the Limit Comparison Test with $\sum \frac{1}{\sqrt{n}}$. The right endpoint $x = 1/2$ gives

$$\sum_{n=0}^{\infty}\frac{(-2x)^n}{\sqrt{n+3}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+3}},$$

which converges by the Alternating Series Test. Thus the interval of convergence is $(-1/2, 1/2]$.

2. Find power series representations for the following functions and determine the radius and interval of convergence for each.

a) $\dfrac{1}{2+x}$

Note that

$$\frac{1}{2+x}=\frac{1}{2\left(1+\frac{x}{2}\right)}=\frac{1}{2}\left(\frac{1}{1-\left(-\frac{x}{2}\right)}\right)=\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{n+1}}.$$

Using the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{2^{n+2}}\cdot\frac{2^{n+1}}{(-1)^n x^n}\right|=\left|\frac{x}{2}\right|.$$

This converges when $\left|\frac{x}{2}\right|<1$ or $|x|<2$, so the radius of convergence is 2. If $x=-2$, we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{n+1}}=\sum_{n=0}^{\infty}\frac{(-1)^n(-2)^n}{2^{n+1}}=\sum_{n=0}^{\infty}\frac{(-1)^n(-1)^n 2^n}{2^{n+1}}=\sum_{n=0}^{\infty}\frac{2^n}{2^{n+1}}=\sum_{n=0}^{\infty}\frac{1}{2},$$

which diverges. If $x=2$ we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{n+1}}=\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{2^{n+1}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2}=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n,$$

which also diverges. Thus the interval of convergence is $(-2,2)$.

b) $\dfrac{1}{1+5x^2}$

We have

$$\frac{1}{1+5x^2}=\frac{1}{1-(-5x^2)}=\sum_{n=0}^{\infty}(-5x^2)^n=\sum_{n=0}^{\infty}(-1)^n 5^n x^{2n}.$$

Using the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}5^{n+1}x^{2(n+1)}}{(-1)^n 5^n x^{2n}}\right|=\lim_{n\to\infty}\left|\frac{(-1)5^{n+1}x^{2n+2}}{5^n x^{2n}}\right|=\lim_{n\to\infty}\left|5x^2\right|=5x^2.$$

This converges when $5x^2<1$, or $-1<\sqrt{5}x<1$, or $-1/\sqrt{5}<x<1/\sqrt{5}$. When $x=-1/\sqrt{5}$ we have, since $(-1)^{2n}=1$,

$$\sum_{n=0}^{\infty}(-1)^n 5^n x^{2n}=\sum_{n=0}^{\infty}(-1)^n 5^n\left(\frac{-1}{\sqrt{5}}\right)^{2n}=\sum_{n=0}^{\infty}(-1)^n\left(\frac{5^n}{5^n}\right)=\sum_{n=0}^{\infty}(-1)^n,$$

which diverges. The same holds true for $x=1/\sqrt{5}$, so the interval of convergence is $(-1\sqrt{5},1/\sqrt{5})$.

c) $\dfrac{1}{x^2+2x}$

Completing the square in the denominator we have

$$\frac{1}{x^2+2x}=\frac{1}{x^2+2x+1-1}=\frac{-1}{1-(x+1)^2}=-\frac{1}{1-(x+1)^2}=-\sum_{n=0}^{\infty}(x+1)^{2n}.$$

Using the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+1)^{2(n+1)}}{(x+1)^{2n}}\right| = \lim_{n\to\infty}\left|(x+1)^2\right| = (x+1)^2.$$

This converges when $(x+1)^2 < 1$, or $-1 < x+1 < 1$, or $-2 < x < 0$ (the radius of convergence is 1). When $x = -2$ we have

$$\sum_{n=0}^{\infty}(x+1)^{2n} = \sum_{n=0}^{\infty}(-1)^{2n} = \sum_{n=0}^{\infty}1^n = \infty.$$

The same is true when $x = 0$. Thus the interval of convergence is $(-2, 0)$.

3. Find the first four terms of the Maclaurin series for $f(x) = \sqrt{1+x}$. Can you come up with a closed-form (i.e. summation) representation of the series?

To start, we compute $f(0)$ and the first few derivatives of f evaluated at 0:

$$\begin{aligned}
f(x) &= \sqrt{1+x} & f(0) &= 1\\
f'(x) &= \frac{1}{2\sqrt{1+x}} & f'(0) &= \frac{1}{2}\\
f''(x) &= \frac{-1}{2^2(1+x)^{3/2}} & f''(0) &= -\frac{1}{2^2}\\
f'''(x) &= \frac{1\cdot 3}{2^3(1+x)^{5/2}} & f'''(0) &= \frac{1\cdot 3}{2^3}\\
f^{(4)}(x) &= \frac{-1\cdot 3\cdot 5}{2^4(1+x)^{7/2}} & f^{(4)}(0) &= -\frac{1\cdot 3\cdot 5}{2^4}\\
&\vdots & &\vdots\\
f^{(n)}(x) &= \frac{(-1)^{n+1}\cdot 3\cdot 5\cdots(2n-3)}{2^n(1+x)^{(2n-1)/2}} & f^{(n)}(0) &= (-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}
\end{aligned}$$

Notice that the sign pattern for the first term is different than for the rest. Thus, we can give the Maclaurin series as

$$\sqrt{1+x} = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)x^n}{n!} = 1 + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-3)x^n}{n!2^n} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots.$$

(Note that this assumes the "empty" product $1\cdot 3\cdot 5\cdots(2n-3) = 1$ when $n = 1$. Otherwise just write the first two terms and start the series at $n = 2$.)

4. Describe (in words) two ways (other than using the definition) to obtain the Maclaurin series for $f(x) = \cos^2 x$, and compute the first four terms of the series.

One way this could be done would be by taking the series for $\cos x$ and squaring it:

$$\cos^2 x = \left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\right)^2 = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)^2.$$

Another way would be to use the identity $\cos^2 x = \dfrac{1+\cos 2x}{2}$:

$$\cos^2 x = \frac{1+\cos 2x}{2} = \frac{1}{2}\left(1+\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n}}{(2n)!}\right) = \frac{1}{2}\left(1+\sum_{n=0}^{\infty}(-1)^n\frac{4^n x^{2n}}{(2n)!}\right) = \frac{1}{2}\left(1+1-\frac{4x^2}{2!}+\frac{16x^4}{4!}-\frac{64x^6}{6!}+\cdots\right).$$

5. The indefinite integral $\displaystyle\int e^{-x^2}\,dx$ cannot be evaluated using traditional means. Find the Maclaurin series for the function $f(x) = e^{-x^2}$ and use it to derive a series representation of $\displaystyle\int f(x)\,dx$.

First, we find the series representation of $f(x) = e^{-x^2}$:

$$e^{-x^2} = \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots.$$

Then we have

$$\int e^{-x^2}\,dx = \int \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}\,dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)n!}$$

6. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{4^{2n+1}(2n+1)!}$. (Hint: this is a known power series evaluated at a particular value of x.)

Note that

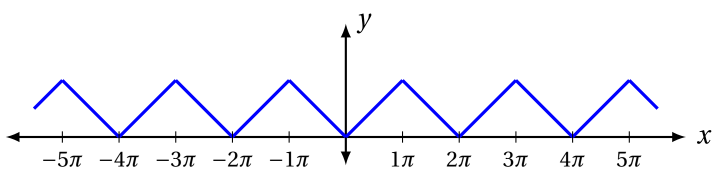
$$\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{\pi^{2n+1}}{4^{2n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{4}\right)^{2n+1} = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

B.20 Section 3.7 Exercises

1. Find the Fourier coefficients of the "triangular wave" function $f(x)$ given by

$$f(x) = \begin{cases} -x & \text{if } -\pi < x < 0, \\ x & \text{if } \quad 0 < x < \pi, \end{cases} \quad \text{and} \quad f(x+2\pi) = f(x).$$

Let's first look at the plot of this function:



Note that this is an even function, so the Fourier series will be a Fourier cosine series. Thus we need to find a_0 and a_n. We have

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ &= \frac{1}{2\pi}\left(\int_{-\pi}^{0}(-x)\,dx + \int_{0}^{\pi} x\,dx\right) \\ &= \frac{1}{2\pi}\left(\frac{-x^2}{2}\bigg|_{-\pi}^{0} + \frac{x^2}{2}\bigg|_{0}^{\pi}\right) \\ &= \frac{1}{2\pi}\left(0 - \frac{-\pi^2}{2} + \frac{\pi^2}{2} - 0\right) \\ &= \frac{1}{2\pi}\cdot\frac{2\pi^2}{2} = \frac{\pi}{2}. \end{aligned}$$

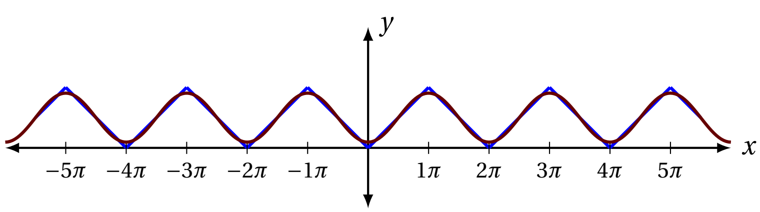
Now

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0}(-x\cos nx)\,dx + \int_{0}^{\pi} x\cos nx\,dx\right) \\
&= \frac{1}{\pi}\left[\left(\frac{-x\sin nx}{n} - \frac{\cos nx}{n^2}\right)\Big|_{-\pi}^{0} + \left(\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right)\Big|_{0}^{\pi}\right] \\
&= \frac{1}{\pi}\left[\left(-\frac{\cos 0}{n^2} + \frac{\cos(-n\pi)}{n^2}\right) + \left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right)\right] \\
&= \frac{1}{\pi}\left[\frac{2\cos n\pi - 2}{n^2}\right] \\
&= \begin{cases} -\frac{4}{n^2\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

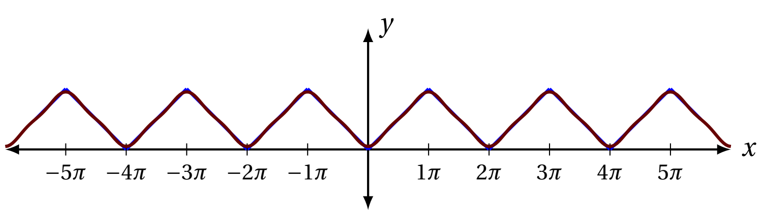
Using $2n-1$ to represent any odd number, we have

$$f(x) = \begin{cases} -x & \text{if } -\pi < x < 0, \\ x & \text{if } \quad 0 < x < \pi, \end{cases} \quad = \quad \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2\pi}\cos((2n-1)x).$$

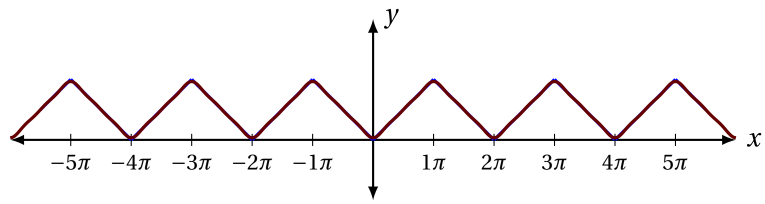
Below we have a plot of $f(x)$ (blue) with $S_m(x) = \frac{\pi}{2} - \sum_{n=1}^{m} \frac{4}{(2n-1)^2\pi}\cos((2n-1)x)$ (dark red) for the first few terms.



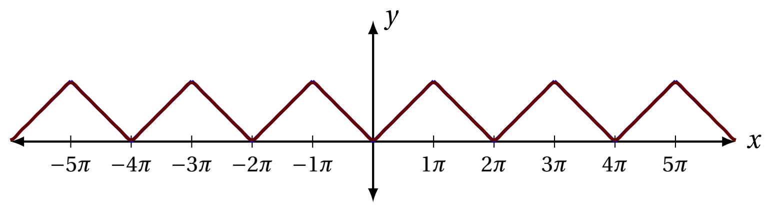
$S_1(x)$



$S_2(x)$



$S_3(x)$



$S_4(x)$

B.21 Section 4.1 Exercises

1. Find the indicated partial derivatives:

a) $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ where $f(x, y) = ae^{-b(x^2+y^2)} + c\sin(x^2 y)$

$$\frac{\partial f}{\partial x} = ae^{-b(x^2+y^2)}(-2xb) + c\cos(x^2 y)(2xy)$$
$$\frac{\partial f}{\partial y} = ae^{-b(x^2+y^2)}(-2yb) + c\cos(x^2 y)(x^2)$$

b) $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y^2}$ where $f(x, y) = \cos(x/y)$

First we find the first partials:

$$f_x = -\sin(x/y)\left(\frac{1}{y}\right) = \frac{-\sin(x/y)}{y}, \qquad f_y = -\sin(x/y)\left(-\frac{x}{y^2}\right) = \frac{x\sin(x/y)}{y^2}$$

Then we have

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{-\sin(x/y)}{y}\right) = -\frac{\cos(x/y)}{y}\cdot\frac{1}{y} = -\frac{\cos(x/y)}{y^2}$$
$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{-\sin(x/y)}{y}\right) = -\frac{y\cos(x/y)\left(\frac{-1}{y^2}\right) - \sin(x/y)}{y^2} = \frac{\cos(x/y) + y\sin(x/y)}{y^3}$$
$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{x\sin(x/y)}{y^2}\right) = \frac{y^2\cdot x\cos(x/y)\left(\frac{-x}{y^2}\right) - x\sin(x/y)(2y)}{y^4} = -\frac{x^2\cos(x/y) + 2xy\sin(x/y)}{y^4}$$

c) $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ where $yz = \ln(x+z)$

Let $F(x, y, z) = yz - \ln(x+z) = 0$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-\frac{1}{x+z}}{y - \frac{1}{x+z}} = \frac{1}{1 - y(x+z)}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{z}{y - \frac{1}{x+z}} = \frac{xz + z^2}{1 - y(x+z)}$$

d) $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ where $z = e^{x+2y}$, $x = s/t$, and $y = t/s$

By the chain rule, we have

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \left(e^{x+2y}\right)\left(\frac{1}{t}\right) + \left(2e^{x+2y}\right)\left(\frac{-t}{s^2}\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \left(e^{x+2y}\right)\left(\frac{-s}{t^2}\right) + \left(2e^{x+2y}\right)\left(\frac{1}{s}\right)$$

2. Test the following differentials for exactness:

a) $(4x+3y)\,dx+(3x+8y)\,dy$

$$\frac{\partial}{\partial y}\left(4x+3y\right)=3, \qquad \frac{\partial}{\partial x}\left(3x+8y\right)=3$$

so the differential is exact.

b) $y\cos x\,dx+\sin x\,dy$

$$\frac{\partial}{\partial y}\left(y\cos x\right)=\cos x, \qquad \frac{\partial}{\partial x}(\sin x)=\cos x$$

so the differential is exact.

c) $y\ln x\,dx+x\ln y\,dy$

$$\frac{\partial}{\partial y}\left(y\ln x\right)=\ln x, \qquad \frac{\partial}{\partial x}\left(x\ln y\right)=\ln y$$

so the differential is not exact.

3. For a gas obeying the van der Waals equation of state

$$\left(P+\frac{n^2a}{V^2}\right)(V-nb)=nRT,$$

Find the partials $\left(\dfrac{\partial V}{\partial P}\right)_{T,n}$, $\left(\dfrac{\partial V}{\partial T}\right)_{P,n}$, $\left(\dfrac{\partial T}{\partial P}\right)_{V,n}$, and $\left(\dfrac{\partial V}{\partial n}\right)_{P,T}$ (here R is always a constant).

$$\left(P+\frac{n^2a}{V^2}\right)(V-nb)=nRT \quad\Longrightarrow\quad P+\frac{n^2a}{V^2}=\frac{nRT}{V-nb} \quad\Longrightarrow\quad F(P,V,n,T)=P+\frac{n^2a}{V^2}-\frac{nRT}{V-nb}=0$$

$$\left(\frac{\partial V}{\partial P}\right)_{T,n}=-\frac{F_P}{F_V}=-\frac{1}{-\frac{2n^2a}{V^3}+\frac{nRT}{(V-nb)^2}}=\frac{V^3(V-nb)^2}{2n^2a(V-nb)^2-nRTV^3}$$

Note that the reciprocal of the above result can be used for F_V.

$$\left(\frac{\partial V}{\partial T}\right)_{P,n}=-\frac{F_T}{F_V}=-\frac{-nR}{V-nb}\cdot\frac{V^3(V-nb)^2}{2n^2a(V-nb)^2-nRTV^3}=\frac{nRV^3(V-nb)}{2n^2a(V-nb)^2-nRTV^3}$$

$$\left(\frac{\partial T}{\partial P}\right)_{V,n}=-\frac{F_P}{F_T}=-\frac{1}{\frac{-nR}{V-nb}}=\frac{V-nb}{nR}$$

$$\left(\frac{\partial V}{\partial n}\right)_{P,T}=-\frac{F_n}{F_V}=-\left(\frac{2na}{V^2}-\frac{(V-nb)(RT)-(nRT)(-b)}{(V-nb)^2}\right)\left(\frac{V^3(V-nb)^2}{2n^2a(V-nb)^2-nRTV^3}\right)$$

4. For a certain system, the thermodynamic energy U is given as a function of S, V, and n by

$$U(S,V,n) = Kn^{5/3}V^{-2/3}e^{2s/(3nR)},$$

where S is the entropy, V is the volume, n is the number of moles, K is a constant, and R is the gas constant. Find the differential dU in terms of dS, dV, and dn.

$$\begin{aligned} dU &= U_S\,dS + U_V\,dV + U_n\,dn \\ &= \left(Kn^{5/3}V^{-2/3}e^{2S/(3nR)}\left(\frac{2}{3nR}\right)\right)dS + \left(-\frac{2}{3}Kn^{5/3}V^{-5/3}e^{2S/(3nR)}\right)dV \\ &\quad + \left(\frac{5}{3}Kn^{2/3}V^{-2/3}e^{2S/(3nR)} + Kn^{5/3}V^{-2/3}e^{2S/(3nR)}\left(-\frac{2S}{(3nR)^2}(3R)\right)\right)dn \end{aligned}$$

5. If $z = f(x,y)$, $x = r\cos\theta$, and $y = r\sin\theta$, find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$ and show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$$

$$\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sin\theta\cos\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta$$

$$\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 r^2\sin^2\theta - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}r^2\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta$$

$$\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta$$

$$\begin{aligned} &\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 \\ &\quad = \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sin\theta\cos\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta + \left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta \\ &\quad = \left(\frac{\partial z}{\partial x}\right)^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} + \left(\frac{\partial z}{\partial y}\right)^2\underbrace{(\sin^2\theta + \cos^2\theta)}_{=1} = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \end{aligned}$$

6. Using the $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ you found in the previous exercise, find (in terms of r and θ and partials of z with respect to those variables) the partials $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$. Functions that satisfy the Laplace's equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

are known as **harmonic functions** and they arise in many applications in the physical sciences, including fluid flow, heat conduction, and gravitational and electrostatic potential theory.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right), \qquad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

We will find the second partials of z with respect to r and θ:

$$\begin{aligned}
\frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) \\
&= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right) \\
&= \cos\theta \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + \sin\theta \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) \\
&= \cos\theta \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial r}\right) + \sin\theta \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial r}\right) \\
&= \cos\theta \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right) + \sin\theta \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right) \\
&= \cos^2\theta \frac{\partial^2 z}{\partial x} + 2\sin\theta\cos\theta \frac{\partial^2 z}{\partial x \partial y} + \sin^2\theta \frac{\partial^2 z}{\partial y^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial \theta}\right) \\
&= \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)\right) \\
&= -r\sin\theta \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\right) - r\cos\theta \frac{\partial z}{\partial x} + r\cos\theta \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial y}\right) - r\sin\theta \frac{\partial z}{\partial y} \\
&= -r\sin\theta \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \theta}\right) - r\cos\theta \frac{\partial z}{\partial x} + r\cos\theta \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \theta}\right) - r\sin\theta \frac{\partial z}{\partial y} \\
&= -r\sin\theta \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)\right) - r\cos\theta \frac{\partial z}{\partial x} + r\cos\theta \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)\right) - r\sin\theta \frac{\partial z}{\partial y} \\
&= r^2\sin^2\theta \frac{\partial^2 z}{\partial x^2} - r^2\sin\theta\cos\theta \frac{\partial^2 z}{\partial x \partial y} - r\cos\theta \frac{\partial z}{\partial x} - r^2\sin\theta\cos\theta \frac{\partial^2 z}{\partial x \partial y} + r^2\cos^2\theta \frac{\partial^2 z}{\partial y^2} - r\sin\theta \frac{\partial z}{\partial y} \\
&= -r\underbrace{\left(\cos\theta \frac{\partial z}{\partial x} + \sin\theta \frac{\partial z}{\partial y}\right)}_{=\frac{\partial z}{\partial r}} + r^2\left(\sin^2\theta \frac{\partial^2 z}{\partial x^2} - 2\sin\theta\cos\theta \frac{\partial^2 z}{\partial x \partial y} + \cos^2\theta \frac{\partial^2 z}{\partial y^2}\right)
\end{aligned}$$

Note that

$$\frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2} = -\frac{1}{r}\frac{\partial z}{\partial r} + \left(\sin^2\theta\frac{\partial^2 z}{\partial x^2} - 2\sin\theta\cos\theta\frac{\partial^2 z}{\partial x\partial y} + \cos^2\theta\frac{\partial^2 z}{\partial y^2}\right)$$

Adding $\frac{\partial^2 z}{\partial r^2}$ gives

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2} = -\frac{1}{r}\frac{\partial z}{\partial r} + \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \qquad \Longrightarrow \qquad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}.$$

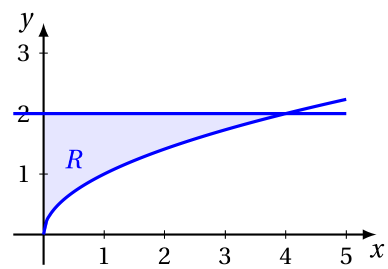
B.22 Section 4.2 Exercises

1. Evaluate the following multiple integrals:

a) $\displaystyle\int_0^2 \int_y^{2y} xy\,dx\,dy$

$$\int_0^2 \int_y^{2y} xy\,dx\,dy = \int_0^2 \left(\frac{x^2 y}{2}\right)\Big|_y^{2y} dy = \int_0^2 \left(\frac{(2y)^2 y}{2} - \frac{(y)^2 y}{2}\right) dy$$
$$= \int_0^2 \left(2y^3 - \frac{y^3}{2}\right) dy = \int_0^2 \frac{3y^3}{2}\,dy = \frac{3y^4}{8}\Big|_0^2 = \frac{3\cdot 2^4}{8} = 6.$$

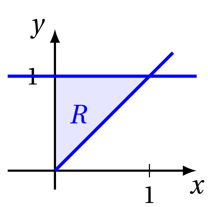
b) $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{1+y^3}\,dy\,dx$



Since there is no easy antiderivative of $\dfrac{1}{1+y^3}$, we first change the order of integration. If we integrate with respect to x first, the lower limit is 0 and the upper limit is $x = y^2$.

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{1+y^3}\,dy\,dx = \int_0^2 \int_0^{y^2} \frac{1}{1+y^3}\,dx\,dy = \int_0^2 \frac{y^2}{1+y^3}\,dy = \frac{1}{3}\ln|1+y^3|\Big|_0^2 = \frac{1}{3}\ln 9.$$

c) $\displaystyle\int_0^1 \int_x^1 e^{x/y}\,dy\,dx$



Since there is no easy antiderivative of $e^{x/y}$ with respect to y, we first change the order of integration. If we integrate with respect to x first, the lower limit is 0 and the upper limit is $x = y$. Then, letting $u = x/y$, we have that $du = dx/y$ so $y\,du = dx$.

$$\int_0^1 \int_x^1 e^{x/y}\,dy\,dx = \int_0^1 \int_0^y e^{x/y}\,dx\,dy = \int_0^1 ye^{x/y}\Big|_0^y dy = \int_0^1 y\left(e^1 - e^0\right) dy = (e-1)\,\frac{y^2}{2}\Big|_0^1 = \frac{e-1}{2}.$$

d) $\displaystyle\int_0^3 \int_0^1 \int_0^{\sqrt{1-z^2}} ze^y\,dx\,dz\,dy$

$$\int_0^3\int_0^1\int_0^{\sqrt{1-z^2}} ze^y\,dx\,dz\,dy = \int_0^3\int_0^1 xze^y\Big|_{x=0}^{x=\sqrt{1-z^2}}\,dz\,dy = \int_0^3\int_0^1 z\sqrt{1-z^2}e^y\,dz\,dy$$
$$= \left(\int_0^3 e^y\,dy\right)\left(\int_0^1 z\sqrt{1-z^2}\,dz\right) = \left(e^3-1\right)\left(-\frac{1}{2}\cdot\frac{2}{3}(1-z^2)^{3/2}\Big|_0^1\right) = \frac{e^3-1}{3}.$$

e) $\displaystyle\int_0^1\int_x^{2x}\int_0^y 2xyz\,dz\,dy\,dx$

$$\int_0^1\int_x^{2x}\int_0^y 2xyz\,dz\,dy\,dx = \int_0^1\int_x^{2x} xyz^2\Big|_{z=0}^{z=y}\,dy\,dx = \int_0^1\int_x^{2x} xy^3\,dy\,dx = \int_0^1 \frac{xy^4}{4}\Big|_{y=x}^{y=2x}\,dx$$
$$= \int_0^1 \frac{x}{4}\left(16x^4-x^4\right)\,dx = \int_0^1 \frac{15x^5}{4}\,dx = \frac{15}{24}x^6\Big|_0^1 = \frac{15}{24} = \frac{5}{8}.$$

f) $\displaystyle\int_0^1\int_0^z\int_0^{x+z} 6xz\,dy\,dx\,dz$

$$\int_0^1\int_0^z\int_0^{x+z} 6xz\,dy\,dx\,dz = \int_0^1\int_0^z 6xyz\Big|_{y=0}^{y=x+z}\,dx\,dz = \int_0^1\int_0^z 6x(x+z)z\,dx\,dz$$
$$= 6\int_0^1\int_0^z \left(x^2z+xz^2\right)\,dx\,dz = 6\int_0^1\left(\frac{x^3z}{3}+\frac{x^2z^2}{2}\right)\Big|_{x=0}^{x=z}\,dx\,dz = 6\int_0^1\left(\frac{z^4}{3}+\frac{z^4}{2}\right)\,dz$$
$$= 6\int_0^1 \frac{5z^4}{6}\,dz = z^5\Big|_0^1 = 1.$$

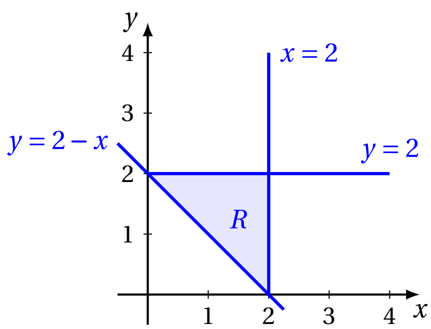
g) $\displaystyle\int_0^{\sqrt{\pi}}\int_0^x\int_0^{xz} x^2\sin y\,dy\,dz\,dx$

$$\int_0^{\sqrt{\pi}}\int_0^x\int_0^{xz} x^2\sin y\,dy\,dz\,dx = \int_0^{\sqrt{\pi}}\int_0^x \left(-x^2\cos y\right)\Big|_{y=0}^{y=xz}\,dz\,dx = \int_0^{\sqrt{\pi}}\int_0^x \left(-x^2(\cos xz-\cos 0)\right)\,dz\,dx$$
$$= \int_0^{\sqrt{\pi}}\int_0^x \left(x^2(1-\cos xz)\right)\,dz\,dx = \int_0^{\sqrt{\pi}}\left(x^2\left(z-\frac{\sin xz}{x}\right)\right)\Big|_{z=0}^{z=x}\,dx$$
$$= \int_0^{\sqrt{\pi}}\left(x^3-\underbrace{x\sin(x^2)}_{u=x^2,\,du=2x\,dx}\right)\,dx = \left(\frac{x^4}{4}+\frac{\cos(x^2)}{2}\right)\Big|_0^{\sqrt{\pi}} = \frac{\pi^2}{4}+\frac{\cos\pi}{2}-0-\frac{\cos 0}{2} = \frac{\pi^2}{4}-1.$$

h) $\displaystyle\int_1^e\int_1^x\int_0^{x+y} \frac{1}{x}\,dz\,dy\,dx$

$$\int_1^e \int_1^x \int_0^{x+y} \frac{1}{x}\,dz\,dy\,dx = \int_1^e \int_1^x \frac{z}{x}\Big|_{z=0}^{z=x+y} dy\,dx = \int_1^e \int_1^x \frac{x+y}{x}\,dy\,dx$$
$$= \int_1^e \frac{2xy+y^2}{2x}\Big|_{y=1}^{y=x} dx = \int_1^e \left(\frac{2x^2+x^2}{2x} - \frac{2x+1}{2x}\right) dx = \int_1^e \left(\frac{3}{2}x - 1 - \frac{1}{2x}\right) dx$$
$$= \left(\frac{3}{4}x^2 - x - \frac{1}{2}\ln x\right)\Big|_1^e = \left(\frac{3}{4}e^2 - e - \frac{1}{2}\right) - \left(\frac{3}{4} - 1\right) = \frac{3}{4}e^2 - e - \frac{1}{4}.$$

2. Let R be the region in the xy-plane bounded by the line $x = 2$, the line $y = 2$, and the line $y = 2 - x$. Set up the double integral $\iint_R f(x,y)\,dA$ two different ways.



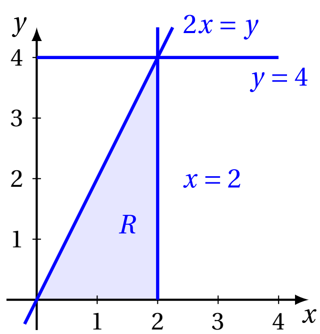
Integrating with respect to x first:

$$\iint_R f(x,y)\,dA = \int_0^2 \int_{2-y}^2 f(x,y)\,dx\,dy$$

Integrating with respect to y first:

$$\iint_R f(x,y)\,dA = \int_0^2 \int_{2-x}^2 f(x,y)\,dy\,dx$$

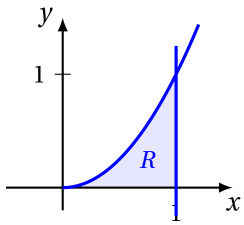
3. Change the order of integration of the integral $\int_0^4 \int_{y/2}^2 e^{x^2}\,dx\,dy$ and evaluate the integral.



We have

$$\int_0^4 \int_{y/2}^2 e^{x^2}\,dx\,dy = \int_0^2 \int_0^{2x} e^{x^2}\,dy\,dx$$
$$= \int_0^2 2xe^{x^2}\,dx = e^{x^2}\Big|_0^2 = e^4 - 1.$$

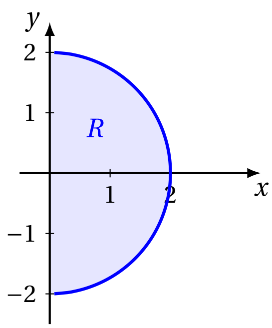
4. Evaluate $\iint_R x\cos y\,dA$ where R is the region bounded by $y = 0$, $y = x^2$, and $x = 1$.



We have

$$\iint_R x\cos y\, dA = \int_0^1 \int_0^{x^2} x\cos y\, dy\, dx$$
$$= \int_0^1 x\sin y\Big|_0^{x^2} dx = \int_0^1 x\sin x^2\, dx = -\frac{1}{2}\cos x^2\Big|_0^1$$
$$= -\frac{1}{2}(\cos 1 - 1).$$

5. Evaluate $\iint_R e^{-x^2-y^2}\, dA$ where R is the region bounded by the semicircle $x = \sqrt{4-y^2}$ and the y-axis.



Note that in polar coordinates, the region R is represented by the rectangle $0 \le r \le 2$, $-\pi/2 \le \theta \le \pi/2$. Thus we have

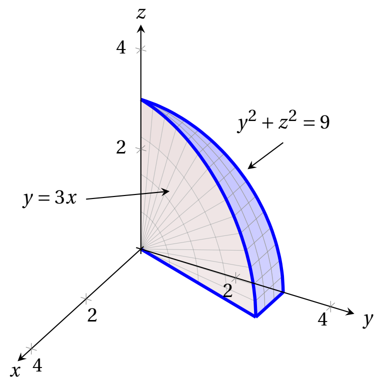
$$\iint_R e^{-x^2-y^2}\, dA = \int_{-\pi/2}^{\pi/2}\int_0^2 e^{-r^2}\, r\, dr\, d\theta$$
$$= \left(\int_{-\pi/2}^{\pi/2} d\theta\right)\left(\int_0^2 e^{-r^2}\, r\, dr\right) = \pi\left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^2 = \frac{\pi}{2}\left(1-e^{-4}\right).$$

6. Evaluate $\iiint_V y\, dV$ where V is the region bounded by the planes $x=0$, $y=0$, $z=0$, and $2x+2y+z=4$.

This integral is straightforward to set up, you simply need to use the plane equation to solve for the upper limit on the inner integrals:

$$\iiint_V y\, dV = \int_0^2\int_0^{2-x}\int_0^{4-2x-2y} y\, dz\, dy\, dx = \int_0^2\int_0^{2-x} yz\Big|_{z=0}^{z=4-2x-2y} dy\, dx$$
$$= \int_0^2\int_0^{2-x}\left(4y-2xy-2y^2\right) dy\, dx = \int_0^2\left(2y^2-xy^2-\frac{2y^3}{3}\right)\Big|_{y=0}^{y=2-x} dx$$
$$= \int_0^2\left(2(2-x)^2 - x(2-x)^2 - \frac{2(2-x)^3}{3}\right) dx = \int_0^2 \frac{4(2-x)^3}{3}\, dx$$
$$= -\frac{(2-x)^4}{3}\Big|_0^2 = -\frac{(2-2)^4}{3} + \frac{(2-0)^4}{3} = \frac{16}{3}.$$

7. Evaluate $\iiint_V z\, dV$ where V is the region bounded by the cylinder $y^2+z^2=9$ and the planes $x=0$, $y=3x$, and $z=0$ in the first octant.



Let R be the region in the first quadrant of the yz-plane inside the circle $y^2+z^2=9$. Then we have

$$\iiint_V z\,dV = \iint_R \left(\int_0^{y/3} z\,dx\right) dA = \frac{1}{3}\iint_R yz\,dA.$$

Using the change of variable $y = r\cos\theta$, $z = r\sin\theta$, we have

$$dA = dy\,dz = r\,dr\,d\theta,$$

thus

$$\begin{aligned}\iiint_V z\,dV = \frac{1}{3}\iint_R yz\,dA = \int_0^{\pi/2}\int_0^3 (r\cos\theta)(r\sin\theta)r\,dr\,d\theta &= \left(\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\right)\left(\int_0^3 r^3\,dr\right)\\ &= \left(-\frac{1}{2}\cos^2\theta\Big|_0^{\pi/2}\right)\left(\frac{r^4}{4}\Big|_0^3\right) = \frac{1}{2}\cdot\frac{3^4}{4} = \frac{81}{8}.\end{aligned}$$

8. Evaluate $\iiint_V xyz\,dV$ where V is the region between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=16$ and above the cone $z^2=x^2+y^2$.

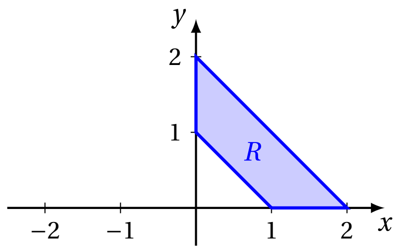
Using the change of variable to spherical coordinates, we see that $2\le\rho\le 4$, $0\le\theta\le 2\pi$, and, since the cone $z^2=x^2+y^2$ is represented by $\phi=\pi/4$, the region above the cone is $0\le\phi\le\pi/4$. Thus we have

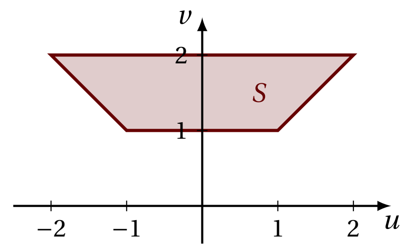
$$\begin{aligned}\iiint_V xyz\,dV &= \int_0^{2\pi}\int_0^{\pi/4}\int_2^4 (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\\ &= \left(\int_0^{2\pi}\cos\theta\sin\theta\,d\theta\right)\left(\int_0^{\pi/4}\sin^3\phi\cos\phi\,d\phi\right)\left(\int_2^4\rho^5\,d\rho\right) = \left(-\frac{1}{2}\cos^2\theta\Big|_0^{2\pi}\right)\left(\frac{\sin^4\phi}{4}\Big|_0^{\pi/4}\right)\left(\frac{\rho^6}{6}\Big|_2^4\right) = 0,\end{aligned}$$

as $\cos^2 0 = \cos^2 2\pi = 1$.

9. Evaluate $\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$ where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2)$, and $(0,1)$.

Here we will need to make a change of variables in order to produce an integral that we can evaluate. Let $u=y-x$ and $v=y+x$. First, note that the trapezoidal region is actually bounded by four lines: $y=1-x$, $y=2-x$, $x=0$, and $y=0$. Note that since we let $v=y+x$, then the lines $y=1-x$ and $y=2-x$ correspond to $v=1$ and $v=2$, respectively. Also, note that $y=\frac{1}{2}(u+v)$ and $x=\frac{1}{2}(v-u)$. Thus, the line $x=0$ corresponds to the line $v=u$, and $y=0$ corresponds to $u=-v$.





The determinant of the Jacobian of the transformation is given by

$$\det(\boldsymbol{J}(u,v)) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -1/2 & 1/2 \\ \\ 1/2 & 1/2 \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

Thus we have

$$\begin{aligned}\iint_R \cos\left(\frac{y-x}{y+x}\right) dA = \iint_S \cos\left(\frac{u}{v}\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA &= \int_1^2 \int_{-v}^{v} \cos\left(\frac{u}{v}\right)\left|-\frac{1}{2}\right| du\,dv \\ &= \frac{1}{2}\int_1^2 v\sin\left(\frac{u}{v}\right)\Big|_{u=-v}^{u=v} dv = \frac{1}{2}\int_1^2 v\,(\sin(1)-\sin(-1))\,dv \\ &= \sin(1)\int_1^2 v\,dv = \sin(1)\frac{v^2}{2}\Big|_1^2 = \sin(1)\left(\frac{4}{2}-\frac{1}{2}\right) = \frac{3\sin(1)}{2}.\end{aligned}$$

10. One way to view the triple integral over all of $\mathbb{R}^3$ is to view it as the triple integral over a sphere as the radius of the sphere goes to infinity. Using this idea, convert $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dx\,dy\,dz$ to an integral in spherical coordinates and use it to evaluate

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}\,e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz.$$

For a sphere of radius ρ, we have that $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$. Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Thus we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{x^2+y^2+z^2}\,e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\left(\rho e^{-\rho^2}\right)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin\phi\,d\phi\right)\left(\int_0^{\infty}\left(\rho e^{-\rho^2}\right)\rho^2\,d\rho\,d\phi\,d\theta\right) = (2\pi)\left(-\cos\phi\Big|_0^{\pi}\right)\left(-\frac{1}{2}e^{(}-\rho^2)(\rho^2+1)\Big|_0^{\infty}\right)$$

$$= 2\pi\,(1+1)\left(-\frac{1}{2}\,(0-1)\right) = 2\pi.$$